Ideals

I. Primes and maximal ideals

Throughout we let R denote a commutative ring.

Def$^{\underline{n}}$: An ideal $I \subseteq R$ is called prime if I is properly contained in R, and for each $a, b \in R$, $ab \in I \Leftrightarrow a \in I$ or $b \in I$.

We call I maximal if $I \subsetneq R$ and any intermediate ideal $I \subseteq I' \subseteq R$ we have either $I' = I$ or $I' = R$.

Lemma 1: i) An ideal $I \subseteq R$ is prime if and only if R/I is a (nonzero) domain.

ii) An ideal $I \subseteq R$ is maximal if and only if R/I is a (nonzero) field.

Proof: (i) Consider $\bar{a}, \bar{b} \in R/I$ with reps $a, b \in R$. Then $\bar{a} \cdot \bar{b} = 0$ if and only if $a \cdot b \in I$. Hence, if I is prime we have one of $\bar{a}$ or $\bar{b} = 0$. Conversely, if R/I is a domain then $\bar{a} \cdot \bar{b} = 0$ implies one of $\bar{a} = 0$ or $\bar{b} = 0$, so that for arbitrary elements $a \cdot b \in I$ implies $a \in I$

or $b \in I$. So I is prime.

(ii) If I is maximal R/I has no proper nonzero ideals [Lemma 11, Smoke]. Hence for any nonzero $\bar{a} \in R/I$ $(\bar{a}) = R$, and hence there exists $\bar{b}$ such that $\bar{a} \cdot \bar{b} = 1$. So we see R/I is a field. Conversely, if R/I is a field then the quotient contains no proper nonzero ideals, and thus any intermediate ideal $I \subseteq I' \subseteq R$ has $I' = I$ or $I' = R$ [Lemma 11, Smoke]. ∎

Corollary 2: Any maximal ideal $m \subseteq R$ is prime.

Proposition 3: Any proper ideal $I \subseteq R$ is contained in a maximal ideal $I \subseteq m$.

Proof: Zorn Lemma. ∎

Example: For $(x) \subseteq k[x,y]$, we have the map $\phi : k[x,y] \longrightarrow k[y]$, $\phi(f(x,y)) = f(0,y)$, has kernel

$$\ker(\phi) = \{f(x,y) : x \text{ divides } f(x,y)\} = (x).$$

Hence, by the universal property of the quotient ring, we obtain an algebra isomorphism $k[x,y]/(x) \cong k[y]$.

Since $k[x]$ is a domain (x) is prime.

Example: Let $f(x_1, \dots, x_n) \in k[x_1, \dots, x_n]$ be irreducible. For $a, b \in k[x_1, \dots, x_n]$ we have

$$a \cdot b \in (f) \iff f \mid a \cdot b.$$

If one of a or b is a unit then either $f \mid a$ or $f \mid b$. If neither is a unit then we have unique decomps into products of irreducibles

$$a = p_1 \cdots p_s, \quad b = q_1 \cdots q_t, \quad a \cdot b = p_1 \cdots p_s q_1 \cdots q_t$$

and, via uniqueness, we have $u \cdot f = p_i$ or $u \cdot f = q_j$ for some unit u, giving either $f \mid a$ or $f \mid b$. Hence $a \in (f)$ or $b \in (f)$. We conclude that (f) is a prime ideal.

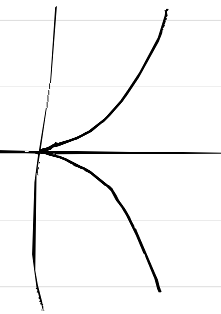
Proposition 4: For any UFD R, and nonzero $f \in R$, the ideal (f) is prime if and only if f is irreducible.

Proof: We've already argued in the example above that (f) is prime when f is irreducible. Conversely, if (f) is prime then under any factoring $f = a_1 \cdot a_2$ we have $a_1 \cdot a_2 \in (f)$ and either $f \mid a_1$ or $f \mid a_2$. Supposing arbitrarily $f \mid a_1$, then $a_1 = f \cdot u$ giving $f = (a_2 \cdot u) \cdot f \Rightarrow$

$a_2 \cdot x = 1$, since R is a domain. So a_2 is a unit.
We conclude that f is irreducible. ∎

Example: Consider the poly $y^2 - x^3$ in $\mathbb{C}[x,y]$.

$y = \sqrt{x^3}$



Supposing $f_1(x,y) \cdot f_2(x,y) = y^2 - x^3$ with neither f_i a unit, we have

$$f_i(x,y) = c_i x^{a_i} + d_i y^{b_i}$$

due to the absence of mixed terms in $y^2 - x^3$, giving

$$f_1 \cdot f_2 = c_1 c_2 x^{a_1 + a_2} + d_1 d_2 y^{b_1 + b_2} + c_1 d_2 x^{a_1} y^{b_2} + c_2 d_1 x^{a_2} y^{b_1}.$$

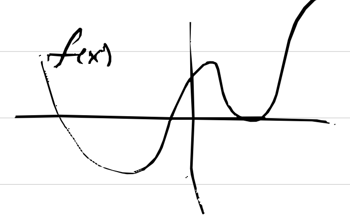
$\Rightarrow$ $c_1 d_2 = c_2 d_1 = 0$ $\Rightarrow$ one of c_1 or $c_2 = 0$

$\Rightarrow$ $y^2 - x^3 = d_1 d_2 y^{b_1 + b_2}$, a contradiction.

So we see $y^2 - x^3$ is an irreducible polynomial in $\mathbb{C}[x,y]$, and $(y^2 - x^3)$ is prime in $\mathbb{C}[x,y]$, and

$$\mathbb{C}[x,y] / (y^2 - x^3) \text{ is a domain!}$$

Example: For $y = f(x)$



we can argue as in the previous example to find that $y-f(x)$ is irreducible. Hence $(y-f(x))$ is prime. Alternatively, we have the algebra map

$$\phi : k[x,y] \to k[x], \quad \phi(x) = x, \ \phi(y) = f(x)$$

with $\ker(\phi) = (y-f(x))$. Let's argue this point. We have the inclusion $k[x] \to k[x,y]$ for which composite

$$\begin{tikzcd} k[x] \arrow[dr] \arrow[drr, bend left, "id"] & & \\ & k[x,y] \arrow[r, "\phi"'] & k[x] . \end{tikzcd}$$

$\Rightarrow$

$$\begin{tikzcd} k[x] \arrow[dr] \arrow[drr, bend left, "id"] & & \\ & k[x,y]/(y-f(x)) \arrow[r, "\bar{\phi}"'] & k[x] \end{tikzcd}$$

so that the induced map from the quotient $\bar{\phi}$, which is obviously surjective, is an $\cong$ if and only if $\bar{\phi}$ is injective. For this it suffices to show that all $\overline{g(x,y)} = \bar{h}(x)$ for some poly $h \in k[x]$ in the quotient, since in this case

$$\bar{\phi}(\bar{g}) = \bar{\phi}(\bar{h}(x)) = h(x) \Rightarrow$$

$$\bar{\phi}(\bar{g}) = 0 \text{ implies } \bar{h}(x) = 0 \text{ implies } \bar{g} = 0.$$

But now this is clear.

Indeed, in the quotient $\bar{y} = \overline{f(x)}$, giving $\bar{y}^m = \overline{f(x)}^m$ for all $m \geq 0$, giving finally

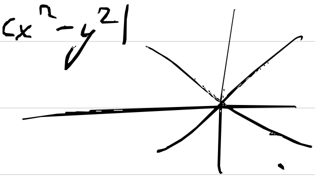
$$\overline{g(x,y)} = \sum_i' \overline{g_i(y)}\,\bar{x}^i$$
$$= \sum' g_i(\overline{f(x)})\,\bar{x}^i = h(x).$$

So we see

$\bar{\phi}$ injective, and thus $\ker(\phi) = (y - f(x))$.

Example: For any field k and $\alpha, \beta \in k$, $(x-\alpha, y-\beta)$ is a maximal ideal in $k[x,y]$.

Example: (x^2-y^2) is <u>not</u> prime, as $(x-y)\cdot(x+y) = x^2-y^2$ while $x-y \notin (x^2-y^2)$ and $x+y \notin (x^2-y^2)$.



Example: (x^3, y) is <u>not</u> prime in $k[x,y]$.

Question: Is every non-maximal prime ideal $I \leq k[x,y]$ principle? I.e. of the form $I = (f)$ for some poly $f(x,y)$?

Question: Does there exist a principle ideal in $k[x,y]$ which is maximal?

Question: Are there maximal ideals in $k[x,y]$ other than those of the form $(x-\alpha, y-\beta)$?

~ II. Zariski's Lemma

Theorem (Zariski's Lemma): For any field k, and any maximal ideal $\mathfrak{m}$ in $k[x_1, \ldots, x_n]$, the quotient $k[x_1, \ldots, x_n]/\mathfrak{m}$ is a finite field extension of k.

~ II ½ Proving Zariski pt 1: Rational functions

For any field k, we let $k(x_1, \ldots, x_n)$ denote the field of "rational functions" over k. Elements in $k(x_1, \ldots, x_n)$ are equiv classes of fractions f/g where $g \neq 0$ and $f_1/g_1 = f_2/g_2 \Leftrightarrow f_1 g_2 = f_2 g_1$; addition

$$f_1/g_1 + f_2/g_2 = (f_1 g_2 + f_2 g_1)/g_1 \cdot g_2$$

and mult.

$$(f_1/g_1) \cdot (f_2/g_2) = (f_1 \cdot f_2)/(g_1 \cdot g_2).$$

Lemma 5: For any field k, $k(x_1, \ldots, x_n)$ is a well-defined ring. All nonzero elem's in $k(x_1, \ldots, x_n)$ are invertible, and the apparent map

$$k[x_1, \ldots, x_n] \to k(x_1, \ldots, x_n), \quad f \mapsto f/1,$$

is an injective ring homomorphism.

Proof: Exercise. ∎

Lemma 6: Take $R = k[x_1,\dots,x_n]$ and $K = k(x_1,\dots,x_n)$, and let $k \to F$ be any field extension. Restriction along the inclusion $R \to K$ induces a bijection

$\mathrm{Hom}_{k\text{-alg}}(K,F) \to \{\text{Injective } k\text{-alg maps } R \to F\}$

Proof: For $\phi : K \to F$ and $\bar\phi : R \to F$ given by restriction, we have at any nonzero poly f

$1 = \phi(1) = \phi(f \cdot f^{-1}) = \bar\phi(f) \cdot \phi(f^{-1}) \Rightarrow \bar\phi(f) \in F - \{0\}$

so that $\bar\phi$ has trivial kernel, and

$\phi(f/g) = \phi(f \cdot g^{-1}) = \bar\phi(f)\,\phi(g^{-1}) = \bar\phi(f)\,\bar\phi(g)^{-1}$

so that ϕ is determined uniquely by its restriction $\bar\phi$. So we see that $(*)$ is a well-defined injection.

For surjectivity, take any inj alg map $\psi : R \to F$. We propose a map

$\bar\psi : K \to F$ defined by

$\bar\psi(f/g) := \psi(f) \cdot \psi(g)^{-1}$.

When $f_1/g_1 = f_2/g_2$ we have $f_1 g_2 = f_2 g_1$ giving

$\psi(f_1)\,\psi(g_2) = \psi(f_2)\,\psi(g_1)$

$\Rightarrow \psi(f_1) \cdot \psi(g_1)^{-1} = \psi(f_2) \cdot \psi(g_2)^{-1}$.

So $\tilde{\psi}$ is well-def as a map of sets. For addition and k-linearity

$$\tilde{\psi}((f_1 g_2 + f_2 g_1)/g_1 g_2) = (\psi(f_1 g_2) + \psi(f_2 g_1)) \cdot \psi(g_1 g_2)^{-1}$$

$$= (\psi(f_1)\psi(g_2) + \psi(f_2)\psi(g_1)) \cdot \psi(g_1)^{-1} \cdot \psi(g_2)^{-1}$$
$$= \psi(f_1)\psi(g_1)^{-1} + \psi(f_2) \cdot \psi(g_2)^{-1}$$
$$= \tilde{\psi}(f_1/g_1) + \tilde{\psi}(f_2/g_2),$$

and for any $c \in k$

$$\tilde{\psi}(c \cdot f/g) = \psi(c \cdot f) \cdot \psi(g)^{-1}$$
$$= c \cdot \psi(f)\psi(g)^{-1} = c\,\tilde{\psi}(f/g).$$

Considering the cases $c = \pm 1$ we see $\tilde{\psi}$ is additive group map, and hence also a k-module map.

For the product, finally, we have

$$\tilde{\psi}(f_1 f_2/g_1 g_2) = \psi(f_1)\psi(f_2)\psi(g_1)^{-1}\psi(g_2)^{-1}$$
$$= \psi(f_1)\psi(g_1)^{-1}\psi(f_2)\psi(g_2)^{-1}$$
$$= \tilde{\psi}(f_1/g_1) \cdot \tilde{\psi}(f_2/g_2). \quad \blacksquare$$

Def$^{\underline{n}}$: For a field extension $k \to F$, we say elements $a_1, \ldots, a_n$ are algebraically independent over k if the induced k-alg map

$$\phi_a : k[x_1, \ldots, x_n] \longrightarrow F, \quad x_i \mapsto a_i,$$

is injective.

By Lemma 6 and freeness of the poly ring [Thm 6, PolyT] we see that there is a bijection:

$$\mathrm{Hom}_{k\text{-alg}}(k[x_1,\dots,x_n], F) \xrightarrow{\sim} \{n\text{-tuples of alg. indep. elem in } F\}$$

$$\phi \longmapsto (\phi(x_1),\dots,\phi(x_n)).$$

~ II. 2/2 Proving Zariski pt 2: Some lemmas

Lemma 7: Let k be a Noeth. comm. ring and $k \to R \xrightarrow{\phi} S$ be a sequence of comm ring maps for which ϕ is injective and the following hold:

(a) S is of finite type over k (fin. gen'd k-alg).

(b) S is finitely gen'd as an R-module.

Then R is of finite type over k.

Proof: Let $x_1,\dots x_n$ be generators for S, as a k-algebra, and $y_1,\dots,y_m$ be gen for S as a R-module. Then we have

$$x_i = \sum_{j=1}^{m} a_{ij} y_j, \quad y_r y_s = \sum_t b_{rs}^t y_t \quad \text{for } a_{ij}, b_{rs}^t \in R.$$

Consider the subalgebra $R' = k[a_{ij}, b_{rs}^t : i,j,r,s,t] \subseteq R$. Since k is Noetherian R' is a Noetherian ring, and we have that S is a finite module over R'.

It follows by Noetherianity that R is a finite module over $R^{\wedge}$, and hence that R is a finite type alg over k. ∎

Lemma 8: Any PID R has infinitely many distinct irreducible elements.

To be more clear, for $\mathrm{Irred}(R)$ the collection of irreducibles, and $R^{\times}$ acting on $\mathrm{Irred}(R)$ by scaling, the set $\mathrm{Irred}(R)/R^{\times}$ is infinite.

Proof: For $p_1, \dots, p_n$ any finite collection of irreducibles and $p = u p_i$ for some $i \in \{1, \dots, n\}$ and unit u, if $p \mid (p_1 \cdots p_n - 1)$ then

$$p \cdot a = (ua) \cdot p_i = p_1 \cdots p_n - 1$$
$$\Rightarrow \quad p_i(ua - p_1 \cdots p_{i-1} p_{i+1} \cdots p_n) = 1$$
$$\Rightarrow \quad p_i \text{ is a unit, a contradiction.}$$

Hence there is an irreducible p which is not an associate to any of the p_i. ∎

~ II 3/2 Proof of Zariski

Recall the statement: For any maximal ideal $m \subseteq k[x_1, \dots, x_n]$, the quotient

$$k(m) = k[x_1, \dots, x_n]/m$$

is a finite field extension of k

$$\begin{tikzcd} & k[x_1,\dots,x_n] \arrow[dr, dashed] & \\ k \arrow[ur, dashed] \arrow[rr] & & k(m). \end{tikzcd}$$

Proof: This is equivalent to the claim that all elem in $k(m)$ are algebraic. Suppose this is not the case, and (after permutation) we assume that

$$k[x_1,\dots,x_m] \to k(m) =: K$$

is injective, giving a field injection $k(x_1,\dots,x_m) \to K$ by Lemma 6, and that all remaining

$$\alpha_i = \text{image } x_i \in K$$

are algebraic over the subfield $k(x_1,\dots,x_m) \subseteq K$. Then K is finite over $k(x_1,\dots,x_m)$, and since K is of finite type over k, $k(x_1,\dots,x_m)$ is of finite type over k by Lemma 7.

Take finite generators

$$\xi_1,\dots,\xi_t \in k(x_1,\dots,x_m), \quad \xi_i = f_i/g_i,$$

and let p be irreducible in $k[x_1,\dots,x_m]$ with

$p \nmid g_i$ for all i. Such p exists by Lemma 8. Since $p^{-1} \in k(x_1,\dots,x_m)$ we have now

$$p^{-1} = \sum_{\vec{i}} c_{\vec{i}}\, \xi^{\vec{i}} = f(x_1,\dots,x_m)/g(x_1,\dots,x_m)$$

for $g(x_1,\dots,x_m)$ a product of the g_i, which gives

$$p \cdot f/g = 1 \Rightarrow p \cdot f = g \Rightarrow p \mid g \Rightarrow p \mid g_i,$$

a contradiction. Thus, K is algebraic and of fin.

type over k, giving K finite dim over k. ∎

As an immediate corollary we have the following.

Theorem (Alt. Zariski): For any field k, any finite type k-algebra R, and any maximal ideal $m \in R$, the quotient $\kappa(m)$ is a finite field extension of k.

~ III. Zariski and algebraic closure

For any finite field extension K/k, we have finitely many $\alpha_1, \ldots, \alpha_n$ in $\bar{k}$ for which
$$K \cong k(\alpha_1, \ldots, \alpha_n) \subseteq \bar{k},$$
as a k-algebra.

Corollary 9: Every maximal ideal $m \subseteq k[x_1, \ldots, x_n]$ appears as the kernel of an algebra map
$$\phi_\alpha : k[x_1, \ldots, x_n] \to \bar{k}, \quad x_i \mapsto \alpha_i.$$
More generally for any max ideal $m \subseteq R$ in a finite type k-alg, m appears as the kernel of an algebra map
$$\phi : R \to \bar{k}.$$

Proof: The case of $R = k[x_1, \dots, x_n]$ is clear, by the above arguments. For general R we have a surj alg map $\pi: k[x_1, \dots, x_n] \to R$ and for $\tilde{m} = \pi^{-1}(m)$ we have that $\tilde{m} \subseteq k[x_1, \dots, x_n]$ is maximal and we have an induced algebra isomorphism

$$k(\tilde{m}) \xrightarrow{\sim} R/m .$$

Hence from an algebra embedding $k(\tilde{m}) \to \bar{k}$ we obtain an algebra embedding $R/m \to \bar{k}$. ∎

Corollary 10: For any alg closed field $k = \bar{k}$, any maximal ideal $m \subseteq k[x_1, \dots, x_n]$ is of the form $m = (x_1 - \alpha_1, \dots, x_n - \alpha_n)$ for uniquely determined $\alpha_i \in k$.

Proof: By Zariski / Cor 9 $m = \ker(\varphi)$ for an alg map $\varphi: k[x_1, \dots, x_n] \to k$, via alg closure. Hence for each x_i we have $x_i - \alpha_i \in \ker \varphi = m$ for $\alpha_i = \varphi(x_i)$. Thus

$$m_\alpha = (x_i - \alpha_i : 1 \le i \le n) \subseteq m .$$

Since m_α is already maximal this gives $m = m_\alpha$. As for uniqueness, since the composite $k \to k[x_1, \dots, x_n] \to k$ is injective, if $x_i - \alpha \in \ker(\pi) = m$ then

$$\alpha = \pi(\alpha) = \pi(x_i) = \alpha_i .$$ ∎

Alt Proof: By Zariski, the induced map

$$k \to k[x_1,\dots,x_n]/m \qquad (*)$$

is an isomorphism, so that $\bar{x}_i = a_i$ for some $a_i \in k$, at each $i = 1,\dots,n$. Thus

$\bar{x}_i - a_i = 0 \Rightarrow x_i - a_i \in m$.

If $x_i - a \in m$ for some $a \in k$ then, for uniqueness, we get

image of $a - a_i$ in $k[x_1,\dots,x_n]/m$

$= (\bar{x}_i - a_i) - (\bar{x}_i - a)$

$= 0$.

By injectivity of $(*)$ we have $a = a_i$. $\blacksquare$

Below we let $\mathbb{A}^n_k \overset{\text{a fancy not for}}{=} k^n$.

Corollary 11: For any alg closed field $k = \bar{k}$, like $\overline{\mathbb{F}_p}$, $\overline{\mathbb{Q}}$, $\mathbb{C}$, $\overline{\mathbb{C}(t)}$, etc., the set map

$$\mathbb{A}^n_k \longrightarrow \{\text{max ideals in } k[x_1,\dots,x_n]\}$$

$$\alpha = (a_1,\dots,a_n) \longmapsto m_\alpha = (x_i - a_i : 1 \le i \le n)$$

is a bijection!

~ IV The maximal spectrum

Def$^{\underline{n}}$: For any commutative ring R, define

$$\mathrm{mSpec}(R) = \left\{\begin{array}{l}\text{The collection of all}\\ \text{max'l ideals } \mathfrak{m} \subseteq R\end{array}\right\}.$$

Take more generally,

$$\mathrm{Spec}(R) = \left\{\begin{array}{l}\text{The collection of all}\\ \underline{\text{prime}} \text{ ideals } \mathfrak{p} \subseteq R\end{array}\right\}.$$

We just saw, for $k = \mathbb{C}$ for example,

$$\mathrm{mSpec}(\mathbb{C}[x_1, \dots, x_n]) \cong \mathbb{C}^n.$$

Over $k = \mathbb{R}$, alternatively,

$$\mathrm{mSpec}(\mathbb{R}[x_1, \dots, x_n]) = \mathbb{R}^n \sqcup \left\{\begin{array}{l}\text{a bunch of}\\ \text{other stuff}\\ \text{having to do w/}\\ \mathbb{C}\end{array}\right\}.$$

For example

$$\mathrm{mSpec}(\mathbb{R}[x]) = \mathbb{R} \sqcup \left\{\begin{array}{l}(x^2 - bx - a)\\ \text{such that}\\ \frac{1}{4}b^2 < a\end{array}\right\}$$

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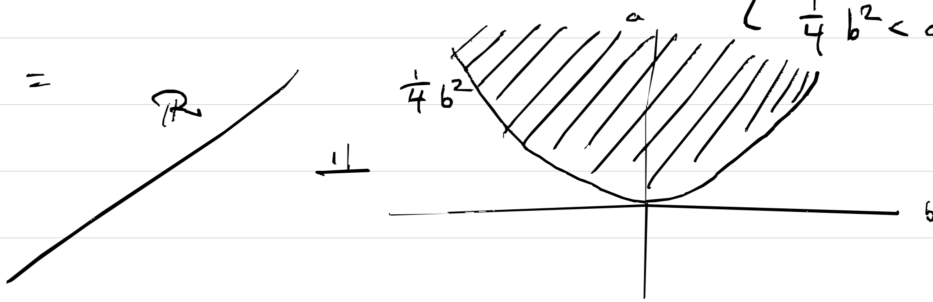


~ IV Further analyses of mSpec(R)

Let's look at some examples of quotients $k[x_1, \dots, x_n]/I$.

Consider the quotient, for example,

$$R = \mathbb{C}[x,y] \big/ (y^2 - x^3)$$

By Zariski each max ideal $\mathfrak{m} \subseteq R$ has $R/\mathfrak{m} = \mathbb{C}$ and restriction along the projection $\pi : \mathbb{C}[x,y] \to R$ gives an injection

$$\pi^* : \mathrm{mSpec}(R) \hookrightarrow \mathrm{mSpec}(\mathbb{C}[x,y]) = \mathbb{C}^2$$

$$\mathfrak{m} \longmapsto \pi^{-1}(\mathfrak{m})$$

$$\mathrm{mSpec}(R) \subseteq \mathbb{C}^2 .$$

Each max ideal $\mathfrak{m}_\alpha \subseteq \mathbb{C}[x,y]$ appears as the corresponding kernel of the evaluation map $ev_\alpha : \mathbb{C}[x,y] \to \mathbb{C}$.

Observe: A max ideal $\mathfrak{m}_\alpha \in \mathrm{mSpec}(\mathbb{C}[x,y])$ is in the image of the inclusion π^*

$\iff$ (a) $y^2 - x^3 \in \ker(ev_\alpha)$

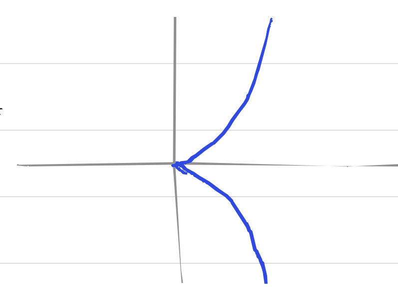
$\iff$ (b) $\alpha_2^2 - \alpha_1^3 = 0$

$\iff$ (c) (α_1, α_2) is a point on the complex curve $y^2 - x^3 = 0$

$\subseteq \mathbb{C}^2$

Hence

$$\mathrm{mSpec}\left(\mathbb{C}[x,y] \big/ (y^2 - x^3)\right) =$$



More generally, for any finite type k-algebra with $k=\bar{k}$ and a given presentation

$$\pi: k[x_1,\dots,x_n] \longrightarrow R, \quad R \cong k[x_1,\dots,x_n]/\ker\pi,$$

Noetherianity gives

$$\ker(\pi) = (f_1,\dots,f_r)$$

for some functions $f_i(x_1,\dots,x_n)$. We again have the inclusion

$$\pi^*: m\mathrm{Spec}(R) = m\mathrm{Spec}\left(k[x_1,\dots,x_n]/(f_1,\dots,f_r)\right)$$
$$\hookrightarrow m\mathrm{Spec}(k[x_1,\dots,x_n]) = \mathbb{C}^n.$$

Theorem 12: Let R be a finite type k-alg over an algebraically closed field $k=\bar{k}$. For a chosen presentation $R \cong k[x_1,\dots,x_n]/(f_1,\dots,f_r)$ and corresp. surj. alg map $\pi: k[x_1,\dots,x_n] \longrightarrow R$, taking the preimage along π provides an inclusion

$$\pi^*: m\mathrm{Spec}(R) \hookrightarrow m\mathrm{Spec}(\mathbb{C}[x_1,\dots,x_n]) \underset{\text{Zar}}{=} k^n$$
$$m \longmapsto \pi^{-1}(m).$$

The image of this inclusion consists of all points $(a_1,\dots,a_n) \in k^n$ such that $f_i(a_1,\dots,a_n)=0$ of all n,

$$m\mathrm{Spec}(R) \xrightarrow{\cong} \left\{ \begin{array}{l} \text{Simultaneous sols} \\ \text{to the equations} \\ f_i(x_1,\dots,x_n)=0,\ i=1,\dots,r \end{array} \right\} \subseteq k^n.$$

Warning! A general finite type k-alg does <u>not</u> come equipped with a specified presentation $R \cong$ quotient of a poly ring. So you can think of

$$\underbrace{\mathrm{mSpec}(R)}_{\text{A choice free, coord free object}} \overset{?}{\longrightarrow} \underbrace{\left\{ \begin{array}{c} \text{Sol'n to eq'ns} \\ f_1(x_1,\dots,x_n)=0 \\ \vdots \\ f_r(x_1,\dots,x_n)=0 \end{array} \right\}}_{\text{Choosing coordinates}}$$

Warning! If $k \neq \bar{k}$, then $\mathrm{mSpec}(R)$ is still a thing, its still perfectly fine to think about, but this geometric picture is no longer valid,

$$\mathrm{mSpec}(R) \neq \text{Vanishing loci of some poly's in } k^n \text{ (or just } \mathbb{C}^n)$$

This is due to the existence of field extensions nontrivial finite = fin type field extensions $k \subseteq K$.

Question: If I have some presentation

$$\pi : k[x_1,\dots,x_n] \to R,$$

how do I think about <u>non-maximal</u> primes in $\mathrm{Spec}(R)$, in geometric terms?

HW

1. Prove that an ideal p in a commutative ring R is prime if and only if p is the kernel of a ring map $\phi: R \to S$ to a domain S.

2. (a) Prove that the ideal (y^3-x^2, y^5-x^2) is not prime in $\mathbb{C}[x,y]$.

(b) Prove that the ideal $(z-x^2, z-y^2)$ is not prime in $\mathbb{Q}[x,y,z]$

(c) Prove that the ideal $(y-x^3-2x, x^4+x^3+x^2+x+1)$ is prime in $\mathbb{Q}[x,y]$.

(d) Prove that the ideal (z^2-xy) is prime in $\mathbb{F}_5[x,y,z]$. (Hint (z^2-xy) is the kernel of an algebra map $\mathbb{F}_5[x,y,z] \to \mathbb{F}_5[t,u]$.)

3. For an ideal I in a commutative ring R, define $Van(I) = \{$all primes $p \subseteq R$ s.t. $I \subseteq p\} \subseteq Spec(R)$ and $mVan(I) = \{$all max ideals $m \subseteq R$ with $I \subseteq m\} \subseteq mSpec(R)$. Prove that for any collection of ideals I_λ, $\lambda \in \Lambda$,

$$\bigcap_{\lambda \in \Lambda} Van(I_\lambda) = Van(\sum_\lambda I_\lambda)$$

and for any finite collection $I_1, I_2, \ldots, I_n$

$$\mathrm{Van}(I_1) \cup \cdots \cup \mathrm{Van}(I_n) = \mathrm{Van}\left(\prod_{j=1}^{n} I_j\right).$$

Convince yourself that the analogous equalities hold when we replace Van with mVan.

4. For $\mathcal{U}_I = \mathrm{Spec}(R) - \mathrm{Van}(I)$ and $m\mathcal{U}_I = m\mathrm{Spec}(R) - m\mathrm{Van}(I)$, prove that the collection $\{\mathcal{U}_I : I \text{ an ideal in } R\}$ and $\{m\mathcal{U}_I : I \text{ an ideal in } R\}$ are the open sets in topologies for $\mathrm{Spec}(R)$ and $m\mathrm{Spec}(R)$ respectively. Prove furthermore that the $\mathcal{U}_f := \mathcal{U}_{(f)}$ and $m\mathcal{U}_f = m\mathcal{U}_{(f)}$ provide bases for these topologies.

These topologies are called the Zariski topology on $\mathrm{Spec}(R)$ and $m\mathrm{Spec}(R)$

5. Describe the open sets in $m\mathrm{Spec}(\mathbb{C}[x])$. Prove that $m\mathrm{Spec}(\mathbb{C}[x])$ is a compact topological space. Prove also that $m\mathrm{Spec}(\mathbb{C}[x])$ is non-Hausdorff as a topological space.

6. Prove that $\mathrm{Spec}(R)$ is compact whenever R is Noetherian.