Localization for rings and modules

I. Localization

Def^{n}: A multiplicative subset T in a comm. ring R is a subset with $1 \in T$ and $s \cdot t \in T$ whenever $s, t \in T$.

Given a mult. subset $T \subseteq R$ define the set of equivalence classes

$$R[T^{-1}] = \left\{ \begin{array}{l} \text{fractions } f/t \text{, or } ft^{-1}\text{, where} \\ ft^{-1} = gs^{-1} \text{ whenever} \\ u\cdot(fs - gt) = 0 \text{ for some } u \in T. \end{array} \right.$$

Example: Suppose $f \in R$. Then we have the mult. subset $\{f^n : n \geq 0\}$, and take
$$R_f = R[f^{-1}] = R[\{f^n : n \geq 0\}^{-1}].$$

Lemma 1: The canonical map of sets
$$(*) \quad loc_T : R \to R[T^{-1}], \quad f \mapsto f/1,$$
is injective if and only if T contains <u>no</u> zero divisors.

Proof: Suppose T contains no zero divisors. Then
$$f/1 = g/1 \iff u(f-g) = 0 \text{ at some } u \in T$$
$$\iff f - g = 0 \iff f = g.$$
So $(*)$ is injective. Conversely, if some $u \in T$ is a

zero divisor then for nonzero f with $u \cdot f = 0$ we have $u(f - 0) = 0$ giving

$$loc_T(f) = loc_T(0).$$

∎

Example: Suppose $t \in R$ is nilpotent. Then $0 \in \{t^n : n \geq 0\}$. Say, $0 = t^N$. Then for each $f \in R$ $t^N \cdot f = 0$ giving

$$f/t^m = 0/1 = 0$$

for all $f/t^m \in R_t$. So we find $R_t = \{0\}$.

Example: If R is a domain and $T \subset R$ is a multiplicative set with $0 \notin T$, then

$$loc : R \longrightarrow R[T^{-1}]$$

is injective. In particular

$$f/s = g/t \quad \text{in } R[T^{-1}]$$

if and only if $f \cdot t = g \cdot s$.

Lemma 2: For any mult. set T in comm. R, the set $R[T^{-1}]$ is a ring under the additive and multiplicative operations

$$f/s + g/t := (ft + gs)/st$$

and

$$(f/s) \cdot (g/t) = (fg)/(st),$$

respectively. The unit in $R[T^{-1}]$ is $1/1$ and $0 = 0/1$.

Proof: Associativity of addition is clear, and we have the additive unit $0 = 0/1$ and additive inverses $-f/s = (-f)/s$. Associativity of mult. is also clear, as is the fact that $1 = 1/1$ is a unit for multiplication. All that is left is distributivity of mult. over addition.

(*) We check directly.

$$f/s\,(g_1/t_1 + g_2/t_2) = (f/s)\,(g_1t_2 + g_2t_1)/t_1\cdot t_2$$
$$= (fg_1t_2 + fg_2t_1)/s\cdot t_1\cdot t_2$$

while

(**) $$(fg_1)/st_1 + (fg_2)/st_2 = (fg_1st_2 + fg_2st_1)/s^2t_1t_2.$$

Since

$$s^2t_1t_2\cdot(fg_1t_2 + fg_2t_1) = st_1t_2(fg_1st_2 + fg_2st_1)$$

we see that (*) = (**), as desired. ∎

Lemma 3: Consider any mult. subset $T \subseteq R$, and endow the set $R[T^{-1}]$ with the ring structure from Lemma 2. Under this structure, the set map

$$\mathrm{loc}_T : R \to R[T^{-1}],\quad f \mapsto f/1,$$

is a morphism of commutative rings.

Proof: Just check $f/1 + g/1 = (f1 + g\cdot 1)/1 = (f+g)/1$ and $(f/1)\cdot(g/1) = (fg)/1$. ∎

From now on, when we write $R[T^{-1}]$ we always mean the set of fractions f/s, $s \in T$, endowed with the ring structure from Lemma 2.

Def$^{\underline{n}}$: For a comm. ring R and a mult. subset T in R, the associated ring $R[T^{-1}]$ is called the localization of R at the mult. set T, and the canonical map $loc_T : R \to R[T^{-1}]$ is called the localization map.

Remark: If R is a k-algebra for a field, or more generally a comm. ring k, then $R[T^{-1}]$ inherits an algebra structure via the map

$$\begin{tikzcd} & R \arrow[r, "loc"] & R[T^{-1}] \\ k \arrow[ur, "unit_R"] \arrow[urr, dashed, "unit_{R[T^{-1}]}"'] & & \end{tikzcd}$$

.

~ II. The universal property of localization

Note that, in the localization $R[T^{-1}]$ all elements $t \in T$ become units. Indeed, for $t \in T$ we have $1_{R[T^{-1}]} = 1/1 = t/t$. Hence any ring map $\psi : R[T^{-1}] \to S$ sends $loc(T)$ into the units $S^{\times}$ in S.

Theorem 4: Let $T \subseteq R$ be a multiplicative subset, and S be an arbitrary commutative ring. Restriction along localization map provides a bijection

$$loc^*: Hom_{Rings}(R[T^{-1}], S) \xrightarrow{\sim} \left\{ \begin{array}{c} \text{Ring maps } \phi: R \to S \\ \text{with } \phi(T) \subseteq \text{units in } S \end{array} \right\}.$$

To say things in human terms, there exists a ring map $\tilde{\phi}: R[T^{-1}] \to S$ completing a diagram

$$\begin{tikzcd} & R[T^{-1}] \arrow[dr, dashed, "\exists ?"] & \\ R \arrow[ur, "loc"] \arrow[rr, "\phi"'] & & S \end{tikzcd}$$

if and only if ϕ sends T into $S^\times$. Furthermore, in this case the extension $\tilde{\phi}$ to the localization is <u>unique</u>.

Proof of Thm 4: Given two maps

$$\psi_1, \psi_2 : R[T^{-1}] \to S$$

with $\phi = \psi_1 \circ loc_T = \psi_2 \circ loc_T$, we have

$$\psi_1(f/s) = \psi_1(f)\,\psi_1(s^{-1}) = \psi_1(f)\,\psi_1(s)^{-1}$$
$$= \phi(f)\,\phi(s)^{-1} = \psi_2(f)\,\psi_2(s)^{-1} = \psi_2(f/s).$$

Hence $\psi_1 = \psi_2$. This verifies injectivity of loc_T^*. Now suppose we have a ring map $\phi: R \to S$ which sends T into $S^\times$. Take now $\tilde{\phi}: R[T^{-1}] \to S$ by $\tilde{\phi}(fs^{-1}) := \phi(f)\,\phi(s)^{-1}$.

as in the proof of [Lemma 6, Ideals], one checks directly that $\tilde{\phi}$ is a well defined ring map. This gives surjectivity of loc_T^*. ∎

This is already interesting and important at the ring theoretic level, but we also have consequences for module categories.

Given mult. $T \subseteq R$, and any R-module M, we have the action map $act_M : R \to End_{\mathbb{Z}}(M)$. This map admits a unique lift to the localization $R[T^{-1}]$ if and only if all $t \in T$ act on M as invertible endomorphisms.

For example, for a prime $p \in \mathbb{Z}$, $r \geq 1$, and q coprime to p, we can write $1 = aq + bp^r$ via coprimeness to p^r as well, so that q acts as a unit on the $\mathbb{Z}$-modules $\mathbb{Z}/p^r\mathbb{Z}$, or more generally on any p-torsion $\mathbb{Z}$-module. So we get an induced action $\mathbb{Z}[q^{-1}] \to End_{\mathbb{Z}}(\mathbb{Z}/p^r\mathbb{Z})$.

Anyway, from all these blablabla we see that the restriction functor

$$res : R[T^{-1}]\text{-mod} \to R\text{-mod}$$

has image consisting of all those R-modules on which

T acts invertibly. Further, for two such modules M and N, and any R-module map $\alpha: M \to N$, we have for general fs^{-1} in $R[T]$,

$$s\cdot\alpha(f\cdot s^{-1}\cdot m) = \alpha(sfs^{-1}m) = \alpha(f\cdot m) = f\cdot\alpha(m)$$

giving, by applying $s^{-1}\cdot-$,

$$\alpha(fs^{-1}\cdot m) = (fs^{-1})\cdot\alpha(m).$$

So α is a map of $R[T^{-1}]$-modules.

Since restriction is always faithful, the above info tells us that $\mathrm{res}: R[T^{-1}]\text{-mod} \to R\text{-mod}$ is in fact fully-faithful!

Theorem 5: For any mult. subset $T \subseteq R$, the restriction functor, along localization, provides a fully faithful functor

$$\mathrm{res}: R[T^{-1}]\text{-mod} \to R\text{-mod}$$

which is an equivalence onto the full subcat of modules on which T acts invertibly.

Proof: See above. ∎

III. Localization of modules

Let $T \subseteq R$ be mult. in a comm. ring, and consider any R-module M. Define the new

R-module $M[T^{-1}]$ as the collection of equivalence classes m/t, for $m \in M$ and $t \in T$, where $m/t = m'/s \Leftrightarrow \exists\, u$ in T w/ $u(sm - tm') = 0$.
We endow $M[T]$ with the additive structure

$$m/t + m'/s = (sm + t\cdot m')/st,$$

which is clearly associative w/ additive inverses $-(m/t) = (-m)/t$, and with the R-action

$$a\cdot(m/s) := (am)/s.$$

We note that for any $t \in T$ and m/s in $M[T^{-1}]$,

$$t\cdot(m/st) = (tm)/st = m/s$$

and if $t\cdot(m/s) = 0$ then

$(u\cdot t)\cdot m = 0$ for some $u \in T \Rightarrow m/s = 0$.
Hence each $t\cdot - : M[T^{-1}] \to M[T^{-1}]$ is invertible. We therefore obtain a uniquely det $R[T^{-1}]$-module structure on $M[T^{-1}]$, which is given by the expected formula
$(f/s)\cdot(m/t) = (fm)/(st)$.

Note that we have the R-module map
$\mathrm{loc}_T(M) : M \to M[T^{-1}]$, $m \mapsto m/1$.

We note that the assignment $M \mapsto M[T^{-1}]$ extends to a functor

$$-[T^{-1}]: R\text{-mod} \to R[T^{-1}]\text{-mod}$$
$$M \mapsto M[T^{-1}], \quad \alpha \mapsto \alpha[T^{-1}],$$

where for an R-module map $\alpha: M \to N$ the induced map $\alpha[T^{-1}]: M[T^{-1}] \to N[T^{-1}]$ is the unique $R[T^{-1}]$-module map which completes a diagram

$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[d, "\mathrm{loc}"'] & N \arrow[d, "\mathrm{loc}"] \\ M[T^{-1}] \arrow[r, dashed, "\exists !", "{\alpha[T^{-1}]}"'] & N[T^{-1}] \end{tikzcd}.$$

One can check the formula $\alpha[T^{-1}](m/t) = t^{-1}\cdot\alpha(m)$.

Theorem 6: For any R-module M, and $R[T^{-1}]$-module N, restriction along $\mathrm{loc}_T(M)$ induces a bijection

$$\mathrm{loc}^*: \mathrm{Hom}_{R[T^{-1}]}(M[T^{-1}], N) \xrightarrow{\sim} \mathrm{Hom}_R(M, \mathrm{res}(N)).$$

Proof: Since $M[T^{-1}]$ is generated by the image of M, as an $R[T^{-1}]$-module, injectivity of loc^* is clear. As for surjectivity, given an R-module map $\alpha: M \to \mathrm{res}(N)$ we have the proposed lift $\tilde{\alpha}: M[T^{-1}] \to N$, $\tilde{\alpha}(m/t) := t^{-1}\cdot\alpha(m)$.

To see that $\tilde{\alpha}$ is well-defined as a map of sets, we have $u\cdot(sm - t\cdot m') = 0$ for $u,s,t \in T$

$$\Rightarrow \quad u(s\alpha(m) - t\alpha(m')) = 0$$

$$\Rightarrow \quad s\alpha(m) = t\alpha(m') \quad \text{since } \mathrm{act}_N(u) \in \mathrm{Aut}(N)$$

$$\Rightarrow \quad t^{-1}\alpha(m) = s^{-1}\alpha(m).$$

For additivity $\tilde{\alpha}(m/t) + \tilde{\alpha}(m'/s)$

$$= t^{-1}\alpha(m) + s^{-1}\alpha(m') = s^{-1}t^{-1}(\alpha(sm) + \alpha(tm'))$$

$$= \tilde{\alpha}(m/t + m'/s),$$

and obv. $\tilde{\alpha}(0/1) = 0$. For $R[T^{-1}]$-linearity $\tilde{\alpha}(f_{s^{-1}}\cdot(m/t))$

$$= s^{-1}t^{-1}\cdot\alpha(fm) = f_{s^{-1}}\cdot\tilde{\alpha}(m/t). \quad \blacksquare$$

~IV Localization via base change.

For mult. $T \subseteq R$, and any R-module M, we have the universal map to the base change $\mathrm{univ}_M : M \to R[T^{-1}] \otimes_R M$.
Via the univ prop. of localization we now obtain a unique $R[T^{-1}]$-module map $\xi_M : M[T^{-1}] \longrightarrow R[T^{-1}] \otimes_R M$ which completes a diagram

$$\begin{tikzcd}
 & M \arrow[dl, "\mathrm{loc}_T"'] \arrow[dr, "\mathrm{univ}_M"] & \\
M[T^{-1}] \arrow[rr, "\xi_M"'] & & R[T^{-1}] \otimes_R M
\end{tikzcd} \qquad (*)$$

Theorem 7: For any R-module M, and mult. subset T, the completing map
$$\xi_M : M[T^{-1}] \longrightarrow R[T^{-1}] \otimes_R M$$
is an isomorphism of $R[T^{-1}]$-modules. Furthermore, the ξ_M together define a natural isomorphism of functors $\xi : -[T^{-1}] \xrightarrow{\sim} R[T^{-1}] \otimes_R -$.

Proof: Clear from the respective univ props from Thm 6 and [Sect. 5, Algmod]. (Exercise) ∎

Corollary 8 (Exactness of localization): For any mult. subset $T \subseteq R$, the functor
$$R[T^{-1}] \otimes_R - \quad (\cong -[T^{-1}])$$
is <u>exact</u>.

Proof: From HW we know $R[T^{-1}] \otimes_R -$ is right exact. So we need only prove that it (also) preserves injections. Via the nat $\cong$ of Theorem 7, it suffices to prove that the functor $-[T^{-1}]$ preserves injections.

Let $\alpha : M' \to M$ be an injective R-module map, and consider $m/t \in M'[T^{-1}]$ for which $\alpha[T^{-1}](m/t) = t^{-1}\alpha(m) = 0$. Since t acts as a unit on $M[T^{-1}]$ we then have $\alpha(m) = 0$ in $M[T^{-1}]$. Hence there exists $u \in T$ with

$$0 = u \cdot \alpha(m) = \alpha(um) \Rightarrow um = 0$$

by injectivity of α. Hence $m = 0$ in $M[T^{-1}]$. So we see that $\alpha[T^{-1}]$ is in fact injective. ∎

V Localization and primes

Let $T \subseteq R$ be a mult. subset in R, and consider a prime ideal $\mathfrak{p} \subseteq R$ for which $\mathfrak{p} \cap T = \emptyset$. Then for the ideal

$$R[T^{-1}] \cdot \mathfrak{p} = \{ x/t^{-1} : x \in \mathfrak{p},\ t \in T \}$$

generated by (the image of) $\mathfrak{p}$ in $R[T^{-1}]$, we have

$$f/t \cdot g/s = (f \cdot g)/st \in R[T^{-1}] \cdot \mathfrak{p}$$
$$\Leftrightarrow \quad u \cdot fg \in \mathfrak{p} \text{ for some } u \text{ in } T.$$

By primeness and the fact that $u \notin \mathfrak{p}$ by hypothesis, we have

$$f \cdot g \in \mathfrak{p} \Rightarrow f \in \mathfrak{p} \text{ or } g \in \mathfrak{p}$$
$$\Rightarrow f/t \in R[T^{-1}] \cdot \mathfrak{p} \text{ or}$$
$$g/s \in R[T^{-1}] \cdot \mathfrak{p}.$$

Furthermore, $1 \in R[T^{-1}] \cdot \mathfrak{p}$ implies $u = t \cdot x$ for some u in T and $t \in T \Rightarrow u \in T \cap \mathfrak{p}$, which is a lie! So $R[T^{-1}]\cdot\mathfrak{p} \subsetneq R[T^{-1}]$ in this case. Thus the "extension of scalars op."

$$\mathfrak{p} \longmapsto R[T^{-1}]\cdot\mathfrak{p}$$

provides a map

$$R[T^{-1}]\cdot - : \left\{\begin{array}{c}\text{Primes } \mathfrak{p}\subseteq R \text{ w/} \\ \mathfrak{p}\cap T = \emptyset\end{array}\right\} \longrightarrow \{\text{primes } \mathfrak{q}\subseteq R[T^{-1}]\} = \operatorname{Spec}(R[T^{-1}]) \qquad (*)$$

Theorem 9 (Loc & Primes): Pulling back along localization $loc_T : R \to R[T^{-1}]$ provides a bijection

$$loc_T^{-1} : \operatorname{Spec}(R[T^{-1}]) \xrightarrow{\sim} \left\{\begin{array}{c}\text{Primes } \mathfrak{p}\subseteq R \\ \text{with } \mathfrak{p}\cap T = \emptyset\end{array}\right\} \subseteq \operatorname{Spec}(R).$$

The inverse to loc_T^{-1} is given by the extension of scalars map $(*)$.

Proof: We have, for any prime $\mathfrak{q}\subseteq R[T^{-1}]$,

$$R[T^{-1}]\cdot loc^{-1}(\mathfrak{q}) = \{x/t : x/1 \in \mathfrak{q}\} \subseteq \mathfrak{q}.$$

For the opposite inclusion, for x/t in $\mathfrak{q}$ we have $loc(x) = x/1 = t\cdot(x/t) \in \mathfrak{q}$ giving

$$x/t \in R[T^{-1}]\cdot loc^{-1}(\mathfrak{q}).$$

So $\mathfrak{q}\subseteq R[T^{-1}]\cdot loc^{-1}(\mathfrak{q}) \Rightarrow \mathfrak{q} = R[T^{-1}]\, loc^{-1}(\mathfrak{q})$.

For the opposite composite, given a prime $\mathfrak{p}\subseteq R$ with $\mathfrak{p}\cap T = \emptyset$ we consider the ideal

$\mathfrak{p}' =$

$l_\infty^{-1}(R[T^{-1}]\cdot\mathfrak{p})$ in R. We have

$$x \in \mathfrak{p}' \iff x/1 = y/t \text{ for some } y \in \mathfrak{p} \text{ and } t \text{ in } T$$

$$\iff t \cdot x = y \text{ for some } y \in \mathfrak{p},\ t \in T$$

$$\iff x \in \mathfrak{p} \text{ via primeness of } \mathfrak{p} \text{ and } \mathfrak{p} \cap T = \emptyset.$$

So we see $l_\infty^{-1}(R[T^{-1}]\cdot\mathfrak{p}) = \mathfrak{p}$. Hence we observe

$$(R[T^{-1}]\cdot -) \circ l_\infty^{-1} = \mathrm{id}_{\mathrm{Spec}(R[T^{-1}])}$$

and

$$l_\infty^{-1} \circ (R[T^{-1}]\cdot -) = \mathrm{id}_{\{\text{primes w/ } \mathfrak{p}\cap T = \emptyset\}}. \qquad \blacksquare$$

For example, take any $f \in R$ and consider the localization $R_f = R[f^{-1}]$. For any prime $\mathfrak{p} \in \mathrm{Spec}(R)$ we have $\mathfrak{p} \cap \{f^n : n \geq 0\} \neq 0$ if and only if $f^n \in \mathfrak{p}$ for some $n > 0$ (since $1 \notin \mathfrak{p}$) giving $f \in \mathfrak{p}$ by primeness. Hence

$$\left\{\begin{matrix}\text{primes } \mathfrak{p} \text{ in } R \\ \text{with } \mathfrak{p}\cap\{f^n : n\geq 0\} = \emptyset\end{matrix}\right\} = \left\{\begin{matrix}\text{primes } \mathfrak{p} \text{ in } R \\ \text{with } f \notin \mathfrak{p}\end{matrix}\right\}$$

$$= \mathrm{Spec}(R) \setminus \{\text{primes } \mathfrak{p} \mid f \in \mathfrak{p}\} = \underbrace{\mathrm{Spec}(R) \setminus \mathrm{Van}(f)}_{= \mathcal{U}_f}.$$

Hence pulling back along $l_\infty : R \to R_f$ provides a

bijection onto the basic open
$$\mathrm{Loc}^{-1} : \mathrm{Spec}(R_f) \xrightarrow{\sim} \mathcal{U}_f \subseteq \mathrm{Spec}(R).$$

~ VI Localization at a prime

Now, given a prime $\mathfrak{p} \subseteq R$ we have the complement $R \setminus \mathfrak{p}$. Since $\mathfrak{p} \neq R$, we have $1 \notin \mathfrak{p}$, and for $f, s \notin \mathfrak{p}$ we have $f \cdot s \notin \mathfrak{p}$ by primeness. Thus, $R \setminus \mathfrak{p}$ is a multiplicative subset in R.

Def$^{\underline{n}}$: For any prime $\mathfrak{p} \subseteq R$, we define
$$R_{\mathfrak{p}} := R[(R \setminus \mathfrak{p})^{-1}].$$

Note that for any prime $\mathfrak{q} \subseteq R$, $\mathfrak{q} \cap (R \setminus \mathfrak{p})$ is empty means $\mathfrak{q} \subseteq \mathfrak{p}$. Thus Thm 9 gives a bijection
$$\mathrm{Loc}_{\mathfrak{p}}^{-1} : \mathrm{Spec}(R_{\mathfrak{p}}) \xrightarrow{\sim} \left\{ \begin{array}{c} \text{Primes } \mathfrak{q} \subseteq R \text{ w/} \\ \mathfrak{q} \subseteq \mathfrak{p} \end{array} \right\}$$

Def$^{\underline{n}}$: A comm ring R is said to be local if R has a unique maximal ideal.

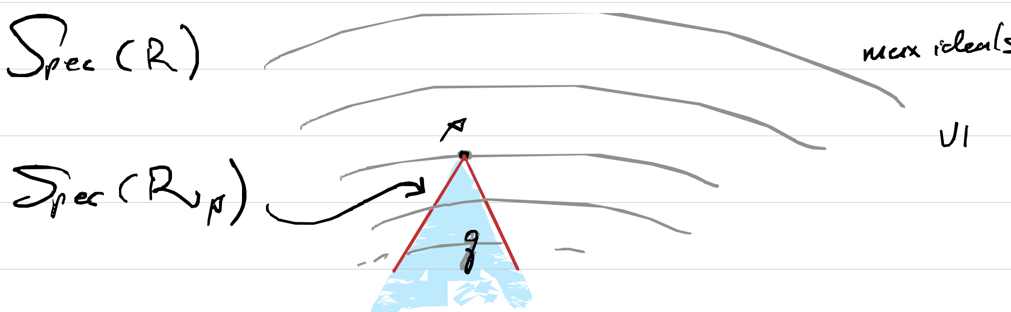
As a corollary to Thm 9 we observe the following.

Proposition 10: For any prime ideal $\mathfrak{p} \subseteq R$, the localization $R_{\mathfrak{p}}$ is a local ring with unique maximal ideal $R_{\mathfrak{p}} \cdot \mathfrak{p} \subseteq R_{\mathfrak{p}}$.

Proof: By Thm 9 we have the inverse map

$$R_{\mathfrak{p}} \cdot - : \{\text{Primes } \mathfrak{q} \subseteq \mathfrak{p}\} \xrightarrow{\sim} \mathrm{Spec}(R_{\mathfrak{p}})$$
$$\mathfrak{q} \longmapsto R_{\mathfrak{p}} \cdot \mathfrak{q}$$

which clearly preserves the orderings by inclusion. From this we see $R_{\mathfrak{p}} \cdot \mathfrak{p}$ is maximal in $\mathrm{Spec}(R_{\mathfrak{p}})$, and indeed the <u>unique</u> max element in this set. ∎



VII The residue field of a prime

Def^n: For a prime ideal $\mathfrak{p}$ in comm. R, the residue field at $\mathfrak{p}$ in $\mathrm{Spec}(R)$ is the field

$$k(\mathfrak{p}) = R_{\mathfrak{p}}/\text{unique max} = R_{\mathfrak{p}}/R_{\mathfrak{p}} \cdot \mathfrak{p}$$

Consider the extreme case of a domain R, and the prime $\{0\}$ in R.

Def$^{\underline{n}}$: For a commutative domain R, the fraction field $\mathrm{Frac}(R)$ is

$$\mathrm{Frac}(R) = R[(R \setminus \{0\})^{-1}] = \{ f/g : g \text{ nonzero} \}.$$

Example: $\mathrm{Frac}(k[x_1,\dots,x_n]) = k(x_1,\dots,x_n)$.
Why? We have all nonzero elem in $R = k[x_1,\dots,x_n]$ mapping to units in $k(x_1,\dots,x_n)$ so that we get an induced map from the localization which completes a diagram

$$\begin{tikzcd}
 & k[x_1,\dots,x_n] \arrow[dl, "\text{include}"'] \arrow[dr, "\text{include}"] & \\
\mathrm{Frac}(k[x_1,\dots,x_n]) \arrow[rr, "\exists!"] & & k(x_1,\dots,x_n)
\end{tikzcd}.$$

Since $\mathrm{Frac}(k[x_1,\dots,x_n])$ is a field, the induced ring map has trivial kernel, and since all elem in $k(x_1,\dots,x_n)$ are products $f(x_1,\dots,x_n)\cdot g(x_1,\dots,x_n)^{-1}$ the induced map is also seen to be surjective. Hence $\mathrm{Frac}(k[x_1,\dots,x_n]) \cong k(x_1,\dots,x_n)$.

Now, for any prime $\mathfrak{p} \subseteq R$, the quotient $R/\mathfrak{p}$ is a domain, so that we have the alt. construction of a field "at $\mathfrak{p}$",

$$R \longrightarrow \mathrm{Frac}(R/\mathfrak{p}).$$

To compare these two fields, we have

$\ker\left(R \xrightarrow{loc} R_{\mathfrak{p}} \to R_{\mathfrak{p}}/(R_{\mathfrak{p}}\cdot\mathfrak{p})\right) = \mathfrak{p}$, by Theorem 9 (loc & primes) so that we obtain an induced injective ring map

$$\mathrm{comp}' : R/\mathfrak{p} \longrightarrow R_{\mathfrak{p}}/R_{\mathfrak{p}}\cdot\mathfrak{p} = k(\mathfrak{p}).$$

Since $k(\mathfrak{p})$ is a field this map localizes to an injective field map

$$\mathrm{comp} : \mathrm{Frac}(R/\mathfrak{p}) \longrightarrow k(\mathfrak{p}).$$

Prop 11: For any prime $\mathfrak{p} \subseteq R$, there is a unique isomorphism of fields

$$\mathrm{comp} : \mathrm{Frac}(R/\mathfrak{p}) \longrightarrow k(\mathfrak{p})$$

which completes a diagram

$$\begin{tikzcd} & R \arrow[dl] \arrow[dr] & \\ \mathrm{Frac}(R/\mathfrak{p}) \arrow[rr, "\cong", "\mathrm{comp}"'] & & k(\mathfrak{p}) \end{tikzcd}.$$

Proof: We saw existence of such a completing map above, and uniqueness follows by the universal prop of the quotient $R \to R/\mathfrak{p}$ and localization

$$R/\mathfrak{p} \to \mathrm{Frac}(R/\mathfrak{p}).$$

Since any map of fields must be injective, we need only establish surjectivity.

For surjectivity, we note that every elem ξ in

$R_{\mathfrak{p}}$ is a product $f \cdot t^{-1}$ of an elem f in (the image of) R with the inverse of an element t in (the image of) $R \setminus \mathfrak{p}$. Hence, every element in the quotient $\kappa(\mathfrak{p})$ admits such a factorization $\bar{\xi} = \bar{f} \cdot \bar{t}^{-1}$. This gives

$$\bar{\xi} = \bar{f} \cdot \bar{t}^{-1} = \mathrm{comp}(\bar{f})\, \mathrm{comp}(\bar{t})^{-1}$$
$$= \mathrm{comp}(\bar{f})\, \mathrm{comp}(\bar{t}^{-1})$$
$$= \mathrm{comp}(\bar{f} \cdot \bar{t}^{-1}).$$

We now observe surjectivity of comp, establishing the claimed isomorphism. ∎

VIII Local vanishing vs. global vanishing

Below we take $M_{\mathfrak{p}} = M[(R \setminus \mathfrak{p})^{-1}]$ for any R-module M and prime $\mathfrak{p} \subseteq R$.

Theorem 12: For a module M over a comm ring R, the following are equivalent:

(a) $M = 0$

(b) For each prime $\mathfrak{p} \in \mathrm{Spec}(R)$, $M_{\mathfrak{p}} = 0$.

(c) For each max ideal $\mathfrak{m} \in \mathrm{mSpec}(R)$, $M_{\mathfrak{m}} = 0$.

Proof: The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are clear.

We prove (c) $\Rightarrow$ (a), or more precisely not (a) $\Rightarrow$ not (c). Suppose M is a nonzero R-module and take x nonzero in M. Since x is nonzero, $\mathrm{Ann}_R(x)$ is a proper ideal in R, and we can chose a maximal ideal $\mathfrak{m}$ in R with $\mathrm{Ann}_R(x) \subseteq \mathfrak{m}$. Then for each $a \in R - \mathfrak{m}$ $a \cdot x \neq 0$ and hence the corresponding element in the localization $x/1 \in M_{\mathfrak{m}}$ is nonzero. In particular, the localization $M_{\mathfrak{m}}$ is nonzero. ∎

Note that when M is finitely generated each localization $M_{\mathfrak{p}}$ is finitely generated over $R_{\mathfrak{p}}$. This follows, for example, from the identification $M_{\mathfrak{p}} \cong R_{\mathfrak{p}} \otimes_R M$ from Theorem 7 and right exactness of base change. We also note the following basic fact.

Lemma 18: For any pair of ring maps $R \to S_1 \xrightarrow{\varphi} S_2$, and R-module M, the natural map

$$S_2 \otimes_{S_1} (S_1 \otimes_R M) \longrightarrow S_2 \otimes_R M, \quad s' \otimes s \otimes m \mapsto s'\varphi(s) \otimes m,$$

is an S_2-module isomorphism.

Proof: Both modules have the same universal property [Exercise]. ∎

As a particular example, we have the natural isomorphism of $\kappa(\mathfrak{p})$-modules

$$\kappa(\mathfrak{p}) \otimes_{R_\mathfrak{p}} M_\mathfrak{p} = \kappa(\mathfrak{p}) \otimes_{R_\mathfrak{p}} (R_\mathfrak{p} \otimes_R M) \xrightarrow{\sim} \kappa(\mathfrak{p}) \otimes_R M$$

at each prime $\mathfrak{p} \in \mathrm{Spec}(R)$.

Theorem 14: For any finitely generated R-module M, the following are equivalent.

(a) $M = 0$.

(b) For each prime $\mathfrak{p}$, $\kappa(\mathfrak{p}) \otimes_R M = 0$

(c) For each maximal $\mathfrak{m}$,

$$\kappa(\mathfrak{m}) \otimes_R M \ (= M/\mathfrak{m}.M) = 0.$$

Proof: In this case all of the localizations $M_\mathfrak{p}$ are finitely generated and

$$\kappa(\mathfrak{p}) \otimes_R M = \kappa(\mathfrak{p}) \otimes_{R_\mathfrak{p}} M_\mathfrak{p} = \cancel{M_\mathfrak{p}/\max . M_\mathfrak{p}}.$$

Now, since each $R_\mathfrak{p}$ is local, by Prop 10, Nakayama's Lemma [Sect 6, Trad] says

$$\kappa(\mathfrak{p}) \otimes_R M = 0 \iff M_{\mathfrak{p}} = 0.$$

Hence Theorem 12 implies the claimed equivalences (a) $\iff$ (b) $\iff$ (c). ∎

Anti-Example: For $k(x)$ over $k[x]$, $k(x)$ is obviously nonzero, but $\kappa(\mathfrak{m}) \otimes_{k[x]} k(x) = 0$ at each maximal $\mathfrak{m} \subseteq k[x]$. But, this is OK, since $k(x)$ is not finitely generated.

IX Local vanishing in the Zariski topology

For an R-module M and f in R, take also $M_f = M[f^{-1}]$.

Proposition 15: Let R be a commutative ring and M be a finitely generated R-module. For any prime $\mathfrak{p} \in \operatorname{Spec}(R)$

$$M_{\mathfrak{p}} = 0 \iff \left\{ \begin{array}{l} \text{There is an element } f \notin \mathfrak{p} \\ \text{with } M_f = 0 \end{array} \right. .$$

Proof: Consider $m_1, \ldots, m_r$ with $M = R \cdot \{m_1, \ldots, m_r\}$. From the vanishing of each m_i in $M_{\mathfrak{p}}$ we can find $f_i \notin \mathfrak{p}$ with $f_i \cdot m_i = 0$. Since $\mathfrak{p}$

$\mathfrak{p}$ is prime the product $f = f_1 \cdots f_r$ is not in $\mathfrak{p}$ as well, and

$$f \cdot m_i = 0 \qquad \text{for all } i.$$

Thus all m_i vanish in the localization M_f, and since these elements generate M_f over R_f (?) we find $M_f = 0$. ∎

How should you think about this stuff? Consider finite type alg R over $\mathbb{C}$ and a fin gen'd R-module M.

So $\quad m\mathrm{Spec}(R) =: \mathrm{Var}_R \subseteq \mathbb{C}^n$

and M specifies some type of "bundle with singularities" over Var_R

$$\tilde{M} \longrightarrow \mathrm{Var}_R \subseteq \mathbb{C}^n$$

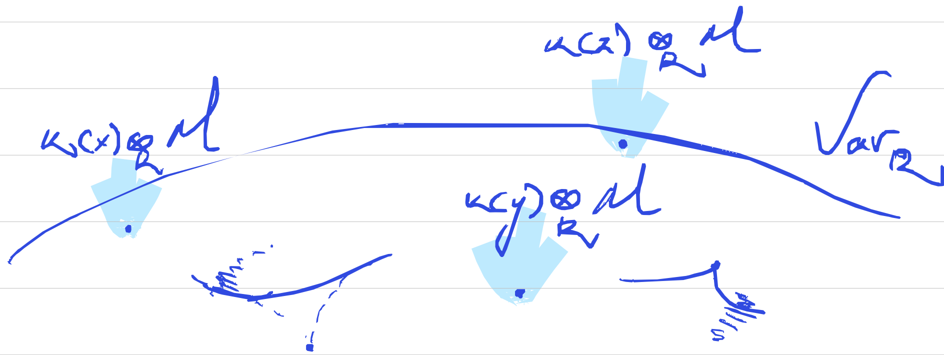
The fibers of this bundle at points x in Var_R, are the finite dimensional $\mathbb{C}$-vector spaces

$$\tilde{M}|_x = \kappa(x) \otimes_R M$$

and sections over each basic open $U_f \subseteq \mathrm{Var}_R$ given by the R_f-module M_f.

Proposition 15 says that, if the fiber $\tilde{M}|_x$

vanishes at a given point x in Var_R, then $\tilde{M}$ vanishes in a Zariski neighborhood of x in Var_R.



This local vanishing is what one expects from a vector bundle over Var_R.

HW

1. For a mult. subset $T \subseteq R$, and an R-mod M, prove that $M = \operatorname{res}(M')$ for a (uniquely associated) $R[T^{-1}]$-module M' if and only if each $t \in T$ acts on M by a $\mathbb{Z}$-linear automorphism.

2. Let R be a comm. ring, T be a mult. subset in R, and $\mathrm{loc}: R \to R[T^{-1}]$ be the loc. map.

(a) For ideals I and J in $R[T^{-1}]$, prove that $I = J$ if and only if $\mathrm{loc}^{-1}(I) = \mathrm{loc}^{-1}(J)$.

(b) Prove that $R[T^{-1}]$ is Noetherian whenever R is Noetherian.

3. Let R be Noetherian and M be a fin. gen'd R-module. Fix a prime $\mathfrak{p}$ and take

$$n = \dim_{k(\mathfrak{p})} k(\mathfrak{p}) \otimes_R M.$$

Prove that there is an element $f \notin \mathfrak{p}$ and elements $m_1, \ldots, m_n \in M$ for which the localization M_f is generated by the (images of) the n elements m_i.

4. Let R be a local ring and M be a finitely generated projective R-module. Prove that M is free, i.e. that $M \cong R^{\oplus n}$ for some n.

5. Let R be a Noetherian ring and M be a finitely generated projective R-module. Prove that for each $\mathfrak{p}$ in $\mathrm{Spec}(R)$ there is an element $f \notin \mathfrak{p}$ for which M_f is a free module over R_f. In particular, the dimension of the fibers $\kappa(\mathfrak{p}) \otimes_R M$ are locally constant across $\mathrm{Spec}(R)$, or equivalently the dimension function
$\dim_M : \mathrm{Spec}(R) \to \mathbb{Z}$, $\mathfrak{p} \mapsto \dim \kappa(\mathfrak{p}) \otimes_R M$
is continuous.

6. Provide an example of a finitely generated R-module M, over a Noetherian ring R, for which the dimension function $\dim_M$ (defined as above) is <u>not</u> continuous, i.e. for which the dimension of the fibers are not locally constant over $\mathrm{Spec}(R)$.