Radicals and vanishing loci

~I Radicals of ideals

Def$^{\text{n}}$: For an ideal $I \leq R$ in a commutative ring, the radical of I is the subset
$$\sqrt{I} := \{f \in R : f^n \text{ is in } I \text{ for some } n \geq 1\}.$$

Lemma 1: For any ideal $I \leq R$, the radical $\sqrt{I}$ is an ideal in R which contains I.

Proof: For $f, g \in \sqrt{I}$ choose n and $m \geq 1$ with $f^n, g^m \in I$. Then
$$(f+g)^{n+m} = \sum_{i \leq n+m} \binom{n+m}{i} f^{n+m-i} g^i \in I.$$
We also have for $c \in R$, $cf \in \sqrt{I}$ whenever $f \in \sqrt{I}$. So we see $\sqrt{I}$ is an ideal. ∎

Proposition 2: For any ideal $I \leq R$,
$$\sqrt{I} = \bigcap_{\mathfrak{p} \in Van(I)} \mathfrak{p}.$$

Recall here that
$$Van(I) = \{\mathfrak{p} \in Spec(R) : \mathfrak{p} \supseteq I\}.$$

Proof: For any prime $\mathfrak{p} \supseteq I$, if $f^n \in I$ for some $n \geq 1$ then $f^n \in \mathfrak{p}$ and by primeness $f \in \mathfrak{p}$. Thus

$$\sqrt{I} \subseteq \bigcap_{\mathfrak{p} \in \mathrm{Van}(I)} \mathfrak{p}. \qquad (*)$$

For the converse, suppose $f \notin \sqrt{I}$. Then the class $\bar{f}$ is non nilpotent in the quotient $\bar{R} = R/I$, and thus the localization $\bar{R}_{\bar{f}}$ is nonzero.

Choose now a prime ideal $\bar{q}$ in $\bar{R}_{\bar{f}}$ and let q be the premiage of $\bar{q}$ along the ring map $R \to \bar{R} \to \bar{R}_{\bar{f}}$. Then q is a prime in R which contains I, but does not contain f. So $f \notin q \supseteq \bigcap_{\mathfrak{p} \in \mathrm{Van}(I)} \mathfrak{p}$.

This shows

$$f \notin \sqrt{I} \Rightarrow f \notin \bigcap_{\mathfrak{p} \in \mathrm{Van}} \mathfrak{p}$$

and we see that the inclusion $(*)$ is an equality. ∎

Lemma 3: For any field k, any map of finite type k-algebras $\phi: R \to S$, and any maximal ideal $\mathfrak{m} \subseteq S$, the premiage $\phi^{-1}(\mathfrak{m})$ is maximal in R.

Proof: The premiage $\phi^{-1}(\mathfrak{m})$ is the kernel of the map $R \to S \to S/\mathfrak{m}$, and by Zariski $S/\mathfrak{m}$ is a fin dimensional field extension of k [Sect 2, Ideals]. Hence the image $R/\phi^{-1}\mathfrak{m} \cong$ image of R in $S/\mathfrak{m}$ is a finite dimensional k-alg which is a domain. Thus, by HW, $R/\phi^{-1}\mathfrak{m}$ is a field, and $\phi^{-1}\mathfrak{m}$ is maximal [Lemma 1, Ideals]. ∎

Corollary 4: Any map between fin type k-alg's $\phi : R \to S$ induces a well-defined map on maximal spectra $\phi^* : \mathrm{mSpec}(S) \to \mathrm{mSpec}(R)$, $\mathfrak{m} \mapsto \phi^{-1}\mathfrak{m}$.

Lemma 5: If R is a finite type over k then, for any $f \in R$, R_f is also of finite type over k.

Proof: We have the map $ev_{f^{-1}} : R[x] \to R_f$, $x \mapsto f^{-1}$, which reduces to a map

$$\phi : R[x]/(fx - 1) \longrightarrow R_f .$$

Similarly, via univ prop of localization we have a map of R-algebras $\psi : R_f \to R[x]/(xf-1)$ which sends f^{-1} to x. One can check directly that $\psi \circ \phi = id$ and $\phi \circ \psi = id$, or observe that ϕ and ψ are inverses via univ properties. ∎

Recall our definition $\mathrm{mVar}(I) = \{\mathfrak{m} \in \mathrm{mSpec}(R) : \mathfrak{m} \supseteq I\}$.

Theorem 6: Let k be a field. For any finite type k-algebra R, and ideal $I \subseteq R$, we have

$$\sqrt{I} = \bigcap_{\mathfrak{p} \in \mathrm{Var}(I)} \mathfrak{p}$$
$$= \bigcap_{\mathfrak{m} \in \mathrm{mVar}(I)} \mathfrak{m}.$$

Proof: We already know

$$(*) \quad \sqrt{I} = \left(\bigcap_{\mathfrak{p} \in \mathrm{Var}} \mathfrak{p}\right) \subseteq \left(\bigcap_{\mathfrak{m} \in \mathrm{mVar}} \mathfrak{m}\right).$$

For $f \notin \sqrt{I}$ we again consider the localization $\bar{R}_{\bar{f}}$, $\bar{R} = R/I$ and $\bar{f}$ = the class of f. As before $\bar{f}$ is non-nilpotent in the quotient, so that the localization $\bar{R}_{\bar{f}}$ is a nonzero ring.

Take maximal $\bar{\mathfrak{n}}$ in $\bar{R}_{\bar{f}}$ and let $\mathfrak{n}$ be the preimage of $\bar{\mathfrak{n}}$ along the map $R \to \bar{R}_{\bar{f}}$.

Since R is of fin type over k, $\bar{R}$ is also of fin type as is $\bar{R}_{\bar{f}}$ by Lemma 5. Hence $\mathfrak{n}$ is maximal by Lemma 3. We note that $I \subseteq \mathfrak{n}$ while $f \notin \mathfrak{n}$, by construction, so that

$$f \notin \mathfrak{n} \subseteq \bigcap_{\mathfrak{m} \in \mathrm{mVar}} \mathfrak{m}.$$

Thus the inclusion $(*)$ is seen to be an equality. ∎

~ II. Radicals vs. Vanishing loci

Def$^{\underline{n}}$: For a subset $Z \subseteq \mathrm{Spec}(R)$ we take $I_Z := \{f \in R : f \in \mathfrak{p} \text{ for all } \mathfrak{p} \text{ in } Z\}$.

Note that for $Z \subseteq \mathrm{mSpec}(R)$ we have $\mathrm{mSpec}(R) \subseteq \mathrm{Spec}(R)$, so that it makes sense to speak of I_Z in this case as well;

$$I_Z = \{f \in R : f \in \mathfrak{m} \text{ for all } \mathfrak{m} \text{ in } Z\}.$$

Example: For a subset

$$Z \subseteq \mathbb{A}^n_{\mathbb{C}} = \mathrm{mSpec}(\mathbb{C}[x_1, \dots, x_n])$$

we have

$$I_Z = \{\text{All polys } f(x_1, \dots, x_n) \text{ w/ } f \in \mathfrak{m}_a\}.$$

when $a \in Z$

As $f \in \mathfrak{m}_a = \ker(ev_a : \mathbb{C}[x_1 - x_n] \to \mathbb{C})$ if and only if $f(a_1, \dots, a_n) = 0$ this gives

$$I_Z = \left\{\begin{array}{l}\text{the collection of polys } f \\ \text{w/ } f(a_1, \dots, a_n) = 0 \text{ for all } a \in Z\end{array}\right\}$$

$$= \left\{\begin{array}{l}\text{the collection of functions} \\ \text{which vanish on } Z\end{array}\right\}.$$

Def$^{\underline{n}}$: We call an ideal $I \subseteq R$ radical if $I = \sqrt{I}$.

Lemma 7: For any subset $Z \subseteq \mathrm{Spec}(R)$, I_Z is a radical ideal in R.

Proof: For any f, g in I_Z and $\mathfrak{p} \in Z$, we have $f, g \in \mathfrak{p}$ by definition and hence $f \pm g \in \mathfrak{p}$. Also for each $a \in R$, $f \in I_Z$, and $\mathfrak{p}$ in Z,

$$f \in \mathfrak{p} \Rightarrow a \cdot f \in \mathfrak{p}.$$

Ranging across all $\mathfrak{p}$ in Z we find that I_Z is closed under addition and the action of R, and hence an ideal.

For radicality, if $f^n \in I_Z$ for $n > 0$ then $f^n \in \mathfrak{p}$ for all $\mathfrak{p}$ in $Z \Rightarrow f \in \mathfrak{p}$ for all $\mathfrak{p}$ in $Z \Rightarrow f \in I_Z$. This shows $I_Z = \sqrt{I_Z}$. ∎

Theorem 8: For any ideal $I \subseteq R$,

$$I_{\mathrm{Van}(I)} = \sqrt{I}.$$

Proof: Clearly, $I \subseteq I_{\mathrm{Van}(I)}$ since, by definition of

$$\mathrm{Van}(I) = \{\text{All primes containing } I\}$$

we have $I \subseteq \mathfrak{p}$ for each $\mathfrak{p}$ in $\mathrm{Van}(I)$. Furthermore for each f with $f^n \in I$ at large

$n>0$ we have

$$f^n \in \mathfrak{p} \text{ for all } \mathfrak{p} \text{ in } \mathrm{Van}(I)$$
$$\Rightarrow f \in \mathfrak{p} \text{ for all } \mathfrak{p} \text{ in } \mathrm{Van}(I)$$
$$\Rightarrow f \in I_{\mathrm{Van}(I)}.$$

So $\sqrt{I} \subseteq I_{\mathrm{Van}(I)}$.

For the converse, any $g \in I_{\mathrm{Van}(I)}$ has $g \in \mathfrak{p}$ for each $\mathfrak{p}$ in $\mathrm{Van}(I)$ giving

$$g \in \bigcap_{\mathfrak{p} \in \mathrm{Van}} \mathfrak{p} = \sqrt{I},$$

by Proposition 2. So $I_{\mathrm{Van}(I)} \subseteq \sqrt{I}$. ∎

Lemma 9: $\mathrm{Van}(I) = \mathrm{Van}(\sqrt{I})$ for each ideal I in commutative R.

Proof: Just follows from the fact that any prime which contains I must contain all of $\sqrt{I}$. ∎

Theorem 10: For any commutative ring R, taking the vanishing locus provides a bijection

$$\mathrm{Van}: \left\{ \begin{array}{c} \text{Radical ideals} \\ I \subseteq R \end{array} \right\} \xrightarrow{\cong} \left\{ \begin{array}{c} \text{Closed subsets} \\ V \subseteq \mathrm{Spec}(R) \end{array} \right\}.$$

The inverse to Van is provided by taking ideals of vanishing functions $V \mapsto I_V$.

Proof: We claim that Van and

$I_? : \{\text{Closed subsets}\} \longrightarrow \{\text{Radical ideals}\}$

are mutually inverse. For the composite $\mathrm{Van} \circ I_?$, any closed subset V in $\mathrm{Spec}(R)$ is of the form $V = \mathrm{Van}(I)$ for some ideal I. Hence, for general closed V,

$$\begin{aligned} \mathrm{Van}(I_V) &= \mathrm{Van}(I_{\mathrm{Van}(I)}) \\ &= \mathrm{Van}(\sqrt{I}) \quad [\text{Thm } 8] \\ &= \mathrm{Van}(I) \quad [\text{Lem } 9] \\ &= V. \end{aligned}$$

So $\mathrm{Van} \circ I_? = \mathrm{id}$. Going the other way,

$$I_? \circ \mathrm{Van}(I) = I_{\mathrm{Van}(I)} = \sqrt{I} = I,$$

whenever I is radical, by Theorem 8. Hence $I_? \circ \mathrm{Van} = \mathrm{id}$ as well. ∎

~ III Radicals and vanishing loci under finiteness

Nullstellensatz

Theorem (Hilbert): Suppose R is a finite type alg over a field, and let $V \subseteq \mathrm{Spec}(R)$ be a closed subset. Consider the corresponding closed subset in the max spectrum

$mV = V \cap \mathrm{mSpec}(R)$. We have

$$I_{mV} = I_V.$$

In particular, $I_{m\mathrm{Van}(I)} = \sqrt{I}$ whenever R is of fin. type over a field.

Proof: Write $V = \mathrm{Van}(I)$ for an ideal $I \subseteq R$, so that

$$mV = \{ \mathfrak{m} \subseteq R \text{ max} : I \subseteq \mathfrak{m} \}.$$

Then

$$\begin{aligned} I_{mV} &= \{ f \in R : f \in \mathfrak{m} \text{ for each } \mathfrak{m} \text{ in } mV \} \\ &= \{ f \in R : f \in \mathfrak{m} \text{ for each maximal } \mathfrak{m} \text{ containing } I \} \\ &= \bigcap_{\mathfrak{m} \in m\mathrm{Van}(I)} \mathfrak{m} \\ &= \sqrt{I}, \end{aligned}$$

by Theorem 6. By Theorem 8 $I_V = \sqrt{I}$ as well. ∎

Theorem 12: Let k be any field and R be a finite type k-algebra. Taking vanishing loci provides a bijection

$$\mathrm{Van} : \{\text{radical ideals in } R\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{Closed subsets} \\ mV \subseteq \mathrm{mSpec}(R) \end{array} \right\}.$$

The inverse to Van is provided by taking ideals of vanishing functions $mV \mapsto I_{mV}$.

Proof: Follows from the equalities

$$\mathrm{mVan}(I_{\mathrm{mVan}(I)}) = \mathrm{mVan}(\sqrt{I}) = \mathrm{mVan}(I)$$

and $I_{\mathrm{mVan}(I)} = \sqrt{I} = I$ of radical I. ∎

So, if we fix our favorite field k, and consider a fin type k-alg R, any property of a (radical) ideal should be reflected in the behaviour of the corresp. vanishing locus, either in $\mathrm{Spec}(R)$ or $\mathrm{mSpec}(R)$.

~ IV Consequences : Primes

Theorem 13: A radical ideal I is prime if and only if, under any decomp

$$\mathrm{Van}(I) = V_1 \cup V_2$$

into a union of closed subsets $V_i \subseteq \mathrm{Spec}(R)$, one one of $V_1 = \mathrm{Van}(I)$ or $V_2 = \mathrm{Van}(I)$.

Proof: Exercise. ∎

Def^{t}: Call a closed subset $V \subseteq \operatorname{Spec}(R)$ irreducible if in any decomposition of V into a union of closed subspaces

$$V = V_1 \cup V_2$$

one of V_1 or V_2 is equal to V.

Corollary 14: Taking vanishing loci provides a bijection

$$\operatorname{Van} : \left\{ \begin{array}{c} \text{Prime ideals} \\ \text{in } R \end{array} \right\} \xrightarrow{\cong} \left\{ \begin{array}{c} \text{Irreducible closed} \\ \text{subspaces in } \operatorname{Spec}(R) \end{array} \right\}.$$

Proof: Exercise. ∎

~ VI More consequences

Let R be of finite type over $\mathbb{R}$. Then via Theorems 10 and 12 we have bijections

$$\begin{tikzcd} & \left\{ \begin{array}{c} \text{Closed sets in} \\ \operatorname{Spec}(R) \end{array} \right\} \arrow[dd, leftrightarrow, dashed, "\cong"] \\ \left\{ \begin{array}{c} \text{Rad. ideals} \\ \text{in } R \end{array} \right\} \arrow[ur, leftrightarrow, "\cong"] \arrow[dr, leftrightarrow, "\cong"'] & \\ & \left\{ \begin{array}{c} \text{closed sets in} \\ \operatorname{mSpec}(R) \end{array} \right\} \end{tikzcd}$$

This completing bijection is given by intersecting

$$-\cap \mathrm{mSpec}(R) : \left\{\begin{array}{c}\text{Closed}\\ \text{subsets in}\\ \mathrm{Spec}(R)\end{array}\right\} \xrightarrow{\cong} \left\{\begin{array}{c}\text{Closed}\\ \text{subsets in}\\ \mathrm{mSpec}(R)\end{array}\right\}$$

Taking complements we observe another bijection

$$-\cap \mathrm{mSpec}(R) : \left\{\begin{array}{c}\text{Open subsets}\\ \text{in } \mathrm{Spec}(R)\end{array}\right\} \xrightarrow{\cong} \left\{\begin{array}{c}\text{Open subsets}\\ \text{in } \mathrm{mSpec}(R)\end{array}\right\}.$$

Def$^{\underline{n}}$: For any topological space X take $\mathrm{Top}(X) = \left\{\begin{array}{l}\text{the partially ordered set of opens } U \subseteq X,\\ \text{ordered under inclusion}.\end{array}\right.$

Corollary 15: For any finite type k-alg R, over a field k, intersection provides an order preserving bijection between the topologies on $\mathrm{Spec}(R)$ and $\mathrm{mSpec}(R)$,

$$-\cap \mathrm{mSpec}(R) : \mathrm{Top}(\mathrm{Spec}\, R) \xrightarrow{\cong} \mathrm{Top}(\mathrm{mSpec}\, R).$$

This bijection preserves unions and intersections. So, for many purposes there is "no difference" between $\mathrm{Spec}(R)$ and $\mathrm{mSpec}(R)$.

~ VI More-more consequences

Theorem 16: For any commutative ring R, and ideals I and J in R, the following are equiv:

(a) $\sqrt{I} = \sqrt{J}$.

(b) $\mathrm{Van}(I) = \mathrm{Van}(J)$.

Furthermore, when R is of finite type over a field, (a) and (b) are furthermore equivalent to

(c) $m\mathrm{Van}(I) = m\mathrm{Van}(J)$.

Proof: Immediate by Theorem 8 and Lemma 11. ∎

~ VII Primary decomposition

We have, in the finite type setting the expression of the radical $\sqrt{I}$ as a huge intersection over maximal ideals $\sqrt{I} = \bigcap_{m\mathrm{Van}} \mathfrak{m}$. Consider for example $\{0\} = \sqrt{\{0\}}$ in $\mathbb{C}[x_1, \dots, x_n]$. We have $\{0\} = \bigcap_{a \in \mathbb{C}^n} \mathfrak{m}_a$ and each finite intersection <u>properly</u> contains $\{0\}$

$$\mathfrak{m}_{a_1} \cap \dots \cap \mathfrak{m}_{a_r} \left[\ni \left(\prod_{i=1}^{r} (x_1 - a_{i,1}) \right) \right] \supsetneq \{0\}.$$

So this intersection <u>needs</u> to be infinite.

At the other extreme, we might think of expressing the radical $\sqrt{I}$ as a finite (?)

intersection of minimal (?) primes over I. This is like prime expansions of integers. We want to prove that such decompositions exist.

Example: Take a radical nonzero ideal $I \subseteq \mathbb{Z}$, which we write uniquely $I = (a)$ for positive integer a. We have $a = p_1 \cdots p_r$ for primes p_i and for $b \in (p_1) \cap \cdots \cap (p_r)$, $p_i \mid b$ for each i. Thus $b = p_1 \cdots p_r \cdot c = a \cdot c$, giving

$$(a) \supseteq (p_1) \cap \cdots \cap (p_r).$$

Obviously $a \in (p_1) \cap \cdots \cap (p_r)$, so that

$$a = (p_1) \cap \cdots \cap (p_r).$$

Def$^{\underline{n}}$: Call an ideal $I \subseteq R$ primary if, for any f, g in R with $fg \in I$, either $f \in I$ or $g^n \in I$ for some $n > 0$.

Observe that any primary ideal I has $\sqrt{I}$ prime but the converse is not necessarily true.

Example: (x, y^3) is primary in $k[x,y]$. Indeed $(k[x,y]/(x,y^3)) \cong k[y]/(y^3)$ and for $f(y), g(y)$ with $f \cdot g = 0$ in $k[y]/(y^3)$ one of f or g has vanishing scalar term. If g has non-vanishing scalar term $g = c_0 + O(y)$ then

$$f \cdot g = c_0 \cdot f(y) + f(y) \cdot O(y)$$

and if $y^3 \nmid f(y)$ then $y^3 \nmid f g$. So we find either $y^3 \mid f(y)$ or g has vanishing scalar term. Hence

either $f = 0$ in $k[y]/(y^3)$ or $g^3 = 0$ in $k[y]/(y^3)$

Pulling back to $k[x,y]$ along the map

$$k[x,y] \to k[y]/(y^p),$$

we see that (x, y^p) is primary.

Example [Atiyah-MacDonald] $R = k[x,y,z]/(xy-z^2)$ then $\mathfrak{p} = (\bar{x}, \bar{z}) \leq R$ is prime as

$$(xy - z^2) \leq (x,z) \leq k[x,y,z]$$

giving $R/\mathfrak{p} = k[x,y,z]/(x,z) \cong k[y]$.

But now $\bar{x}\bar{y} = \bar{z}^2 \in \mathfrak{p}^2$ while $\bar{x} \notin \mathfrak{p}^2$ and $\bar{y}^n \notin \sqrt{\mathfrak{p}^2} = \mathfrak{p}$ for all $n > 0$. Hence $\mathfrak{p}^2$ has prime radical $\sqrt{\mathfrak{p}^2} = \mathfrak{p}$ but is not primary.

We have a generic class of examples, which we want use explicitly.

Lemma 17: If $\sqrt{I}$ is maximal, then I is primary.

For example for any maximal $\mathfrak{m}$ and $n>0$ $I = \mathfrak{m}^n$ is primary.

For the moment let's call an ideal I irreducible if any decomposition

$$I = J_1 \cap J_2 \Rightarrow I = J_1 \text{ or } I = J_2.$$

Lemma 18: If R is Noetherian, any irreducible ideal $I \subseteq R$ is primary.

Proof: If I is irreducible and $f g \in I$ with $f \notin I$. Suppose $g^n \notin I$ for all $n>0$ and consider the sequence of annihilators in the quotient R/I, $\mathrm{Ann}_{R/I}(\bar{g}) \subseteq \mathrm{Ann}_{R/I}(\bar{g}^2) \subseteq \cdots$. By Noetherianity of R, and hence of R/I, these annihilators stabilize, giving $\mathrm{Ann}(\bar{g}^{n+l}) = \mathrm{Ann}(\bar{g}^n)$ for all $l \geq 0$ at large n. Consider now the intersection $(R\cdot f + I) \cap (R \cdot g^n + I)$ and a in their intersection,

$$a = a_0 \cdot f + x = a_1 \cdot g^n + y \quad \text{for } x, y \in I$$

$$\Rightarrow a_1 g^{n+1} = a_0 f g + x g - y g \in I$$

$$\Rightarrow a_1 \in \mathrm{Ann}(\bar{g}^{n+1}) = \mathrm{Ann}(\bar{g}^n).$$

So $a_1 g^n \in I$, giving $a \in I$. Hence

$$I = (R\cdot f + I) \cap (R \cdot g^n + I).$$

Since $f \notin I$, $(R\cdot f + I) \supsetneq I$. Hence by irreducibility of I, $I = (R\cdot g^n + I)$ and thus $g^n \in I$. So we see I is primary. ∎

Theorem 19: For any ideal I in a Noetherian ring R,

$$I = I_1 \cap \cdots \cap I_m$$

for radical ideals I_t.

Proof: It suffices to show that all I can be written as a finite intersection of irreducible I_t. Suppose not, and consider the (nonempty) collection $\mathcal{X}$ of ideals in R which admit no such decomposition. By Noetherianity, all ascending chains in $\mathcal{X}$ achieve a maximum in $\mathcal{X}$, so that $\mathcal{X}$ has a maximal element by Zorn.

For maximal M in $\mathcal{X}$, M is itself not irreducible, and we can write $M = J_1 \cap J_2$ with $J_1, J_2 \supsetneq M$. But now each J_i admits such a decomp into irreducibles, implying such a decomposition for M, a contradiction. So we see $\mathcal{X}$ must be empty. ∎

Theorem 20: For any ideal I in a Noetherian ring R,

$$\sqrt{I} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$$

where $\{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$ is the collection of minimal primes over I. In particular, this collection is finite.

Proof pt 1: We have $\sqrt{I} = J_1 \cap \cdots \cap J_r$ for primary J_i. We are free to assume (*)

$$(J_1 \cap \cdots \cap J_{i-1} \cap J_{i+1} \cap \cdots \cap J_r) \supsetneq \sqrt{I}$$

for all i. Take now $\mathfrak{p}_i = \sqrt{J_i}$ for each i, recalling each $\mathfrak{p}_i$ is prime. We have

$$(\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r) \supseteq (J_1 \cap \cdots \cap J_r) = \sqrt{I}$$

Now for $x \in (\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r)$ there is $n_i > 0$ with $x^{n_i} \in J_i$ for each i, giving

$$x^n \in (J_1 \cap \cdots \cap J_r) = \sqrt{I} \quad \text{for } n = \max\{n_i\}.$$

By radicality, $x \in \sqrt{I}$, so that

$$\sqrt{I} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r. \qquad (**)$$

We can assume (**) has no redundance, as in (*).

Proof pt 2: We claim each $\mathfrak{p}_i$ is minimal over I. Indeed, for a prime $\mathfrak{q}$ with

$$I \subseteq \mathfrak{q} \subseteq \mathfrak{p}_i.$$

Take for convenience $i=1$. Then

$$\mathfrak{q} \supseteq (\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r) \supseteq \mathfrak{p}_1 \cdots \mathfrak{p}_r$$

and by primeness $\mathfrak{q} \supseteq \mathfrak{p}_i$ for some i, giving $\mathfrak{p}_i \subseteq \mathfrak{p}_1$. If $i \neq 1$ then

$$\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r = \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_r,$$

violating our assumption of non redundance. Hence $i = 1$, and the inclusion $\mathfrak{p}_1 \subseteq \mathfrak{q}$ implies $\mathfrak{q} = \mathfrak{p}_1$. So we see each $\mathfrak{p}_i$ is minimal over I.

Now for any minimal prime $\mathfrak{q}$ over I, the inclusion $\mathfrak{q} \supseteq \mathfrak{p}_1 \cdots \mathfrak{p}_r$ again gives $\mathfrak{p}_i \subseteq \mathfrak{q}$, implying $\mathfrak{q} = \mathfrak{p}_i$ by minimality. Thus

$$\min \operatorname{Var}(I) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\},$$

as claimed. ∎

For an ideal I, call an expression of the radical as a finite intersection of primes $\sqrt{I} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$ <u>reduced</u> if omitting any $\mathfrak{p}_i$ results in a strictly larger ideal $\sqrt{I} \subsetneq (\mathfrak{p}_1 \cap \cdots \mathfrak{p}_{i-1} \cap \mathfrak{p}_{i+1} \cdots \cap \mathfrak{p}_r)$. We've already covered the following in the proof of Theorem 20.

Proposition 21: Given an ideal I in a Noetherian ring R, and two reduced expressions

$$(\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s) = \sqrt{I} = (\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r),$$

we have $s = r$ and unique $\sigma \in S_r$ for which

$q_i = \mathfrak{p}_{\sigma(i)}$ at each $i = 1, \ldots, r$.

VIII Geometry of the primary decomposition.

Theorem 22: Let R be a Noetherian ring. For any closed subset $Z \subseteq \mathrm{Spec}(R)$ there is a unique expression of Z as a union of irreducible subvarieties

$$Z = Z_1 \cup \cdots \cup Z_r$$

in which each omission of some Z_i results in a proper containment

$$(Z_1 \cup \cdots Z_{\hat{i}} \cup Z_{i+1} \cdots \cup Z_r) \subsetneq Z.$$

Proof: We have the primary decomp of the rad ideal I_Z, $I_Z = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$, take $Z_i = \mathrm{Var}(\mathfrak{p}_i)$, and note

$$\sqrt{\Pi_i \mathfrak{p}_i} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$$

to get

$$Z_1 \cup \cdots \cup Z_r = \mathrm{Var}(\Pi_i \mathfrak{p}_i) = \mathrm{Var}(\cap_i \mathfrak{p}_i)$$
$$= Z. \quad \blacksquare$$

Theorem 23: Let R be a finite type algebra over a field. For any closed subset $Z \subseteq \mathrm{mSpec}(R)$

there is a unique decomposition

$$Z = Z_1 \cap \cdots \cap Z_r$$

into irreducible closed subsets $Z_i \subseteq \mathrm{mSpec}(R)$.

Example: In $\mathbb{C}[x,y]$ the only prime ideals are $\{0\}$ and $(p(x,y))$ for an irreducible poly p and $\hat{m}_\alpha$ for $\alpha \in \mathbb{C}^n$.
For a general closed subset

$$Z \subseteq \mathbb{A}^2_{\mathbb{C}} = \mathrm{mSpec}(\mathbb{C}[x,y]).$$

Then we have

$$Z = \mathrm{Van}(I), \quad I = (f_1, \ldots, f_t),$$

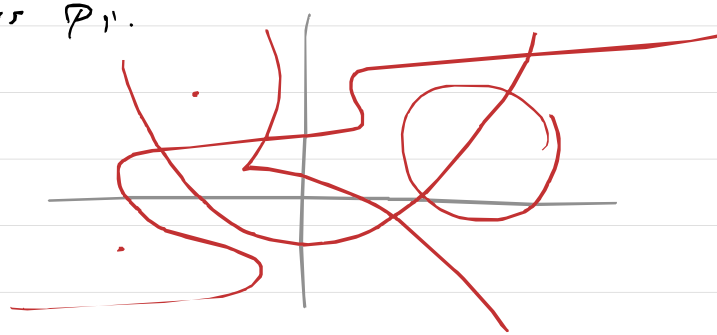
for a radical ideal I, giving

$$Z = \mathrm{Van}(f_1) \cap \cdots \cap \mathrm{Van}(f_t).$$

Each $\mathrm{Van}(f_i)$ is a hypersurface in $\mathbb{A}^2_{\mathbb{C}}$. Then 2.3 makes an "opposite" claim, Z is constructible as a finite union of irreducible hypersurfaces and pts

$$Z = \mathrm{Van}(p_1) \cup \cdots \cup \mathrm{Van}(p_r) \cup \{\alpha_1\} \cup \cdots \cup \{\alpha_m\}$$

for irred polys p_i.

$Z =$ 

HW

1. For a point $\mathfrak{p}$ in $\mathrm{Spec}(R)$, prove that $\mathfrak{p}$ is maximal if and only if the singleton $\{\mathfrak{p}\}$ is a closed subset in $\mathrm{Spec}(R)$.

2. Prove that a radical ideal $I \subseteq R$ is prime if and only if its vanishing locus $\mathrm{Van}(I)$ is an irreducible closed subset in $\mathrm{Spec}(R)$. If R is of finite type over a field, prove that a radical ideal I is prime if and only if $m\mathrm{Van}(I)$ is irreducible in $m\mathrm{Spec}(R)$.

3. For any surjective ring map $\phi: R \to S$, prove that the induced map on spectra is a closed embedding $\phi^*: \mathrm{Spec}(S) \to \mathrm{Spec}(R)$, i.e. an injective set map which sends closed subsets in $\mathrm{Spec}(R)$ to closed subsets in $\mathrm{Spec}(R)$.

4. For any ring R take $R_{rad} = R/\sqrt{\{0\}}$. Prove that the reduction $\pi: R \to R_{rad}$ induces a homeomorphism $\pi^*: \mathrm{Spec}(R_{rad}) \to \mathrm{Spec}(R)$.

5. For a closed subset $Z \subseteq \operatorname{Spec}(R)$, prove that Z is irreducible if and only if Z contains a point $p \in Z$ for which Z is the closure $Z = \overline{\{p\}}$.

6. Classify all radical ideals in $\mathbb{C}[x]$. For a general field k, classify all ideals in $k[x]$.