- Algebras, modules, and functors

~ I. Rings and algebras

Def^n: A ring is an additive group R with bilinear (mult) oper $\cdot : R \times R \to R$ and unit $1 \in R$ for which

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) \quad \text{and} \quad 1 \cdot a = a = a \cdot 1$$

at all a, b, c in R.

Note that bilinearity gives

$$\begin{cases} a \cdot (b + b') = ab + ab' \\ (a + a') \cdot b = a \cdot b + a' \cdot b . \end{cases}$$

We call a ring R <u>commutative</u> if $a \cdot b = b \cdot a$ for all a, b in R.

Def^n: A ring homomorphism / map is an additive group map $f : R \to S$ for which $f(a \cdot b) = f(a) f(b)$ at all a, b in R and $f(1_R) = 1_S$.

Def^n: For a ring R the opposite ring R^{op} is the same additive group w/ multiplication $a \cdot_{op} b := b \cdot a$ and unit $1_{R^{op}} = 1_R$.

Def^n: The <u>center</u> of a ring $Z(R)$ is the collection of all $a \in R$ for which $a \cdot b = b \cdot a$ at all b in R. We call a ring map $f : S \to R$

from a commutative ring S <u>central</u> if $f(S) \subseteq Z(R)$.

Def$^{\underline{n}}$: For a fixed commutative ring k, a k-algebra R is a ring R equipped with a central ring map $\mathrm{unit}_R : k \to R$. A map of k-algebras $f : R \to S$ is a ring map for which the diagram

$$\begin{tikzcd} R \arrow[rr, "f"] & & S \\ & k \arrow[ul, "\mathrm{unit}_R"] \arrow[ur, "\mathrm{unit}_S"'] & \end{tikzcd}$$

commutes.

~ I $\frac{1}{3}$ Ideals and quotient rings.

Def$^{\underline{n}}$: An ideal $I \subseteq A$ in a ring A is an additive subgroup for which

$$a \cdot m \text{ and } m \cdot a \in I \quad \text{whenever } m \in I,\ a \in A.$$

Lemma 1: For any ideal $I \subseteq A$ the additive quotient A/I inherits a unique ring structure under which the quotient map $\pi : A \to A/I$ is a ring map. Furthermore, for any ring map $\phi : A \to B$ with $I \subseteq \ker(\phi)$, there is a unique ring map $\bar{\phi} : A/I \to B$

which completes a diagram

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[dr, "\pi"'] & B \\ & A/I \arrow[u, dashed, "\exists ?\, \bar{\phi}"'] \end{tikzcd}$$

.

Construction/Proof: For $\bar{a} = a + I$ ($= \pi(a)$), in A/I we claim that the product $\bar{a}\cdot\bar{b} := \overline{a\cdot b}$ is well defined, i.e. independent of choice of rep's a and b in A. This is clear since I is stable under the action of A on the right and left. The unit in A/I is $1_{A/I} = \overline{1_A}$.

As for $\bar{\phi} : A/I \to B$, we have such a unique map of additive groups, and for $\bar{a}, \bar{b} \in A/I$ we check the products $\bar{\phi}(\bar{a}\cdot\bar{b}) = \bar{\phi}(\overline{ab}) = \bar{\phi}(\pi(ab))$
$= \phi(ab) = \phi(a)\cdot\phi(b) = \bar{\phi}(\bar{a})\cdot\bar{\phi}(\bar{b})$
and also $\bar{\phi}(\bar{1}) = \phi(1) = 1$. So $\bar{\phi}$ is a ring map. ∎

Lemma 2: The kernel of any ring map $\phi : A \to B$ is an ideal $\ker(\phi) \subseteq A$.

Def^{n}: For elements $a_1, \dots, a_n \in A$ the ideal generated by $a_1, \dots, a_n$ is the smallest ideal in A which contains the given elements, i.e. the intersection $\subseteq I$
$(a_1, \dots, a_n) = \bigcap_{\lambda \in \Lambda} I_\lambda$ over the set of ideals I w/ $\{a_1, \dots, a_n\}$.

Lemma 3: For any collection of elements $a_1, \ldots, a_n \in A$ we have

$$(a_1, \ldots, a_n) = \sum_{i=1}^{n} A \cdot a_i \cdot A = \{\sum_{i=1}^{n} b_i a_i b_i' : b_i, b_i' \text{ in } A\}.$$

When A is commutative

$$(a_1, \ldots, a_n) = \sum_{i=1}^{n} A \cdot a_i = \{\sum_{i=1}^{n} b_i a_i : b_i \text{ in } A\}.$$

Example: For any field k, all ideals in $k[x]$ are of the form $(p) \subseteq k[x]$, for p a polynomial in $k[x]$, and $(p) = \{q \in k[x] : p \text{ divides } q\}$. This follows by the division algorithm.

Example [HW]: For any field k, the only ideals I in $M_n(k)$ are 0, and $M_n(k)$ itself.

~ I 2/3 Products of rings

For any collection of rings A_λ, indexed by a set Λ, the additive product $\prod_{\lambda \in \Lambda} A_\lambda$ admits a ring structure w/ product

$$(a_\lambda : \lambda \in \Lambda) \cdot (b_\lambda : \lambda \in \Lambda) = (a_\lambda \cdot b_\lambda : \lambda \in \Lambda)$$

and unit $1 = (1_{A_\lambda} : \lambda \in \Lambda)$.

Note that we have ring maps

$p_\mu : \prod_{\lambda \in \Lambda} A_\lambda \to A_\mu$, $p_\mu(a_\lambda : \lambda \in \Lambda) = a_\mu$

at each μ in Λ.

Exercise: Any collection of ring maps $\phi_\lambda : A \to A_\lambda$, across all λ in Λ, determines a unique ring map $\phi : A \to \prod_{\lambda \in \Lambda} A_\lambda$ for which $p_\lambda \circ \phi = \phi_\lambda$ at each λ in Λ.

Question: Is the inclusion $i_0 : A \to A \times A$, $i_0(a) = (a, 0)$, a ring map?

~ II. Categories and functors

A category is a class of objects $\mathrm{obj}\,\mathcal{C}$ which is paired w/ a set of morphisms $Hom_{\mathcal{C}}(x,y)$ at each x, y in $\mathcal{C}$ and associative composition operations

$\circ : Hom_{\mathcal{C}}(x_1, x_2) \times Hom_{\mathcal{C}}(x_0, x_1) \to Hom_{\mathcal{C}}(x_0, x_2)$

We also require the specification of identity morphisms $id_x : x \to x$ at each x in $\mathcal{C}$ for which $f \circ id_x = f = id_y \circ f$ at each $f \in Hom_{\mathcal{C}}(x,y)$.

A functor between categories $F : \mathcal{C} \to \mathcal{D}$

is a choice of set map $obj(F) : obj(\mathbb{C}) \to obj(\mathbb{D})$ and, for each x, y in $\mathbb{C}$, a set map

$$F_{x,y} : Hom_{\mathbb{C}}(x,y) \to Hom_{\mathbb{D}}(F(x), F(y))$$

for which

$$F(g \circ f) = F(g) \circ F(f)$$

at each sequence $x \xrightarrow{f} y \xrightarrow{g} z$ in $\mathbb{C}$.

Remark: Obv. we've abused notation to take $F = obj(F)$ and/or $F = F_{xy}$ when convenient.

Def^n: For $F, F' : \mathbb{C} \to \mathbb{D}$ functors, a natural transformation $\xi : F \to F'$ is a collection of maps $\xi_x : F(x) \to F'(x)$ in $\mathbb{D}$, at each x in $\mathbb{C}$, for which the diagram

$$\begin{tikzcd} F(x) \arrow[r, "\xi_x"] \arrow[d, "F(f)"'] & F'(x) \arrow[d, "F'(f)"] \\ F(y) \arrow[r, "\xi_y"'] & F'(y) \end{tikzcd}$$

commutes, at each f in $\mathbb{C}$.

Note that we can compose natural transformations, in the apparent way, so that we have a category $Fun(\mathbb{C}, \mathbb{D})$ of functors and transformations.

Def^n: A morphism $f : x \to y$ in a category $\mathbb{C}$ is called an isomorphism if there exists $g : y \to x$

for which $g \circ f = id_X$ and $f \circ g = id_Y$.

Lemma 4: A natural transformation $\xi : F \to F'$ between functors is a natural isomorphism, i.e. an isom in the cat $Fun(\mathcal{C}, \mathcal{D})$, if and only if each $\xi_x : F(x) \to F'(x)$ is an isomorphism in $\mathcal{D}$.

Proof: Exercise. ∎

Def^n: A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be an equivalence of categories if there exists some functor $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms

$$\xi : G \circ F \xrightarrow{\sim} id_{\mathcal{C}} \quad , \quad \xi' : F \circ G \xrightarrow{\sim} id_{\mathcal{D}} .$$

I will call a functor $F : \mathcal{C} \to \mathcal{D}$ an isomorphism of categories if there exists $G : \mathcal{D} \to \mathcal{C}$ for which

$$G \circ F = id_{\mathcal{C}} \quad \text{and} \quad F \circ G = id_{\mathcal{D}} .$$

Def^n: Given a functor $F : \mathcal{C} \to \mathcal{D}$ w/ maps on Hom: $F_{xy} : Hom_{\mathcal{C}}(x, y) \to Hom_{\mathcal{D}}(Fx, Fy)$, we call F

i) Faithful if each map F_{xy} is injective.

ii) Fully faithful if each F_{xy} is bijective.

iii) Essentially surjective if each z in $\mathbb{D}$ admits an x in $\mathbb{C}$ for which there is an isomorphism $z \xrightarrow{\sim} F(x)$ in $\mathbb{D}$.

Theorem 5: A functor $F: \mathbb{C} \to \mathbb{D}$ is an equivalence if and only if F is fully faithful and essentially surjective.

Proof: Exercise. ∎

Def^{t}: For any category $\mathbb{C}$ the opposite category $\mathbb{C}^{op}$ has the same objects and morphisms as $\mathbb{C}$ but the opposite composition

$$f \circ_{op} g := g \circ f.$$

Ex: We have the cat $\mathbf{Ab}$ of abelian groups and group homomorphisms, the cat $\mathbf{Ring}$ of rings and ring homomorphisms, the cat $\mathbf{Alg}_k$ of k-algs and k-alg homomorphisms, the cat $\mathbf{Set}$ of sets and set maps, The cat $\mathbf{Field}_F$ of field extensions of a given field F and maps over F, etc.

Ex: We have the fully faithful inclusion

$$\mathrm{Fields}_F \to \mathrm{Alg}_F, \begin{cases} K \mapsto K \\ f \mapsto f. \end{cases}$$

Ex: We have the free alg functor

$$\mathrm{Free}_{\mathbb{C}} : \mathrm{Set} \to \mathrm{Alg}_{\mathbb{C}},$$

$$X \mapsto \mathbb{C}[x : x \in X], \quad \tau \mapsto (p(x) \mapsto p(\tau(x))).$$

Ex: For any category $\mathcal{C}$, we have the functor

$$\mathrm{Yon} : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$$

$$\mathrm{Yon}(x) = \mathrm{Hom}_{\mathcal{C}}(-, x), \quad \mathrm{Yon}(f) = f \circ - .$$

Ex: For any comm ring k, we have the forgetful functor $\mathrm{forget} : \mathrm{Alg}_k \to \mathrm{Ring}$, $\begin{cases} A \mapsto A \\ f \mapsto f. \end{cases}$

Proposition 6: The forgetful functor

$$\mathrm{forget} : \mathrm{Alg}_{\mathbb{Z}} \to \mathrm{Ring}$$

is an equivalence. In fact, it is an isomorphism.

Construction: The inverse $\mathrm{alg} : \mathrm{Ring} \to \mathrm{Alg}_{\mathbb{Z}}$ takes a ring R to R equipped with the (unique) central ring map $\mathrm{unit}_R : \mathbb{Z} \to R$, $n \mapsto n \cdot 1_R$. ∎

— III. Modules and restriction

Let A be a ring. An A-module is an additive group M equipped with a bilinear map
$$\cdot : A \times M \to M$$
w/ $a\cdot(b\cdot m) = (a\cdot b)\cdot m$ and $1\cdot m = m$ at all a, b in A and m in M. A homomorphism of A-modules is an additive group map $f : M \to N$ for which $f(a\cdot m) = a\cdot f(m)$ at all a in A and m in M.

Def.: The category $\underset{\vee}{A\text{-mod}}$ of A-modules is the collection of all A-modules, with A-module homomorphisms.

Now, for any ring map $\phi : A \to B$ we have the restriction functor
$$\mathrm{res} = \mathrm{res}_\phi : B\text{-mod} \to A\text{-mod}$$
which sends a B-module N to N itself equipped w/ the action $a \cdot_\phi n := \phi(a)\cdot n$. Via the eq
$$f(a\cdot_\phi n) = f(\phi(a)\cdot n) = \phi(a)\cdot f(n)$$
$$= a\cdot_\phi f(n),$$
we see that any B-module map $f : N \to N'$ becomes an A-module map $f : \mathrm{res}_\phi(N) \to \mathrm{res}_\phi(N')$, and so we can define res_ϕ explicitly on obj and morph by

$$\mathrm{res}_\phi : \begin{cases} N \longmapsto \mathrm{res}_\phi(N) \\ f \longmapsto f \end{cases}$$

Example: We have the ring map

$$\mathbb{C}[x_1, x_2, x_3]/(x_1x_2 - x_3^2) \longrightarrow \mathbb{C}[y_1, y_2], \quad \begin{array}{l} x_1 \mapsto y_1^2 \\ x_2 \mapsto y_2^2 \\ x_3 \mapsto y_1y_2 \end{array}$$

so that any $\mathbb{C}[y_1, y_2]$-module restricts to a module over $\mathbb{C}[x_1, x_2, x_3]/(x_1x_2 - x_3^2)$.

Observation 7: For any k-algebra A, one have the restriction functor $A\text{-mod} \to k\text{-mod}$. In particular, when k is a field, any A-module is (naturally) a vector space, and any A-module map is (naturally) a vector space over k, and any A-algebra map is a map of vect spaces over k.

~ IV. The abelian structure on A-mod

We have the zero module 0 w/ $\mathrm{Hom}_A(M, 0) = * = \mathrm{Hom}_A(0, M)$ at all M.

A <u>submodule</u> $M' \subseteq M$ in an A-module M is an abelian subgroup for which $a \cdot m \in M'$ whenever m is in M', and $a \in A$. We note that in this case M' itself is an A-module and the inclusion

$$i : M' \to M$$

is a morphism of A-modules.

Furthermore, for any submodule $M' \subseteq M$

The additive quotient M/M' inherits a unique A-module structure under which the quotient

$$(*) \quad \pi: M \to M/M', \quad m \mapsto \pi(m) = \bar{m},$$

as an A-module map. Explicitly,

$$a \cdot \bar{m} := \overline{a \cdot m}.$$

The projection $(*)$ is universal amongst A-module maps $f: M \to N$ for which $f|_{M'} = 0$.

(ii') Image is submod

Lemma 8: i) For any A module hom $f: M \to N$, the kernel $\ker(f) \subseteq M$ is an A-submodule.

ii) For any two maps $f, f': M \to N$, and central elements $z, z' \in Z(A)$ the linear combo

$$(z \cdot f + z' \cdot f'): m \mapsto z f(m) + z' f'(m)$$

is also a map of A-modules.

iii) The zero map $0: M \to N$ is always an A-module map.

Proof: (i) For m in $\ker(f)$ and a in A we have $f(a \cdot m) = a f(m) = a \cdot 0 = 0$. Since the kernel is known to be an additive subgroup, we have that $\ker(f)$ is a submodule. (ii) Additivity is clear. For A-linearity we just check

$$(z f + z' f')(a \cdot m) = z(a f(m)) + z' \cdot (a f'(m))$$

$$\overset{\text{Central}}{=} a \cdot (z f(m) + z' f'(m)) = a \cdot (zf + z' f')(m).$$

(iii) Trivial

Corollary 9: For any k-algebra A, and A-modules M and N, the set of A-module maps

$$\mathrm{Hom}_{A\text{mod}}(M, N)$$

is naturally a k-module.

Example: For $A = \mathbb{Q}[x]$, $M = \mathbb{Q}[x]/(x^2)$, $\mathrm{Hom}_A(A, M)$ is 2-dimensional w/ basis spanned by the maps $\pi_0 : A \to M$, $\pi_0(a) = \bar{a}$ and $\pi_1 : A \to M$, $\pi_1(a) = \bar{a} \cdot \bar{x}$.

Example: For any field k we have the matrix algebra $M_n(k) = \mathrm{End}_k(k^n)$ and the "standard module" $V = k^n$ with action $A \cdot v$ given by matrix action.

Then $\mathrm{End}_{M_n(k)}(V)$ is 1-dimensional and spanned by the identity. This follows by considering eigenvectors for the action of the diagonal matrices in conj w/ the actions of the superdiagonals [HW].

Def$^{\underline{n}}$: Call an A-module M <u>simple</u> (aka irreducible) if any submodule $M' \leq M$ is either zero, or equal to M.

Question: We have the field inclusion $\mathbb{R} \hookrightarrow \mathbb{C}$ under which $\mathbb{C}$ becomes an $\mathbb{R}$-module. Is $\mathbb{C}$ simple over $\mathbb{R}$?

HW: Classify simple modules over $\mathbb{Q}[x]$.

Example: Any ideal $I \leq A$ is an A-submodule in A. For the poly ring $k[x]$ over a field, all ideals $I \leq k[x]$ are isomorphic as modules. Indeed, for $I = (p)$ we have the module isomorphism

$$f_p : k[x] \xrightarrow{\cong} (p), \quad a \longmapsto a \cdot p.$$

For injectivity ..., for surjectivity ... .

~ IV 1/3 Generating submodules

For any A-module M and subset $X \subseteq M$ we have the submodule $A \cdot X \leq M$ generated by X. This submodule can be defined, for example, as the intersection of all submodules which contain X.

Lemma 10: For any subset $X \subseteq M$, the submodule gen'd by X in M is explicitly the subset

$$A \cdot X = \left\{ \sum_{i=1}^{n} a_i \cdot x_i \; : \; n \geq 0,\ a_i \in A,\ x_i \in X \right\} \leq M.$$

~IV 2/3 Sums of modules

Given a collection of A-modules M_λ, indexed by a set Λ, the product module $\prod_{\lambda\in\Lambda} M_\lambda$ is the product of the additive groups, along w/ the A-action $a\cdot(m_\lambda : \lambda\in\Lambda) := (a\cdot m_\lambda : \lambda\in\Lambda)$.
The direct sum $\bigoplus_{\lambda\in\Lambda} M_\lambda$ is the submodule in $\prod_{\lambda\in\Lambda} M_\lambda$ consisting of all tuples $(m_\lambda : \lambda\in\Lambda)$ in which all but finitely many of the m_λ are 0.

When all of the $M_\lambda = A$, the regular A-mod, we call the sum $\bigoplus_{\lambda\in\Lambda} A$ the <u>free module</u> for A on the basis Λ. For such a free module we take 1_λ the tuple with 1 in the λ-th position and 0 elsewhere. We have the set inclusion

$$(*) \quad i: \Lambda \longrightarrow \bigoplus_{\lambda\in\Lambda} A, \quad i(\lambda) := 1_\lambda.$$

Lemma 11: For any set Λ, and any A-module M, restriction along the inclusion $(*)$ provides a bijection of sets

$$i^*: Hom_{A\text{mod}}\left(\bigoplus_{\lambda\in\Lambda} A, M\right) \xrightarrow{\cong} Hom_{Set}(\Lambda, M).$$

Proof: For the inverse, take any set map $\xi:\Lambda\to M$ to the module homomorphism $\bar{\xi}: \bigoplus_{\lambda\in\Lambda} A \to M$

defined by $\tilde{\xi}(\sum'_{\lambda\in\Lambda} a_\lambda 1_\lambda) := \sum_\lambda a_\lambda \xi(\lambda)$, where only finitely many of the a_λ here are necessarily 0. The fact that $\tilde{\xi}$ is additive is clear, and A-linearity is also clear. So $\tilde{\xi}$ is an A-module map. Furthermore $i^*(\tilde{\xi}) = \tilde{\xi}|_\Lambda = \xi$ by construction.

Conversely, for an A-mod map $f : \bigoplus_{\lambda\in\Lambda} A \to M$ we have $(\widetilde{i^*(f)} - f)(1_\lambda) = f(1_\lambda) - f(1_\lambda) = 0$ at all $\lambda \in \Lambda$ which gives
$(\widetilde{i^*(f)} - f)(\sum'_{\lambda\in\Lambda} a_\lambda 1_\lambda) = 0$ at all $\sum'_{\lambda\in\Lambda} a_\lambda 1_\lambda$ in $\bigoplus_{\lambda\in\Lambda} A$. Hence $\widetilde{i^*(f)} = f$. So the maps i^* and $\tilde{?}$ are mutually inverse. ∎

V Induction / Base change

Given a ring map $\phi : A \to B$,
We define a "base change" functor which goes in the other direction.

Def^{n}: Given an A-module M and a B-module N, we call an A module map
$u : M \to \mathrm{res}_\phi(N)$ <u>universal</u> if restriction along u provides a bijection
$$u^* : \mathrm{Hom}_{B\text{-mod}}(N, N') \to \mathrm{Hom}_{A\text{mod}}(M, \mathrm{res}_\phi N').$$

$$u^*(g) = \mathrm{res}_\phi(g)\circ u$$

at arbitrary N' in B-mod.

Equivalently, u is universal if, for each A-mod map $w: M \to \mathrm{res}_\phi(N')$ w/ generic target there is a uniquely determined B-module map $f_w : N \to N'$ which completes a diagram

$$\begin{tikzcd} & M \arrow[dl, "u"'] \arrow[dr, "w"] & \\ N \arrow[rr, "\exists!"', "f_w"] & & N' \end{tikzcd}$$

Proposition 13: Let $\phi: A \to B$ be a ring map and M be any A-module.

i) There is a B-module N which admits a universal A-module map $u: M \to \mathrm{res}_\phi(N)$.

ii) For any two universal maps $u_i : M \to N_i$ there is a unique B-module isomorphism $f_{01} : N_0 \to N_1$ which completes a diagram

$$\begin{tikzcd} & M \arrow[dl, "u_0"'] \arrow[dr, "u_1"] & \\ N_0 \arrow[rr, "f_{01}"] & & N_1 \end{tikzcd}.$$

iii) For any A-module map $f: M \to M'$ and univ maps $u: M \to \mathrm{res}_\phi N$, $u': M' \to \mathrm{res}_\phi N'$, there is a unique B-module map ${}_Bf : N \to N'$ which completes a diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "u"'] & M' \arrow[d, "u'"] \\ N \arrow[r, "{}_Bf"'] & N' \end{tikzcd}.$$

Proof / Construction: (i) Consider the free B-module $B\cdot\{M\} := \bigoplus_{m\in M} B$ on basis M, and let 1_m denote the copy of 1 in the m-th position. For $\mathrm{Univ}_B(M) \subseteq B\cdot\{M\}$ the submodule gen'd by the relations

$$(*)\quad \begin{cases} 1_{a\cdot m} - \phi(a)\, 1_m & \text{for } a \text{ in } A \text{ and } m \text{ in } M \\ 1_{m+m'} - 1_m - 1_{m'} & \text{for } m, m' \text{ in } M \end{cases}$$

we define

$$B \otimes_A M := B\cdot\{M\} / \mathrm{Univ}_B(M).$$

For b in B and m in M we take $b\otimes m = b\cdot 1_m$ in $B\otimes_A M$, and the relations $(*)$ tell us that the set map $u: M \to B\otimes_A M$, $u(m) = 1\otimes m$, is a map of A-modules from M to $\mathrm{res}_\phi(B\otimes_A M)$.

Now, for any A-mod map $w: M \to \mathrm{res}_\phi N$ we have the unique B-mod map $\tilde{f}_w: B\cdot\{M\} \to N$ w/ $\tilde{f}_w(1_m) = w(m)$, by Lemma 11, and we have $\tilde{f}_w$ vanishing on the relations $(*)$ via A-linearity of w. Hence we obtain a unique B-module map from the quotient $f_w: B\otimes_A M \to N$ with

$f_w(1\otimes m) = w(m)$ at all b in B and m in M.

Rather, we observe the existence of a unique B-module map which completes a diagram

$$\begin{tikzcd} & M \arrow[dl, "u"'] \arrow[dr, "\omega"] & \\ B\otimes_A M \arrow[rr, "f_\omega"] & & N \end{tikzcd}$$

. We're done.

(ii) We have the uniquely determined B-module maps $f_{ij}: N_i \to N_j$, for $i,j \in \{0,1\}$, which complete the prescribed diagrams. The univ. property then demands

$$f_{ii} = id_{N_i} \quad \text{and} \quad f_{jk} \circ f_{ij} = f_{ik}$$

for each i,j,k in $\{0,1\}$. In particular,

$$f_{01} f_{10} = id_{N_1} \quad \text{and} \quad f_{10} f_{01} = id_{N_0} .$$

(iii) Immediate from the universal property. ∎

Def^{n}: For any A-module M we let $B\otimes_A M$ denote any B-module which is equipped with a universal map

$$u_M : M \to_{res} B\otimes_A M .$$

For any map of A-modules $f: M \to M'$ we let $B\otimes_A f : B\otimes_A M \to B\otimes_A M'$ denote the unique B-module map with $B\otimes_A f|_M = u_{M'} \circ f$.

On generating monomials

$$(B\otimes_A f)(1\otimes m) = 1\otimes f(m)$$

so that $(B\otimes_A f)(b\otimes m) = b\otimes f(m)$.

Theorem 14: For any map of rings $\phi : A \to B$, there is a functor

$$B \otimes_A - : A\text{-mod} \to B\text{-mod} \quad \begin{cases} M \mapsto B \otimes_A M \\ f \mapsto B \otimes_A f. \end{cases}$$

Furthermore, the universal maps define a natural transformation

$$u : id_{A\text{-mod}} \to res_\phi \circ (B \otimes_A -)$$

for which, at each M in A-mod and N in B-mod, the map

$$Hom_{B\text{-mod}}(B \otimes_A M, N) \longrightarrow Hom_{A\text{-mod}}(M, res_\phi(N))$$
$$g \longmapsto res_\phi(g) \circ u_M$$

is an isomorphism.

Proof: (The second assertion just follows by) Prop 13 (iii) and the universal property. We need only check that $B \otimes_A id_M = id_{B \otimes_A M}$ and
$(B \otimes_A g) \circ (B \otimes_A f) = B \otimes_A (g f)$.

The first equality follows by the diagram

$$\begin{tikzcd} M \arrow[r, "id"] \arrow[d, "u"'] & M \arrow[d, "u"] \\ B \otimes_A M \arrow[r, "id"] & B \otimes_A M \end{tikzcd}$$

and the second follows by the diag

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[rr, bend left, "gf"] \arrow[d] & M' \arrow[r, "g"] \arrow[d] & M'' \arrow[d] \\ B \otimes_A M \arrow[r, "B \otimes f"] & B \otimes_A M' \arrow[r, "B \otimes g"] & B \otimes_A M'' \end{tikzcd}$$

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HW

1. Prove that the only ideals in the matrix ring $M_n(k)$, for k a field, are 0 and $M_n(k)$ itself.

2. a) Prove that, for any field k, the standard module $V = k^n$ over $M_n(k)$ is simple. [Hint: Think about rowred.]
b) Prove that $End_{M_n(k)}(V) = k \cdot id_V$. [Hint: Consider eigenvectors for the diagonal matrices, and the action of the superdiagonal.]

3. Classify simple modules over $\mathbb{Q}[x]$.

4. Let k be a commutative ring and A be a k-alg. For a k-module M, prove that the choice of an A-mod structure on M is equivalent to the choice of a k-alg map

$$\phi : A \to End_k(M).$$

5. For any A-module M define the annihilator $Ann_A(M) \subseteq A$ as the subset of a in A w/ $a \cdot m = 0$ at all m in M. Show that $Ann_A(M)$ is an ideal in A.

6. For any quotient map $\pi : A \to A/I$ via an ideal $I \subseteq A$, prove that the restriction functor

$$\mathrm{res}_\pi : A/I\text{-mod} \to A\text{-mod}$$

is fully faithful. Prove, furthermore, that res_π is an equivalence onto the full subcategory in A-mod consisting of all modules M with $I \subseteq \mathrm{Ann}_A(M)$.

7. a) For any ideal $I \subseteq A$ and A-module M, prove that $I \cdot M = \{a \cdot m : a \in I, m \in M\} \subseteq M$ is a submodule in M.

b) Prove that the quotient map $M \to M/I \cdot M$ induces an isomorphism of A/I-modules

$$A/I \otimes_A M \xrightarrow{\cong} M/I \cdot M .$$