Radicals and semisimplicity

To begin we discuss another class of rings / algebras –

~ I : Division algebras

Def: A division ring D, is a ring for which each nonzero $a \in D$ admits $a^{-1} \in D$ with $a^{-1}a = aa^{-1} = 1$. A division algebra over a field k is an algebra which is also a division ring

Def: A ring R is called a domain if for each nonzero a in R an equation $a \cdot b = 0$ or $b \cdot a = 0$, at b arbitrary in R, implies $b = 0$.

Observation 1: Any division ring is a domain.

Ex: $\mathbb{H}$ = quaternions are a division alg over $\mathbb{R}$.

Lemma 2: The center of any division ring is a field.

(Proof: For a in $Z(D)$) we have for a^{-1} and arbitrary b in D,

$$a^{-1}b = a^{-1}baa^{-1} = a^{-1}aba^{-1} = ba^{-1}.$$

So $a^{-1} \in Z(D)$) as well. ∎

Lemma 3: If k is an algebraically closed field, then the only finite dimensional k-algebra is k itself.

Proof: Take D a div. alg over k and suppose there exists $x \in D \setminus k$. Then we have the alg. map $\phi: k[X] \to D$, $X \mapsto x$. If $\ker(\phi) \neq 0$ then $k[X]/\ker(\phi) \subseteq D$ is a domain, and hence $\ker(\phi) = (p(x))$ for an irred poly $p(x)$. Consequently $K = k[X]/\ker(\phi)$ is a finite field extension of k, $k \to K \subseteq D$, and by alg closure $K = k$. Thus $x \in k \subseteq D$, a contradiction. So $\ker(\phi) = 0$ necessarily, and $\dim D \geq \dim k[X] = \infty$.

As a consequence, if $\dim D < \infty$ for div k-alg D then we must have $D = k$. ∎

Corollary 4: The only finite division alg's over $\mathbb{C}$, $\overline{\mathbb{Q}}$, $\overline{\mathbb{F}}_p$ etc. are $\mathbb{C}$, $\overline{\mathbb{Q}}$, $\overline{\mathbb{F}}_p$ themselves, respectively.

Proposition 5: Any finitely generated module M over a div alg D is of the form D^l for some unique $l \in \mathbb{Z}_{\geq 0}$, and if $M' \subseteq M$ then $M' = D^{l'}$ for $l' \leq l$. Equality holds if and only if $l = l'$.

Here note first that any div ring D has no submods besides 0 and 1. Hence D is both Artinian and Noeth, and all of its finitely gen'd D-modules are of finite length [Thm 7, Lemma 10, Findlin].

Proof: Note that every cyclic D-module is isomorphic to D. Take $\{m_1, \ldots, m_\ell\} \subseteq M \setminus \{0\}$ maximal so that the corresponding map

$$\bigoplus_{i=1}^{\ell} D \cong \bigoplus_{i=1}^{\ell} Dm_i \longrightarrow M \qquad (*)$$

is injective, and take $M' =$ image. Note that the size of this set ℓ is bounded above by the length of M. If $M' \subsetneq M$ then for any $m \in M \setminus M'$ we have

$$Dm \cap M' = 0$$

by simplicity of $D \cong Dm$, giving an inclusion

$$\left(\bigoplus_{i=1}^{\ell} Dm_i\right) \oplus Dm \to M,$$

contradicting maximality. Hence $M' = M$ and $(*)$ is necessarily an isomorphism.

Note that under such an $\cong$ $(*)$, we have $\mathrm{length}(M) = \ell$. Since an inclusion of fin length submods $M' \subseteq M$ is an $\cong$ if and only if

$\mathrm{length}(\mathcal{M}') = \mathrm{length}(\mathcal{M})$
(see [Prop 11, Fraction]), when $\bigoplus_{i=1}^{l'} D \cong \mathcal{M}'$
the inclusion $\mathcal{M}' \subseteq \mathcal{M}$ is an equality if and only if $l = l'$.

In this setting we can refer to the rank of a fin. gen'd D-module $\mathcal{M}$ as the unique integer r at which we have an isomorphism $\bigoplus_{i=1}^{r} D \xrightarrow{\sim} \mathcal{M}$. Obviously this is the same thing as the length of $\mathcal{M}$.

II. Matrices over division rings

On the HW we saw k^n is simple over $M_n(k)$, for k a field. Similarly, for any division ring D, D^n is simple module over $M_n(D)$, and $M_n(D)$ admits no ideals other than 0 and $M_n(D)$ itself.

Furthermore we have
$$M_n(D) = \bigoplus_{i=1}^{n} D^n.$$
as a $M_n(D)$-module.

Here, for any ring R, $M_n(R)$ is the ring with $1 = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$ and mult $[a_{ij}][b_{ij}] = [c_{ij}]$ as $c_{ij} = \sum_k a_{ik} b_{kl}$.

III. Semisimple modules

Def^n: A module M over an algebra A is called semisimple if

$$M = \bigoplus_{\lambda \in \Lambda} L_\lambda$$

for simple submodules $L_\lambda \subseteq M$.

Equiv.: M is semisimple if and only if M admits an A-module isomorphism

$$\bigoplus_{\lambda \in \Lambda} L_\lambda \xrightarrow{\sim} M$$

for simple M_λ.

Observation 6: For a semisimple module M the following are equivalent

a) M is finitely generated

b) M is finite length (admits a composition series)

c) For any isomorphism $\bigoplus_{\lambda \in \Lambda} L_\lambda \xrightarrow{\sim} M$ w/ the L_λ simple, the indexing set Λ is finite.

Proof: Exercise. ∎

Lemma (Schur) If L and L' are simple A-modules, then any A-module map $f: L \to L'$ is either

0, or an isomorphism.

Proof: By simplicity either $\ker(f) = 0$ or L. In the first case f is injective with

$$0 \subsetneq \operatorname{im}(f) \subseteq L$$

and by simplicity again $\operatorname{im}(f) = L$. Hence f is bijective as a map of sets, and thus invertible as a map of A-modules. In the second case, $\ker(f) = L$, we have $f = 0$. ∎

Corollary 7: For any simple A-module L, $\operatorname{End}_A(L)$ is a division ring.

Corollary 8: If A is a finite dimensional algebra over an algebraically closed field $k = \bar{k}$, and L is simple over A, then

$$\operatorname{End}_A(L) = k \cdot \mathrm{id}_L \cong k.$$

Proof: Since L is cyclic we have a surjection $A \to L$ and hence $\dim_k(L) \leq \dim_k(A) < \infty$. Thus

$$\dim_k \operatorname{End}_A(L) \leq \dim_k \operatorname{End}_k(L) < \infty.$$

Thus $\operatorname{End}_A(L)$ is a finite-dimensional division alge

over k. Since $k = \bar{k}$, we now have
$$\mathrm{End}_A(L) = k$$
by Lemma 3. ∎

Example: For $L_{\mathbb{C}}(2)$ the 2-dim'l simple mod over $\mathbb{C}S_3$, we have
$$\mathrm{End}_{\mathbb{C}S_3}(L_{\mathbb{C}}(2)) = \mathbb{C}.$$
Also,
$$\mathrm{End}_{\bar{\mathbb{F}}_p S_3}(L_{\bar{\mathbb{F}}}(2)) = \bar{\mathbb{F}}_p \text{ for all } p > 3.$$

More generally, for any finite group G, any simple G-rep L over an alg closed field k has
$$\mathrm{End}_{kG}(L) = k.$$

Proposition 4: Let N be a finitely generated semisimple A-module.

i) Every submodule $M \leq N$ is semisimple and admits a complementary submodule $M' \leq N$ for which $M \oplus M' = N$.

ii) For any quotient module $\pi: N \to M$, M is semisimple and π admits a section, i.e. a module map $i: M \to N$ with $\pi \circ i = id_M$.

Proof: (i) Proceed by induction on the length of N. The statements are clear when $\mathrm{length}(N) \leq 1$. Suppose now that $\mathrm{length}(N) = l$ and that the statement holds for all semisimple modules of length $< l$.

Take an expression $\bigoplus_{i=1}^{l} L_i = N$ for simple L_i and consider a submodule $M \subseteq N$. If $M = N$ then there's nothing to prove. So suppose $M \subsetneq N$. Then we have some simple $L_j \subseteq N$ with $L_j \not\subseteq M$. Then $M \cap L_j = 0$ via simplicity of L_j and the sequence

$$\begin{tikzcd} & N \arrow[r] & N/L_j \\ M \arrow[ur] \arrow[urr] & & \end{tikzcd}$$

provides an injection $M \hookrightarrow N/L_j$. By induction hypothesis we conclude M is semisimple as well.

As for the splitting of N, choose $M_0 \subsetneq N$ maximal with $M \subseteq M_0$. Then $\mathrm{length}(M_0) = \mathrm{length}(N) - 1$ and by induction hyp we have a complement $M_0 = M \oplus M_0'$. Since $M_0 \neq N$ we can again pick L_j with $L_j \cap M_0 = 0$, giving $M_0 \oplus L_j \subseteq N$ and $M_0 \oplus L_j = N$ by equal of the lengths. This gives finally

$$\mathcal{N} = \mathcal{M}_0 \oplus L_j = \mathcal{M} \oplus (\underbrace{\mathcal{M}_0' \oplus L_j}) \\ = \mathcal{M} \oplus \mathcal{M}'.$$

(ii) We have for $K = \ker(\pi)$ a splitting
$$\mathcal{N} = K \oplus \tilde{\mathcal{M}} \qquad (*)$$
by (i), giving an iso $f: \tilde{\mathcal{M}} \xrightarrow{\sim} \mathcal{M}$ via the sequence $\tilde{\mathcal{M}} \to \mathcal{N} \xrightarrow{\pi} \mathcal{M}$. ($f$ is both inj and surj via surjectivity of π and the splitting $(*)$.) So, using (i), we see $\mathcal{M}$ is semisimple, and we split π via the inverse
$$f^{-1}: \mathcal{M} \to \tilde{\mathcal{M}} \subseteq \mathcal{N}.$$
∎

The following is an immediate consequence of Proposition 9 (ii), or Proposition 9 (i), whichever one prefers.

Theorem 10: If $\mathcal{M}$ is semisimple, any exact sequence $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ splits.

Remark: The analog of Prop 9 holds for infinite length semisimple modules as well. One can argue the point from the finite case and some Zorn's lemma argument.

Proposition $9\frac{1}{2}$: For any semisimple A-module N, the following hold. (i) Any submodule $M \subseteq N$ is semisimple and admits a complement. (ii) Any quotient module is semisimple and admits a splitting.

Proof: Omitted. ∎

~ III $\frac{1}{2}$ An example: Matrix rings

Example: Consider any division ring D and $M_n(D)$. We have the simple module
$$L_{\text{standard}} = D^n$$
provided by the columnar module and the decomp. of the regular module
$$M_n(D) = \bigoplus_{i=1}^{n} L_{\text{standard}}.$$
Hence $M_n(D)$ itself is semisimple.

Now, since any fin. gen'd $M_n(D)$-module M admits a surjection
$$\bigoplus_{i=1}^{m} M_n(D) = \bigoplus_{i=1}^{n\cdot m} L_{\text{standard}} \to M$$
Prop 9 (ii) tells us that M is also semisimple, and [Prop 9, Findim] forces (as expected)
$$M = \bigoplus_{i=1}^{t} L_{\text{standard}}.$$

Corollary 11: For any division ring D, any finitely generated $M_n(D)$-module is semisimple, and

The standard column module is the only simple $M_n(D)$ module, up to isomorphism.

IV Radicals for modules

We call a proper submodule $M_0 \subsetneq M$ maximal if any larger module $M_0 \subseteq M_1 \subseteq M$ has either $M_1 = M_0$ or $M_1 = M$.

Lemma 11: A proper submodule $M_0 \subseteq M$ is maximal if and only if M/M_0 is simple.

Proof: Consider the surjection $\pi : M \to N$, with $N = M/M_0$. We have the bijection

$$\left\{\begin{matrix}\text{Submodules } M_1 \\ \text{with } M_0 \subseteq M_1 \subseteq M\end{matrix}\right\} \rightleftarrows \left\{\begin{matrix}\text{Submodules} \\ N_1 \subseteq N\end{matrix}\right\}$$

$$M_1 \longmapsto \pi(M_1)$$

$$\pi^{-1}(N) \longleftarrow N_1 .$$

Indeed, for $m \in M$ with $\pi(m) = \pi(m')$ for m' in M_1, we have $m - m' \in \ker(\pi) = M_0$ so that $m = m' + (m - m') \in M_1$. This gives

$$\pi^{-1}(\pi(M_1)) = M_1 .$$

The equality $\pi(\pi^{-1}(N_1)) = N_1$, at any submodule $N_1 \subseteq N$ follows by surjectivity of π. Hence

The only submodules between $\mathcal{M}_0$ and $\mathcal{M}$ are $\mathcal{M}_0$ and $\mathcal{M}$ itself if and only if the only submodules in N are 0 and N itself. Rather, $\mathcal{M}_0 \overset{\mathcal{M}_1}{\underset{\cup}{\subsetneq}}$ is maximal if and only if $N = \mathcal{M}/\mathcal{M}_0$ is simple. ∎

Def$^{\underline{t}}$: Given a ring A and a nonzero A-mod $\mathcal{M}$, the radical in $\mathcal{M}$ is the intersection

$$\mathrm{Rad}(\mathcal{M}) = \bigcap_{\mathcal{M}_0 \subseteq \mathcal{M}\text{ax}} \mathcal{M}_0 .$$

Example: For an infinite field k (e.g. $\mathbb{Q}$, $\mathbb{C}$, or $\overline{\mathbb{F}_p}$) and any $\alpha \in k$ we have the max ideal / submodule $(x-\alpha)$ in the regular module $k[x]$, and the quotient

$$k[x]/(x-\alpha) \xrightarrow{\sim} k\cdot(\alpha)$$

is the 1-dimensional $k[x]$-module k on which $x \cdot 1_\alpha = \alpha\, 1_\alpha$. Now, for any poly

$$p(x) \in \mathrm{Rad}(k[x])$$

we have $p(x)\cdot 1_\alpha = p(\alpha)\cdot 1_\alpha = 0$ at all $\alpha \in k$. Hence $p(\alpha)$ has as many zeros, and we conclude $p(x) = 0$. So

$$\mathrm{Rad}(k[x]) = 0 .$$

Example: At each $i = 1, \ldots, n$, and k a field, we have the surjection

$$\pi_i : M_n(k) \to L_{\text{standard}} = k^n$$

with kernel

$$\ker(\pi_i) = \left\{ \begin{array}{c} \left[\text{stuff} \,\middle|\, 0 \,\middle|\, \text{stuff} \right] \\ \uparrow \\ i\text{th column} \end{array} \right\}$$

Since L_{standard} is simple this gives

$$\operatorname{Rad}(M_n(k)) \subseteq \bigcap_i \ker(\pi_i) = 0.$$

Similarly, for any division ring D, $\operatorname{Rad}(M_n(D)) = 0$.

Example: For $M = k[x]/(x^{10})$ as a module over $k[x]$, any simple quotient

$$\pi : M \to L$$

we have $x \cdot - : L \to L$ an endomorphism. By Schur either $x \cdot -$ is 0 or an isomorphism. Since

$$(x \cdot -)^{10} = x^{10} \cdot - = 0$$

we conclude $x \cdot - = 0$. Rather, x annihilates L. We see now $x \in \ker(\pi)$ at all such π, giving

$$\bar{x} \in \operatorname{Rad}(M) \Rightarrow k[x]\bar{x} = x \cdot M \subseteq \operatorname{Rad}(M).$$

Since the quotient $M/x \cdot M \cong k$ is simple, we get $\operatorname{Rad}(M) = x \cdot M$.

~ IV ½ Radicals and semisimplicity

Theorem 12: For any finite length module M, $M/\mathrm{Rad}(M)$ is semisimple. Furthermore, for any surjective A-module map $\pi: M \to N$, N is semisimple if and only if $\mathrm{Rad}(M) \subseteq \ker(\pi)$.

Before we prove the result, we note the following

Lemma 13: For any surjective module map $\pi:$ $M \to N$ to nonzero N, $\mathrm{Rad}(M) \subseteq \pi^{-1}(\mathrm{Rad}(N))$.

Proof: For maximal $N_0 \subseteq N$, $\pi^{-1}(N_0)$ is max in M. Hence

$$\pi^{-1}\mathrm{Rad}(N) = \pi^{-1}\Big(\bigcap_{N_0 \max} N_0\Big)$$

$$= \bigcap_{N_0 \max} \pi^{-1}(N_0) \supseteq \bigcap_{M_0 \max} M_0 = \mathrm{Rad}(M). \blacksquare$$

Lemma 14: If N is finite length and semisimple, then $\mathrm{Rad}(N) = 0$.

Proof: Given an expression $N = \bigoplus_{i=1}^{r} L_i$ with the L_i simple, the kernel $R_i \subseteq N$ of each

projection $p_i : \mathcal{N} \to \mathcal{L}_i$ satisfying
$\bigcap_{i=1}^r \mathcal{K}_i = \ker([p_1 \cdots p_r]^t : \mathcal{N} \to \bigoplus_{i=1}^r \mathcal{L}_i)$.
But! $[p_1 \cdots p_r]^t$ just recover the identity on $\mathcal{N}$, and hence $\mathrm{Rad}(\mathcal{N}) \subseteq \bigcap_{i=1}^r \mathcal{K}_i = 0 \Rightarrow \mathrm{Rad}(\mathcal{N}) = 0$. ∎

We now prove our main Theorem.

Proof of Theorem 12: We have - via Artinianness of the quotient $\mathcal{M}/\mathrm{Rad}(\mathcal{M})$ the existence of finitely many maximal submodules $\mathcal{K}_1, \ldots, \mathcal{K}_r \subseteq \mathcal{M}$ for which $\mathrm{Rad}(\mathcal{M}) = \mathcal{K}_1 \cap \cdots \cap \mathcal{K}_r$. Hence, $\mathrm{Rad}(\mathcal{M})$ is the kernel of the map

$$\mathcal{M} \to \bigoplus_{i=1}^r \mathcal{L}_i, \quad \mathcal{L}_i := \mathcal{M}/\mathcal{K}_i,$$

induced by the individual quotients $\mathcal{M} \to \mathcal{L}_i$, giving

$$\mathcal{M}/\mathrm{Rad}(\mathcal{M}) \subseteq \bigoplus_{i=1}^r \mathcal{L}_i \text{ semisimple.}$$

By Proposition 9 we conclude $\mathcal{M}/\mathrm{Rad}(\mathcal{M})$ is semisimple.

As for the second claim, for semisimple $\mathcal{N}$ we have $\mathrm{Rad}(\mathcal{N}) = 0$ by Lemma 14 and hence
$\mathrm{Rad}(\mathcal{M}) \subseteq \ker(\pi) = \pi^{-1}\mathrm{Rad}(\mathcal{N})$ by Lemma 13.
Conversely, if $\pi : \mathcal{M} \to \mathcal{N}$ is surj with $\mathrm{Rad}(\mathcal{M}) \subseteq \ker$ then $\mathcal{N}$ is the quotient of the semisimple module $\mathcal{M}/\mathrm{Rad}(\mathcal{M})$.

By Proposition 9 we conclude that N is semisimple. ∎

~ V Socles for modules

Theorem 15: For a Noetherian module M (e.g. a fin. gen. module over fin. dim. A), the sum

$$\mathrm{soc}(M) = \sum_{\lambda \in \Lambda} L_\lambda \subseteq M \qquad (*)$$

over the collection Λ of all simple submodules $L_\lambda \subseteq M$ is a finite length semisimple submodule in M.

Furthermore, for any semisimple module N, any module map $f: N \to M$ has image in $\mathrm{soc}(M)$.

Proof: Since M is Noetherian, the sum in $(*)$ is finite, $\sum_\lambda L_\lambda = L_1 + \cdots + L_r$ for some simple $L_i \subseteq M$. This sum is the image of the map

$$\bigoplus_{i=1}^r L_i \to M$$

induced by the inclusions, giving $\mathrm{soc}(M)$ as a quotient of a finite length semisimple module. Hence $\mathrm{soc}(M)$ is finite length semisimple, by Proposition 9. For the second claim write $N = \bigoplus_\mu V_\mu$ for simple V_μ to get $\mathrm{im}(N) = \sum_\mu \mathrm{im}(V_\mu)$ with each $\mathrm{im}(V_\mu)$

either simple or zero, by simplicity. Hence $\operatorname{im}(N) \subseteq \operatorname{soc}(M)$. ∎

Def$^{\underline{n}}$: The sum of all simple submodules in a Noeth. A-module M is called the socle in M.

Ex: For $k[x]/(x^n)$, considered as a module over $k[x]$, any simple submod $M \subseteq k[x]/(x^n)$ is annihilated by x^n, and thus simplicity forces $x \cdot M = 0$. This gives

$$\operatorname{soc}(k[x]/(x^n)) = k \cdot \bar{x}^{n-1} \cong k_0.$$

the 1-dim simple on which x acts as 0.

HW

1. For the regular A-module A, prove that there is a canonical ring isomorphism

$$A^{op} \xrightarrow{\sim} \mathrm{End}_A(A).$$

3. Let G be a group and suppose G acts on a k-algebra A by algebra automorphisms. Suppose k is a field, and take $A \rtimes G := A \otimes_k kG$ with the unique bilinear map $\cdot : A \rtimes G \times A \rtimes G \to A \rtimes G$ satisfying

$$(a \otimes g) \cdot (b \otimes h) = a(g * b) \otimes gh$$

on monomials. Prove that $A \rtimes G$ is a k-algebra.

4. Let $\mathbb{Z}/n\mathbb{Z}$ act on $\mathbb{C}[x]$ via the automorphisms $\bar{m} \cdot p(x) = p(q \cdot x)$ for $q \in \mathbb{C}^\times$ an n-th root of unity. Take $A_q = \mathbb{C}[x] \rtimes \mathbb{Z}/n\mathbb{Z}$.

a) Prove that $M_r = A_q / A_q \cdot x^r$ is a free module over $\mathbb{C}\mathbb{Z}/n\mathbb{Z}$ of rank r. Give a basis for M_r over $\mathbb{C}\mathbb{Z}/n\mathbb{Z}$.

b) Calculate $\mathrm{Rad}(M_r)$, $M_r/\mathrm{Rad}(M_r)$, and $\mathrm{soc}(M_r)$.

2. a) Fix $\mathbb{Z}/n\mathbb{Z}$. Prove that $\mathbb{Z}/n\mathbb{Z}$ has precisely n non-isomorphic (simple) 1-dimensional representations L_i over $\mathbb{C}$.

b) Prove that each simple 1-dimensional representation L_i over $\mathbb{Z}/n\mathbb{Z}$ admits an injective $\mathbb{C}\mathbb{Z}/n\mathbb{Z}$-module map $L_i \to \mathbb{C}\mathbb{Z}/n\mathbb{Z}$, and that that map is unique up to rescaling.

c) Provide an isomorphism of $\mathbb{C}\mathbb{Z}/n\mathbb{Z}$-modules

$$\bigoplus_{i=1}^{n} L_i \xrightarrow{\sim} \mathbb{C}\mathbb{Z}/n\mathbb{Z}.$$

d) Prove that every finite-dimensional $\mathbb{Z}/n\mathbb{Z}$-rep M over $\mathbb{C}$ decomposes as

$$M \cong \bigoplus_{i \in I}^{n} m_i \cdot L_i$$

where each $m_i = [L_i : M]$.

5. For any finite length semisimple module M over a ring A, prove that there are division rings $D_1, \dots, D_t$ and integers $n_1, \dots, n_t$ for which

$$\mathrm{End}_A(M) \cong \prod_{i=1}^{t} M_{n_i}(D_i)$$

as rings.