Fin dim algebras and Jordan-Holder

$\sim$ I Aside: Group algebras!

Let's just take a moment to think about some interesting examples.

Let G be a group and k be a comm ring (generally a field)

Def^n: A G-representation is a vector space V equipped with an action $\cdot : G \times V \to V$ which satisfies

$$g\cdot(h\cdot v) = (g\cdot h)\cdot v \quad \text{and} \quad g\cdot(cv + c'v') = c(g\cdot v) + c'(g\cdot v')$$

at all $g, h \in G$, $v, v' \in V$ and $c, c' \in k$.

Equivalently, we specify a group map

$$G \to \mathrm{Aut}_k(V).$$

For example we have S_n and D_n acting on $\mathbb{C}^n$ by permuting coordinates. $\sigma\left(\sum_{i=1}^n c_i e_i\right) = \sum_{i=1}^n c_i e_{\sigma(i)}$.

Also we have the 1-dimensional trivial representation

$\mathbb{C}_{triv} = \mathbb{C}$ with S_n-action $\sigma \cdot 1 = 1$

and the 1-dimensional sign representation

$\mathbb{C}_{sign} = \mathbb{C}$ w/ S_n-action $\sigma \cdot 1 = \mathrm{sign}(\sigma) 1$.

Def$^{\underline{n}}$: A homomorphism of G-reps is a linear map $f: V \to W$ for which $f(g \cdot v) = g f(v)$ of all v in V, $g \in G$.

Note that we have the inclusion of S_n-reps
$$\mathbb{C}_{triv} \to \mathbb{C}^n, \quad 1 \longmapsto \sum_{i=1}^{n} e_i$$
for example.

We can also define the group algebra kG of arbitrary G which is the free vect space with basis G along w/ the expected multiplication
$$\left(\sum_{g \in G} c_g g\right) \cdot \left(\sum_{h \in G} c'_h h\right) \qquad (*)$$
$$= \sum_{g,h \in G} c_g \cdot c'_h (g \cdot h).$$
and unit $1 = 1_G$.

Def$^{\underline{n}}$: For any ring A, a unit in A is an element a which admits a^{-1} so that $a^{-1}a = aa^{-1} = 1$. We let $A^\times = \{a \in A : a \text{ is unit}\}$.

Note that $A^\times$ is a group under mult.

Ex: $M_n(k)^\times = GL_n(k)$, or in basis free notation $End_k(V)^\times = Aut_k(V)$ for any vect space V.

Ex: For each finite group G, we have a group embedding $G \to (kG)^\times$. This is not an isomorphism since, for example $-g$ is invertible at all g.

Obviously any ring map $A \to B$ induces a group map $A^\times \to B^\times$. In particular, any map of k-alg's $kG \to A$ restricts to a group map $G \to A^\times$.

Lemma 1: For any k-alg A and finite group G, restriction provides a bijection

$$\{k\text{-alg maps } kG \to A\} \xrightarrow{\sim} \{\text{group maps } G \to A^\times\}.$$

Proof: The map is obviously injective, since kG is spanned by G as a k-module and any alg map is k-linear. Now, given a group map $\psi : G \to A^\times$ we understand, just via bilinearity of the product on A, that the elements

$$\sum_g c_g \psi(g) \quad \text{in } A \quad \text{multiply according to}$$

the formula $(*)$.

Hence the unique linear map

$$\phi : kG \to A$$

w/ $\phi|_G = \psi$ respects multiplication

$$\phi(x \cdot y) = \phi(x) \cdot \phi(y) \text{ for all } x, y \in G$$

and has

$$\phi(1_G) = \psi(1_G) = 1_{A^\times} = 1_A .$$

So ϕ is an algebra map w/ $\phi|_G = \psi$ and we see that restriction provides the claimed bijection. ∎

Theorem 2: A G-representation over k is the same thing as a kG-module. More precisely, we have a (strictly invertible) equivalence of categories

$$kG\text{-mod} \xrightarrow{\sim} G\text{-rep}_k .$$

$$\left\{ \begin{array}{l} V \text{ w/} \\ \phi : kG \to \mathrm{End}_k(V) \end{array} \right\} \longmapsto \left\{ \begin{array}{l} V \text{ w/} \\ \phi|_G : G \to \mathrm{Aut}_k(V) \end{array} \right\}$$

$$\{ f : V \to W \} \longmapsto \{ f : V \to W \} .$$

Proof: ∎

Corollary 3: The category of G-reps has kernels and cokernels, quotients, subreps, etc. and they behave in the expected way.

Ex: We saw the inclusion $k_{triv} \to k^n$ into the permutation representation over S_n. In the case

$n=3$, we take the quotient to get a 2-dim rep
$$L(2) = k^3 / k_{triv}.$$
This 2-dim rep is actually simple [HW].

In fact, we'll see later that
$$\{ k_{triv}, k_{sign}, L(2) \}$$
provides a complete list of simple kS_3-modules / S_3-reps in characteristic other than 2.

Though $\overline{\mathbb{F}_3} S_3$ and $\mathbb{C}S_3$ have "the same" simples, the module categories
$$\overline{\mathbb{F}_3} S_3\text{-mod} \quad \text{and} \quad \mathbb{C}S_3\text{-mod}$$
are wild different.

Theorem (Maschke's Theorem) Let $k=\bar{k}$ be a field. If $\operatorname{char}(k) \nmid |G|$ then kG-mod is very easy to understand theoretically, but combinatorially interesting. If $\operatorname{char}(k) \mid |G|$ the module category kG-mod can (generally speaking) never be understood in any concrete terms by anyone ever.

Proof: Future. ∎

Rem: $\overline{\mathbb{F}_3} S_3$ is actually not so bad, but like $\overline{\mathbb{F}_3} S_6$ is an absolute disaster...

II. Artinian and Noetherian rings and modules

Def^n: Let A be a ring.

An A-module M, call M Artinian (resp. Noetherian) if any sequence of submodules

$M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$ (resp. $M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$)

stabilizes.

We call A Artinian (resp. Noetherian) if every finitely generated A-module is Artinian (resp. Noetherian.

Fake-Def^n: Call A ring theoretically Artinian (resp. Noetherian) if A is Artinian (resp. Noetherian) as a module over itself.

Future Theorem 4: Any Artinian ring is also Noetherian.

We will focus on a concrete setting where both Artinianity and Noetherianity are apparent.

Example: Any finite dimensional algebra A, i.e. algebra over a field k w/ $\dim_k A < \infty$, is both Artinian and Noetherian. Indeed, to say M is for generic is to say M admits a surj $A^r \to M$, giving $\dim M < \infty$. So M satisfies ACC / DCC for simple dimension reasons.

< Explain extensions

Proposition 5: Given any extension
$$0 \to M' \to N \xrightarrow{\pi} M \to 0,$$
N is Artinian (resp. Noetherian) if and only if M' and M are Artinian (resp. Noetherian).

Proof: Any descending chain $\cdots \subseteq M_3' \subseteq M_2' \subseteq M'$, is a descending chain in N.
Hence stabilization for N implies stabilization for M'.
Similarly, any desc. chain $\cdots \subseteq M_2 \subseteq M_1 \subseteq M$ pulls back to a desc. chain $\cdot \subseteq \pi^{-1}$
Since $\pi\pi^{-1}(M_i) = M_i$, stabilization for N implies stabilization for M. So N Artinian $\Rightarrow$ M Artinian.
Conversely, suppose M and M' Artinian, and take a chain $\cdots \subseteq N_2 \subseteq N_1$ in N. Define
$$M_i' = M' \cap N_i \quad \text{and} \quad M_i = \pi(N_i)$$
to obtain desc. chains $\cdots \subseteq M_2' \subseteq M_1'$ and $\cdots \subseteq M_2 \subseteq M_1$. Take K w/ $M_n = M_K$ and $M_n' = M_K'$ whenever $n \geq K$. Then we

have exact sequences of each $\kappa \in K$

$$\begin{tikzcd} 0 \arrow[r] & M'_\kappa \arrow[r] \arrow[d, "\mathrm{incl}'_\kappa"'] & N_\kappa \arrow[r] \arrow[d, "\widetilde{\mathrm{incl}}_\kappa"] & M_\kappa \arrow[r] \arrow[d, "\mathrm{incl}_\kappa"] & 0 \\ 0 \arrow[r] & M'_K \arrow[r] & N_K \arrow[r] & M_K \arrow[r] & 0 \end{tikzcd}$$

in which incl'_κ and incl_κ are isomorphisms. Hence $\widetilde{\mathrm{incl}}_\kappa$ is an isomorphism by short five lemma, and thus an equality. So we see that the sequence $\cdots \subseteq N_3 \subseteq N_2 \subseteq N_1$ stabilizes, and hence that N is Artinian.

The Noetherian arguments are completely similar. ∎

Corollary 6: i) For M Artinian, any quotient mod or submodule of M is Artinian

ii) Any fin sum $\bigoplus_{i=1}^n M_i$ of Artinian modules is Artinian.

Furthermore, the same result holds when Artinian is rep w/ Noetherian.

Theorem 7: A ring A is Artinian (resp. Noetherian) if and only if A is ring theoretically Artinian (resp. Noeth).

Proof: A module is finitely generated iff M admits a surjection $\bigoplus_{i=1}^n A \twoheadrightarrow M$. So fol by Corollary 3. ∎

Corollary 8: Any principle ideal domain is Noetherian. ∎

Proof: H/W.

Observation 9: If $A \to B$ is a ring map, B is finite as a module over A, and A is Artinian (resp. Noetherian) then B is also Artinian (resp. Noetherian).

Example: $\mathbb{Z}$ is Noetherian, but not Artinian. For example, we have the infinite descending chain of ideals $(p) \supsetneq (p^2) \supsetneq (p^3) \supsetneq \cdots$ of any prime p. Similarly, for k a field, $k[x]$ is Noetherian but not Artinian.

Example: If R is a commutative Noeth. ring then, for any finitely generated R-module M, the R-algebra $\mathrm{End}_R(M)$ is Noetherian. (?)

Example: For any group G and commutative ring R we have the group ring

$$RG = \bigoplus_{g \in G} R \cdot g$$

with mult $\left(\sum_{g \in G} a_g g\right) \cdot \left(\sum_{h \in G} b_h h\right)$
$= \sum_{g,h} a_g \cdot b_h (gh)$.

When G is finite $\mathbb{Q}G$ is finite over $\mathbb{Q}$. Hence $\mathbb{Z}G$ is Noetherian and, for any field k, kG is Artinian and Noetherian.

Example: For X a finite CW complex, the rational cohomology $H^*(X, \mathbb{Q})$ is Artinian and Noetherian, under the cup product.

— VII. Composition series and Jordan-Hölder

Def: A composition series for a module M is a sequence of submodules

composition factors

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_\ell = M \qquad (*)$$

in which each subquotient M_{i+1}/M_i is simple. The number ℓ is called the length of the series $(*)$.

Ex: M has length 0 comp. series $\iff M = 0$

M has length 1 comp. series $\iff M$ is simple.

We do not claim that all modules admit composition series.

Lemma 10: An A-module M admits a composition series $(*)$ if and only if M is both Artinian and Noetherian.

Proof: Suppose M has a composition series of length l, and that the any module w/ a comp series of length $< l$ is both Artinian and Noetherian. From the supposed series

$$0 = M_0 \leq M_1 \leq \cdots \leq M_{l-1} \leq M_l = M$$

we obtain M as an extension

$$0 \to M_{l-1} \to M \to M/M_{l-1} \to 0$$

with M/M_{l-1} Art and Noeth since its simple, and M_{l-1} Art. and Noeth. by our assumption. Then M is both Art and Noeth by Proposition 2. Since every length 0 module is Art and Noeth, naturally, we see that all modules which admit a composition series are both Artinian and Noetherian.

Conversely, suppose M is both Artinian and Noetherian. If $M = 0$ then it clearly has a comp series $0 = M$, so we assume $M \neq 0$. By Artinianness, M admits a simple submodule $M_1 \leq M$. Taking the quotient and noticing that M/M_1 remains Artinian, by Corollary 3, we find a simple module $\bar{M}_2 \leq M/M_1$, OR take $\bar{M}_2 = 0$ if $M/M_1 = 0$ aka $M_1 = M$. Pulling back along the projection $\pi_1 : M \to M/M_1$ we obtain a submodule $M_2 = \pi^{-1}(\bar{M}_2) \leq M$ with

$\mathcal{M}_1 \subseteq \mathcal{M}_2$ and $\mathcal{M}_2/\mathcal{M}_1 = \bar{\mathcal{M}}_2$ simple. Proceeding in this way we obtain an ascending sequence

$$0 \subseteq \mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \mathcal{M}_3 \subseteq \cdots \subseteq \mathcal{M}$$

By Noetherianity there must be an index l at which $\mathcal{M}_l = \mathcal{M}_n$ at all $n \geq l$, and hence at which $\mathcal{M}_l = \mathcal{M}$. We thus obtain a composition series for $\mathcal{M}$,

$$0 = \mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \cdots \subseteq \mathcal{M}_l = \mathcal{M}. \qquad \blacksquare$$

Example: We know each simple module over $\mathbb{Q}[x]$ is finite dimensional (though there is no bound on the dimension). Hence a $\mathbb{Q}[x]$ module $\mathcal{M}$ is both Artinian and Noetherian, equivalently, admits a composition series if and only if $\mathcal{M}$ is finite dimensional.

For a specific example, given distinct irred polys $p(x)$ and $q(x)$ with roots α and β, the mod $\mathcal{M} = \mathbb{Q}[x]/(p^2q^2)$ has composition series

$$0 = (p^2q^2)\cdot\mathcal{M} \subseteq (pq^2)\mathcal{M} \subseteq (pq)\mathcal{M} \subseteq q\cdot\mathcal{M} \subseteq \mathcal{M}$$

$$0 = (p^2q^2)\cdot\mathcal{M} \subseteq (p^2q)\mathcal{M} \subseteq p^2\cdot\mathcal{M} \subseteq p\mathcal{M} \subseteq \mathcal{M}$$

for example, w/ resp. subquotients

$\mathbb{Q}(\alpha)$, $\mathbb{Q}(\beta)$, $\mathbb{Q}(\alpha)$, $\mathbb{Q}(\beta)$ and

$\mathbb{Q}(\beta)$, $\mathbb{R}(\beta)$, $\mathbb{Q}(\alpha)$, $\mathbb{R}(\alpha)$.

So we see, composition series are not unique. Though in this example we find that

(a) The length of the two series are the same

(b) The simple modules which appear as subq[uotients] in these series agree.

Ex: A module M over a finite dim alg A admits a comp. series if and only if M is finite dimensional.

Theorem (Jordan-Hölder): Let M be an A module w/ composition series

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\ell = M$$

and

$$0 = M_0' \subseteq M_1' \subseteq \cdots \subseteq M_r' = M.$$

Then $\ell = r$ and, for some permutation $\sigma \in S_\ell$, we have

$$M_{i+1}/M_i \cong M'_{\sigma(i)+1}/M'_{\sigma(i)}$$

at each $0 \le i < \ell$.

Proof: For a module N which admits a composition series, define the length of N to be the minimal length of a composition series for N. Note that a module is of length 1 if and only if it is simple.

The result holds for any length 0 or length 1 module trivially. Suppose now that the result holds for all modules of length $< l$ and take M of length l. Consider comp. series as in the statement.

If $M_{l-1} = M'_{r-1}$ then $r = l$ as M_{l-1} is of length $l-1$. Otherwise we have proper inclusions

$$\begin{array}{c} M'' \\ \shortparallel \\ M_{l-1} \leftarrow (M_{l-1} \cap M'_{r-1}) \rightarrow M_{r-1} \end{array}$$

and hence nonzero injections

$$M_{l-1}/M'' \rightarrow M/M'_{r-1} \qquad (*)$$

and

$$M'_{r-1}/M'' \rightarrow M/M_{l-1}.$$

By simplicity of the target modules these injections are both isomorphisms, so that both quotients by M'' are simple.

From any comp series for M''

$$0 = M''_0 \subseteq \cdots \subseteq M''_t = M''$$

we obtain comp series

$$0 = M''_0 \subseteq \cdots \subseteq M''_t \begin{array}{l} \subseteq M_{l-1} \\ \subseteq M'_{r-1} \end{array}.$$

This gives
$$r-1 = \text{length}(M'_{r-1}) = \text{length}(M_{\ell-1}) = \ell-1$$
$\Rightarrow$ $r=\ell$.

By our ind. hyp. the comp factors for the resp. series are
$$M''_{i+1}/M''_i, \quad M_{\ell-1}/M'', \quad M/M_{\ell-1}$$
vs.
$$M''_{i+1}/M''_i, \quad M'_{\ell-1}/M'', \quad M/M'_{\ell-1}.$$
We already calculated isomorphisms
$$M_{\ell-1}/M'' \cong M/M'_{\ell-1}$$
and
$$M'_{\ell-1}/M'' \cong M/M_{\ell-1},$$
so that all of the factors are identified (after a permutation).

Def$^{\underline{n}}$: Given finite length M over a ring A, the length of M is the length of any comp. series for M. For any simple A-module L the multiplicity of L in a comp. series for M is the integer
$$[L:M] := \left\{ \begin{array}{l} \text{the number of distinct indices} \\ i \text{ at which } L \cong M_{i+1}/M_i \\ \text{in a given comp. series } M_0 \leq M_1 \leq \cdots \leq M. \end{array} \right.$$
Note that this integer is indep. of the choice of comp series

for M, by Jordan-Holder.

Example: For distinct irreducible $p_1, \dots, p_t$ in $\mathbb{Q}[x]$, and $M = \mathbb{Q}[x]/(p_1^{m_1} \cdots p_t^{m_t})$ has length

$\text{length}(M) = \sum_{i=1}^{t} m_i$ and

$$[\mathbb{Q}(\alpha) : M] = \begin{cases} m_i & \text{if } \alpha \text{ is a root for } p_i \\ 0 & \text{if all } p_i(\alpha) \neq 0. \end{cases}$$

Proposition 4: Given an extension of finite length modules

$$0 \to M' \to N \xrightarrow{\pi} M \to 0$$

we have $\text{length}(N) = \text{length}(M) + \text{length}(M')$ and for any simple module L we have

$$[L : N] = [L : M] + [L : M'].$$

Proof: From comp series $M_0 \subseteq \cdots \subseteq M_l = M$ and $M'_0 \subseteq \cdots \subseteq M'_t = M'$ we obtain a comp series

$$N_0 \subseteq \cdots \subseteq N_t = M' \subseteq N_{t+1} \subseteq \cdots \subseteq N_{t+l} = N$$

w/ $N_i = M'_i$ for $i \leq t$ and $N_{t+j} = \pi^{-1}(M_j)$ and subsequently,

$$N_{i+1}/N_i = M'_{i+1}/M'_i \text{ for } i < t \text{ and}$$

$$N_{t+j+1}/N_{t+j} \cong M_{j+1}/M_j.$$

This gives the proposed result. ∎

HW

1. Let k be a field of characteristic $\neq 2$. Prove that the quotient module $L(2) = k^3 / k_{triv}$ of the permutation module over kS_3 along the inclusion $k_{triv} \to k^3$, $1 \mapsto e_1 + e_2 + e_3$, is a simple module over kS_3.

When $\mathrm{char}(k) \neq 2$,

2. Prove that the action map $kS_3 \to \mathrm{End}_k(L(2))$ is surjective; in particular, observe that the matrix ring

3. a) Prove that any PID is Noetherian.

b) Prove that $\mathbb{Z}$ and $k[x]$ are Noetherian but not Artinian, for any field k.

4. For distinct irreducible polynomials $p_1, \dots, p_r$ in $\mathbb{Q}[X]$, and integers $m_i > 0$, take
$$M = \mathbb{Q}[X] / (p_1^{m_1} \cdots p_r^{m_r}).$$
For $\alpha \in \overline{\mathbb{Q}}$, prove that $[\mathbb{Q}(\alpha) : M] > 0$ if and only if $p_i(\alpha) \neq 0$ at some i, and in this case $[\mathbb{Q}(\alpha) : M] = m_i$.

5. a) For any finite dimensional $\mathbb{C}[x,T]$-module M, prove that $\text{length}(M) = \dim_{\mathbb{C}}(M)$.

b) Prove that there are finitely generated $\overline{\mathbb{F}}_3 S_3$-modules M for which $\text{length}(M) < \dim(M)$.