Algebraic Groups and Representations

Today: Algebraic group basics and examples

I. Complex Lie groups vs. algebraic groups

From an analytic perspective, an affine algebraic group is a complex analytic group (Lie group) which is equipped with a good class of algebraic functions

$$\mathcal{O}_{alg} = \mathcal{O}_{alg}(G) = \left\{ \begin{array}{c} \text{Some specific subcollection of} \\ \text{analytic fun's } f : G \to \mathbb{C} \end{array} \right\}.$$

These functions must be stable under the group op's:

- For $f : G \to \mathbb{C}$ algebraic, the composites
$$G \times G \xrightarrow{m} G \xrightarrow{f} \mathbb{C}$$
and
$$G \xrightarrow{inv.} G \xrightarrow{f} \mathbb{C}$$
must be alg. fun's on $G \times G$, and G, respectively.

(Here a fun. on $G \times G$ is alg if it's a product of alg fun's $f_1 \cdot f_2(z_1, z_2) := f_1(z_1) \cdot f_2(z_2)$ or a sum of such things.)

Ex: $\mathcal{O}_{alg}(GL_n) = \left\{ \begin{array}{c} \text{Polynomial fun's in the} \\ \text{matrix entries } x_{ij} \end{array} \right\}$

$= \mathbb{C}[x_{ij} : 1 \leq i,j \leq n][\det^{-1}]$.

For each coord fun x_{ij} the composite

$$GL_n \times GL_n \xrightarrow{m} GL_n \xrightarrow{x_{ij}} \mathbb{C}$$

is the alg fun

first factor second factor

$$\sum_{k=1}^{n} x_{ik} \cdot x_{kj},$$

and

$$GL_n \xrightarrow{inv} GL_n \xrightarrow{x_{ij}} \mathbb{C}$$

is $(-1)^{i+j} \det^{-1} \cdot (\det [x_{k\ell}]$ minus j-th row and i-th col$)$

$= (i,j)$-th entry in "$[x_{k\ell}]^{-1}$".

Since pulling back along m and inv preserve products and sums, it follows that

$$f \circ m \text{ and } f \circ inv$$

are alg whenever f is algebraic.

Other examples: - $SL_n(\mathbb{C})$ w/ poly's in the x_{ij},
- $Sp_{2n}(\mathbb{C})$ w/ poly's in the x_{ij}.
- $\mathbb{C}^\times$ w/ poly's in z and z^{-1}.

The class of algebraic functions must form a finitely generated $\mathbb{C}$-alg, i.e. must admit some finite generating collection, and must be "complete", in the following sense:

For each point $p \in G$ there must be some collection of funs $f_1, f_2, \ldots, f_r$ in $\mathcal{O}_{alg}$ with

$$\{p\} = V_{an}(f_1, \ldots, f_r) \; (= \{q \text{ in } G \text{ w/ all } f_i(q) = 0\}).$$

Can talk about maps of alg groups as maps which preserve alg fun's, etc.

II. The pure algebraic perspective

Def$^{\underline{n}}$: An affine alg group G, over $\mathbb{C}$, is a group object in the category of affine $\mathbb{C}$-schemes.

So, $G = \mathrm{Spec}(\mathcal{O})$ for some f.g. $\mathbb{C}$-alg $\mathcal{O}$, and G comes equipped w/ an assoc. product map

$$G \times_{\mathrm{Spec}(\mathbb{C})} G \to G \qquad (*)$$

an inv. map $\quad inv: G \to G$ and unit

$$1: \mathrm{Spec}(\mathbb{C}) \to G,$$

all of which are maps of schemes.

Def$^{\underline{n}}$: A map of affine alg groups $\xi: H \to G$ is a map of schemes which preserves the given group structures.

As affine $\mathbb{C}$-schemes are dual to comm (fin. gen) $\mathbb{C}$-algebras,

$$\mathrm{Spec}: (\mathrm{CAlg}_{\mathbb{C}})^{op} \xrightarrow{\sim} \mathrm{AffSch}_{\mathbb{C}},$$

the group structure $(*)$ specifies, and is specified by, a certain structure on $\mathcal{O}$.

Def^n: A (commutative) Hopf algebra is a comm. (for us fin. gen'd) $\mathbb{C}$-alg. $\mathcal{O}$ w/ algebra maps

comult) $\Delta : \mathcal{O} \to \mathcal{O} \otimes \mathcal{O}$

counit) $\epsilon : \mathcal{O} \to \mathbb{C}$

antipode) $S : \mathcal{O} \to \mathcal{O}$

which satisfy

coassoc.) $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$

counit ax) $(\epsilon \otimes 1)\Delta = (1 \otimes \epsilon)\Delta$

antip. ax) $mult(S \otimes id)\Delta = mult(id \otimes S)\Delta = unit \circ \epsilon$.

Ex) • $GL_n = Spec(\mathbb{C}[x_{ij} : 1 \le i,j \le n][det^{-1}])$

w/ structure specified on gen's by

$$\Delta(x_{ij}) = \sum_k x_{ik} \otimes x_{kj}, \quad \epsilon(x_{ij}) = \delta_{ij},$$

$$S(x_{ij}) = \frac{(-1)^{i+j}}{det} \cdot det([x_{kl}] - j\text{-th row } i\text{th col}).$$

• $SL_n = Spec(\mathbb{C}[x_{ij} : 1 \le i,j \le n]/(det - 1))$

w/ same Δ, ϵ and S,

$$S(x_{ij}) = (-1)^{i+j} det([x_{kl}] - j\text{-th row } i\text{th col}).$$

We have the closed embedding

$$SL_n \to GL_n$$

dual to the Hopf alg surjection $\mathcal{O}(GL_n) \to \mathcal{O}(SL_n)$.

- $\mathbb{G}_m = \mathrm{Spec}(\mathbb{C}[x, x^{-1}])$,
$$\Delta(x) = x \otimes x, \quad \Delta(x^{-1}) = x^{-1} \otimes x^{-1}, \quad \epsilon(x^{\pm 1}) = 1,$$
$S(x) = x^{-1}$.

- $\mathbb{G}_a = \mathrm{Spec}(\mathbb{C}[x])$ w/ $\Delta(x) = x \otimes 1 + 1 \otimes x$
$\epsilon(x) = 0$, $\quad S(x) = -x$.

— III. Points in an algebraic group

Proposition 1: For G an affine group scheme, and R any comm. $\mathbb{C}$-alg, the collection of R-points
$$G(R) := \{\text{Scheme maps } \mathrm{Spec}(R) \to G\}$$
$$= \{\text{Alg maps } \mathcal{O}(G) \to R\}$$
naturally forms a discrete group, and for any group hom $\xi : H \to G$, evaluating at R-points provides a discrete group map
$$\xi(R) : H(R) \to G(R).$$

Let's just explain this: Any two maps
$$A, B : \mathrm{Spec}(R) \to G$$
specify a single map to the product $[A\ B] : \mathrm{Spec}(R) \to G \times G$, and we can define
$$A \cdot B = \left(\mathrm{Spec}(R) \xrightarrow{[A\ B]} G \times G \xrightarrow{m} G \right).$$

The unit is given by the composite

$$1_{G(R)} = \left(\mathrm{Spec}(R) \xrightarrow[\text{map}]{\text{structure}} \mathrm{Spec}(\mathbb{C}) \xrightarrow{1} G \right)$$

and inverse is

$$A^{-1} = \left(\mathrm{Spec}(R) \xrightarrow{A} G \xrightarrow{\text{inv}} G \right).$$

Example: • $GL_n(R)$ = invertible matrices over R,

• $\mathbb{G}_m(R)$ = mult. group of units in R. $= R^\times$,

• $\mathbb{G}_a(R)$ = additive group underlying $R = (R, +)$.

Rem: The $\mathbb{C}$-points $G(\mathbb{C})$ have the natural structure of a complex Lie group, and you can just think of G as the pairing of this complex Lie group $G(\mathbb{C})$ with the alg of fun's

$$\mathcal{O}(G) \subseteq \{\text{Analytic fun's on } G(\mathbb{C})\}.$$

We'll elaborate more on this R-point stuff next time

– IV. Points continued

Lets return to the claim:

For R comm $\mathbb{C}$-alg, G affine group scheme,

$$G(R) = \{ \text{Alg maps } \mathcal{O}(G) \to R \}$$

has group structure induced by group str. on G / Hopf str. on $\mathcal{O}(G)$.

Example: $\mathbb{G}_a(R) = (R, +)$. $^{(*)}$ Lets check.

$\mathcal{O}(\mathbb{G}_a) = \mathbb{C}[x]$ w/ $\Delta(x) = x \otimes 1 + 1 \otimes x$.

$$\mathrm{Hom}_{Alg}(\mathbb{C}[x], R) \xrightarrow{\sim} R, \quad \xi \mapsto \xi(x).$$

For two maps ξ_1, ξ_2 w/ $\xi_i(x) = r_i$, we have uniquely def

$$[\xi_1, \xi_2] : \mathbb{C}[x] \otimes \mathbb{C}[x] = \mathbb{C}[x_1, x_2] \to R$$
$$x_i \longmapsto r_i$$

and compose w/ comult to get the group structure

$$\xi_1 \cdot \xi_2 := \left(\mathbb{C}[x] \xrightarrow{\Delta} \mathbb{C}[x] \otimes \mathbb{C}[x] \xrightarrow{[\xi_1, \xi_2]} R \right)$$
$$x \longmapsto x \otimes 1 + 1 \otimes x \longmapsto r_1 + r_2 .$$

$\xi_1 \cdot \xi_2 (x) = r_1 + r_2$. This gives $(*)$.

Anyway, to repr

— V. Representations of algebraic groups

Def$^{\underline{n}}$: A G-representation, for alg group G, is a vector space V equipped with a map of alg groups
$$\rho_V : G \to GL(V).$$

Here we take the coord free expression
$$GL(V) = \mathrm{Spec}\left\{ \mathrm{Sym}\left(\mathrm{End}_{\mathbb{C}}(V)^*\right)[\det^{-1}] \right\}$$

with
$$\Delta : \mathrm{Sym}(\mathrm{End}(V)^*)[\det^{-1}] \to \mathrm{Sym}(\mathrm{End}(V)^*)[\det^{-1}]^{\otimes 2}$$
given on the generators by coevaluation

$$\Delta|_{\mathrm{End}^*} = \mathrm{coev}_V : \mathrm{End}_{\mathbb{C}}(V)^* = V^* \otimes V \longrightarrow V^* \otimes V \otimes V^* \otimes V$$
$$= \mathrm{End}(V)^* \otimes \mathrm{End}(V)^*$$
$$f \otimes v \longmapsto \sum_i f \otimes \underbrace{v_i \otimes v^i}_{\text{doesn't depend on basis}} \otimes v.$$

and ϵ given on gen's by evaluation

$$\Delta|_{\mathrm{End}^*} = \mathrm{ev}_V : \mathrm{End}_{\mathbb{C}}(V)^* = V^* \otimes V \longrightarrow \mathbb{C}$$
$$f \otimes v \longmapsto f(v).$$

Rem: The antipode S on a bialgebra $\mathcal{O}$, if it exists, is uniquely determined. Hence a Hopf alg is, equiv, a bialgebra $(\mathcal{O}, \Delta, \epsilon)$ for which an antipode exists. The antipode is also preserved under Hopf maps. So we generally ignore it.

We'll give a few, equivalent descriptions of the category of G-representations

VI. Corepresentations

Def^{n}. Given a Hopf algebra (or coalg) $\mathcal{O}$ a corepresentation over $\mathcal{O}$ is a vector space V equipped with a linear map (coaction)

$$\rho_V : V \longrightarrow V \otimes \mathcal{O}$$

which is coassociative and counital

$$(1 \otimes \Delta)\rho_V = (\rho_V \otimes 1)\rho_V, \quad (1 \otimes \epsilon)\rho_V = id_V.$$

Example (Universal coaction): For $\mathcal{O}_V = \mathcal{O}(GL(V))$, I claim $\mathcal{O}_V$ always coacts on V in a natural way, giving it the structure of a $\mathcal{O}_V$-corepresentation.

We define

$$\rho_V^{univ} : V \longrightarrow V \otimes V^* \otimes V = V \otimes End_{\mathbb{C}}(V)^* \subseteq V \otimes \mathcal{O}_V$$

by

$$v \longmapsto coev(1) \otimes v = \sum v_i \otimes v^i \otimes v.$$

From the formula for $\Delta|_{\mathrm{End}(V)^*}$ this coaction is clearly coassociative, and for an expression

$v = \sum_i c_i \cdot v_i$

in a chosen basis

$$(1 \otimes \epsilon)\rho^{\mathrm{univ}}(v) = \sum_i v_i(v^i(v))$$

$$= \sum c_i \cdot v_i = v,$$

so counital as well.

VII. Reps vs. coreps

Theorem 2: Let G be an algebraic group over $\mathbb{C}$. For a vector space V, the following data are equivalent:

a) The structure of a G-representation,

$$\phi_V : G \to GL(V).$$

b) The structure of an $\mathcal{O}(G)$-corepresentation,

$$\rho_V : V \to V \otimes \mathcal{O}(G).$$

c) The structure of an R-linear group action of $G(R)$ on the base change $V_R = V \otimes_{\mathbb{C}} R$, of each comm $\mathbb{C}$-alg R, which varies naturally in R.

We'll sketch a proof.

Sketch proof: (a) $\Rightarrow$ (b) Suppose we have a map of algebraic groups $\phi_V : G \to GL(V)$. Then we obtain a Hopf algebra $\psi_V : \mathcal{O}(GL) \to \mathcal{O}(G)$ (via res. of algebraic function along ϕ_V). We compose the universal coaction with ψ_V to obtain a coaction of $\mathcal{O}(G)$ on V,

$$\rho_V := (1 \otimes \psi_V)\rho_V^{univ} : V \to V \otimes \mathcal{O}(GL) \to V \otimes \mathcal{O}(G).$$

(b) $\Rightarrow$ (a) Given a coaction ρ_V we need to produce a Hopf map $\psi_V : \mathcal{O}(GL_V) \to \mathcal{O}(G)$. For this we need to produce a map on the generators

$$\mathcal{O}(GL_V) \supseteq \mathrm{End}(V)^* = V^* \otimes V \longrightarrow \mathcal{O}(G).$$

Now the coaction gives an element

$$\rho_V \in \mathrm{Hom}_{\mathbb{C}}(V, V \otimes \mathcal{O}(G))$$

and via adjunction this determines a corresp. linear function

$$\psi_V|_{\mathrm{End}(V)^*} : \mathrm{End}(V)^* = V^* \otimes V \longrightarrow \mathcal{O}(G).$$

This extends to an algebra map

$$\psi_V : \mathrm{Sym}(\mathrm{End}(V)^*) \longrightarrow \mathcal{O}(G)$$

which I promise sends the determinant to a unit and thus gives $\psi_V : \mathcal{O}(GL_V) \to \mathcal{O}(G)$. This map is a Hopf algebra map, again I promise.

(b) $\Rightarrow$ (c) Given $\xi \in G(R)$, $\xi : \mathcal{O}(G) \to R$, We have the corresponding $\mathbb{C}$-linear function

$$\overline{act}_\xi : V \xrightarrow{\rho_V} V \otimes \mathcal{O}(G) \xrightarrow{1 \otimes \xi} V \otimes R.$$

This function determines a unique R-linear map $act_\xi : V_R \to V_R$ w/ $act_\xi|_V = \overline{act}_\xi$.

This map is invertible with inverse $act_{\xi \circ S}$, and the assignment $\xi \mapsto act_\xi$ determines a natural action

$$G(R) \to Aut_R(V_R).$$

(c) $\Rightarrow$ (a) $GL(V)$ has functor of pts

$$GL(V)(R) = Aut_R(V_R),$$

and (c) determines a transformation

$$G(-) \to GL(V)(-)$$

of functors $CAlg_{\mathbb{C}}^{op} \to Group$. By Yoneda this gives a group map $\chi_V : G \to GL(V)$. ∎

Rem: Note that morphisms of G-reps, in terms of maps $\chi_V : G \to GL(V)$, are somewhat opaque. However, morphisms of coreps, or in terms of (c), are clear.

VIII. The category of G-representations

Def$^{\underline{n}}$: We define, for alg group G, $Rep(G) = Corep(\mathcal{O}(G))$.

Next Time: I'll explain clearly what a map of coreps is, and elaborate more on various things.

$\Rightarrow$ $Lie(G)$ in alg terms

$\Rightarrow$ Examples

$\Rightarrow$ The functor $Rep(G) \to Rep(Lie\,G)$.

IX. More fundamentals

For an alg group G recall that the R-points
$$G(R) = \mathrm{Hom}_{Sch}(\operatorname{Spec} R, G)$$
act on V_R, for $\mathcal{O}(G)$-comod V, as
$$x\cdot - : V_R \to V_R,$$

$$x\cdot(v\otimes r) = \sum_i v_{i_0} \otimes x(v_{i_1})\cdot r$$
where $\Delta(v) = \sum_i v_{i_0} \otimes v_{i_1} \in V \otimes \mathcal{O}(G)$. In particular, the complex points $G(\mathbb{C})$ act as
$$x\cdot v = \sum_i x(v_{i_1})\, v_{i_0}. \qquad (*)$$

Lemma 3: Let $\rho_V^{univ} : V \to V \otimes \mathcal{O}(GL(V))$ be the universal coaction and $A \in GL(V)(\mathbb{C}) = \mathrm{Aut}_{\mathbb{C}}(V)$ be a complex point. Then
$$\underbrace{A\cdot v}_{\text{from } (*)} = \underbrace{A(v)}_{\text{expected action}}$$

Proof: In a basis $\{v_1, \ldots, v_n\}$ we have $A = [a_{ij}]$ and then
$$A\cdot v_j = \sum_{i=1}^n v_i \otimes A(\underbrace{v^i \otimes v_j}_{\text{in } \mathrm{End}(V)^*}) = \sum_i v^i(A(v_j))\cdot v_i$$
$$= \sum_i a_{ij} v_i$$
$$= A(v_j).$$
By linearity then $A\cdot v = A(v)$ at all $v \in V$. ∎

Elaborating on the equivalence between (a) and (b) in Theorem 2.

Lemma 4: Let $\phi : G \to GL(V)$ be a G-rep with corresponding coaction $\rho = (1\otimes\phi^*)\rho_V^{univ} : V \to V\otimes\mathcal{O}(G)$. For each $x\in G(\mathbb{C})$ and $v \in V$ we have

$$x\cdot v = \phi(x)(v).$$

Proof: We have $\phi(x) = \left(\mathcal{O}(GL(V)) \xrightarrow{\phi^*} \mathcal{O}(G) \xrightarrow{x} \mathbb{C}\right)$ and $x\cdot - : V \to V$ is given by the composite

$$V \xrightarrow{\rho_V^{univ}} V\otimes\mathcal{O}(GL(V)) \xrightarrow{1\otimes\phi^*} V\otimes\mathcal{O}(G) \xrightarrow{1\otimes x} V\otimes\mathbb{C} \cong V$$

$$= \phi(x)\cdot - : V \to V.$$

Thus, by Lemma 3, $x\cdot v = \phi(x)\cdot v = \phi(x)(v)$. ∎

– X. Morphisms of G-repr via closed points

By a map of $\mathcal{O}$-coreps, $f : V \to W$, we mean the expected thing: A linear map which fits into a diagram

$$\begin{tikzcd} V \arrow[r, "\rho_V"] \arrow[d, "f"'] & V\otimes\mathcal{O} \arrow[d, "f\otimes 1"] \\ W \arrow[r, "\rho_W"'] & W\otimes\mathcal{O} \end{tikzcd}.$$

Proposition 5: Let $\phi_V : G \to GL(V)$ and $\phi_W : G \to GL(W)$ be G-representations, with corresponding comodules (V, ρ_V) and (W, ρ_W). A linear map $f : V \to W$ is a map of $\mathcal{O}(G)$-comods if and only if, at each complex point $x \in G(\mathbb{C})$, we have

$$f(x \cdot v) = x \cdot f(v).$$

Before the proof we need an algebra fact.

Theorem (Reducedness): Any affine algebraic group G over $\mathbb{C}$ is reduced, i.e. has no nilpotent elements in its algebra of functions.

OK.

Proof of Proposition 5: If f is a map of $\mathcal{O}(G)$-comods then

$$(*) \quad f(x \cdot v) = f\left(\sum_i x(v_{i_1}) v_{i_0}\right) = \sum x(v_{i_1}) f(v_{i_0})$$

and by comodness

$$\sum_i f(v_{i_0}) \otimes v_{i_1} = \rho_W(f(v)),$$

so that $(*) = x \cdot f(v)$ as well.

Suppose conversely $f(x \cdot v) = x \cdot f(v)$ at all complex points. In a basis $\{w_1, \ldots, w_m\}$ for W, we have at each $v \in V$

$$(f \otimes 1)\rho_V(v) = \sum_{j=1}^m w_j \otimes \xi_j$$

$$\rho_W(f(v)) = \sum_{j=1}^m w_j \otimes \xi'_j$$

for some functions $\xi_j, \xi_j' \in \mathcal{O}(G)$. Now the difference is given by

$$(f\otimes 1)(\rho_V(v)) - \rho_W(f(v)) = \sum_j w_j \otimes (\xi_j - \xi_j') \quad (*)$$

and we know that at each $\mathbb{C}$-point $x : \mathrm{Spec}(\mathbb{C}) \to G$

$$f(x\cdot v) - x\cdot f(v) = (1\otimes x)\cdot(*) = \sum_j x(\xi_j - \xi_j')\, w_j = 0,$$

by assumption.

Hence $x(\xi_j - \xi_j') = 0$ at each j and all x. Thus the vanishing locus of $\xi_j - \xi_j'$ is all of $G(\mathbb{C})$,

$$\mathrm{Van}(\xi_j - \xi_j') = G(\mathbb{C}),$$

which gives by Nullsensatz (sp?) $\xi_j - \xi_j' \in \sqrt{0} \subseteq \mathcal{O}$, at each j. But finally by reducedness of G this implies $\xi_j - \xi_j' = 0$ at all j, and thus

$$(f\otimes 1)(\rho_V(v)) = \rho_W(f(v)).$$

Since this holds at all $v \in V$, we conclude that f is a map of $\mathcal{O}(G)$-corepresentations. ∎

Theorem 6: Let G be an affine algebraic group, and $\mathrm{Rep}(G)$ be the category whose objects are fin-dim vect spaces V equipped with a map of alg groups $\rho : G \to GL(V)$, and whose morphisms are linear maps $f : V \to W$ which satisfy $f(x\cdot v) = x\cdot f(v)$ at all $v \in V$ and $x \in G(\mathbb{C})$.

There is a canonical isomorphism of categories

$$t : \mathrm{Rep}(G) \xrightarrow{\sim} \mathcal{O}(G)\text{-corep}$$
$$(V, \phi) \longmapsto (V, (1 \otimes \phi^*)\rho_V^{\mathrm{univ}})$$
$$f \longmapsto f.$$

Proof: By Theorem 2 t is bijective on objects and by Proposition 5 t is bijective on morphisms.

The fact that t respects composition is automatic by construction $t(f \circ g) = f \circ g = t(f) \circ t(g)$. ∎

Exercise: Prove that for an affine algebraic group G, the forgetful functor
$$\mathrm{Rep}(G) \longrightarrow \mathrm{Rep}(\underbrace{G(\mathbb{C})})$$
is fully faithful.

Considered as a discrete group!

[This is just me asking if you payed attention in Arvind's AG class.]

– XII. The tangent at the identity (pre-Lie)

Let G be an analytic group. We would think of the tangent space here as equivalence classes of functions
$$g : \mathbb{C} \supseteq D \to G$$
for a small disk D around 0 with $g(0) = 1$. Here
$$[g] \sim [h] \quad \text{if} \quad g'(0) = h'(0),$$

If we let ε be the parameter on D, we have an expansion in coords around 1

$$g(\varepsilon) = 1 + g_1 \varepsilon + O(\varepsilon^2)$$

and $[g] = [h]$ in $T_1 G$ iff $g_1 = h_1$. So, only the linear term matters.

In algebraic perspective we replace the infinitesimally small disk with the "dual number" $\mathbb{C}[\varepsilon]/\varepsilon^2$. We note that $\mathbb{C}[\varepsilon]/\varepsilon^2$ has only one map to $\mathbb{C}$, namely $\varepsilon \to 0$, and hence a canonical map for the functor of points

$$G(\mathbb{C}[\varepsilon]/\varepsilon^2) \to G(\mathbb{C})$$

at algebraic G. The tangent space at 1 is then

$$T_1 G := \{1_G\} \underset{G(\mathbb{C})}{\times} G(\mathbb{C}[\varepsilon]/\varepsilon^2)$$

$$= \left\{ \begin{array}{l} \text{Algebra maps } \mathcal{O}(G) \to \mathbb{C}[\varepsilon]/\varepsilon^2 \\ \text{whose composite along } \mathbb{C}[\varepsilon]/\varepsilon^2 \to \mathbb{C} \\ \text{recovers } 1_G = \epsilon \end{array} \right\}$$

where $\mathfrak{m}_G \subseteq \mathcal{O}$ = the kernel of counit ϵ.

Example: For GL_n

$$GL_n(\mathbb{C}[\varepsilon]/\varepsilon^2) = \text{Inv. } n\times n\text{-mat over } \mathbb{C}[\varepsilon]/\varepsilon^2$$

and

$$T_1 GL_n = \left\{ \begin{array}{l} \text{Inv. mat. } [a'_{ij} + a_{ij}\varepsilon] \text{ over} \\ \mathbb{C}[\varepsilon]/\varepsilon^2 \text{ w/ } [a'_{ij}] = I_n \end{array} \right\}$$

$$= \left\{ \begin{array}{l} I_n + A\varepsilon \text{ where } A \text{ is arbitrary} \\ \text{in } M_{n\times n}(\mathbb{C}) \end{array} \right\}.$$

Thus we have

$$M_{n\times n}(\mathbb{C}) \xrightarrow{\cong} T_1 GL_n, \quad A \mapsto I_n + A\varepsilon.$$

Remark:

Note that, if we consider ε a complex number of sufficiently small magnitude, $I_n + A\varepsilon$ is actually invertible. So we literally have corresponding complex paths

$$I_n + A\varepsilon : D \longrightarrow GL_n(\mathbb{C})$$

which realize an identification between algebraic $T_1 GL_n$ and analytic $T_1 GL_n(\mathbb{C})$.

Example: We have $SL_n(\mathbb{C}[\varepsilon]/\varepsilon^2) \subseteq GL_n(\mathbb{C}[\varepsilon]/\varepsilon^2)$ as matrices of determinant 1. Thus

$$T_1 SL_n = \left\{ \begin{array}{l} I_n + A\varepsilon \in T_1 GL_n \text{ with} \\ \det(I_n + A\varepsilon) = 1 \end{array} \right\}.$$

We check now

$$\det(I_n + A\varepsilon) = \det \begin{bmatrix} 1 + a_{11}\varepsilon & & a_{ij}\varepsilon \\ & \ddots & \\ a_{ji}\varepsilon & & 1 + a_{nn}\varepsilon \end{bmatrix} = 1 + \operatorname{tr}(A)\varepsilon.$$

Thus
$$T_1 SL_n = \{1 + A\varepsilon : \ tr(A) = 0\} \subseteq gl_n(\mathbb{C})$$
$$= sl_n(\mathbb{C})$$

Example : $Sp_{2n}(\mathbb{C}[\varepsilon]/\varepsilon^2) \subseteq GL_{2n}(\mathbb{C}[\varepsilon]/\varepsilon^2)$
is given by matrices $\tilde{A}$ over $\mathbb{C}[\varepsilon]/\varepsilon^2$ with
$$\langle \tilde{A}\cdot v, w\rangle = \langle v, \tilde{A}^{-1}\cdot w\rangle \qquad (*)$$
where
$$\langle\ ,\ \rangle : \mathbb{C}^{2n} \otimes \mathbb{C}^{2n} \to \mathbb{C}$$
is the anti-symmetric form $\langle v_i, v_j\rangle = \delta_{j\,i+n} - \delta_{i\,j+n}$.
Now for an element of the tangent space
$$\tilde{A} = I_n + A\varepsilon \quad \in T_1 GL_n$$
we have
$$\tilde{A}^{-1} = I_n - A\varepsilon$$
so that $(*)$ becomes
$$\langle v, w\rangle + \langle A\cdot v, w\rangle = \langle v, w\rangle - \langle v, A\cdot w\rangle$$
$$\iff \langle A\cdot v, w\rangle = -\langle v, A\cdot w\rangle.$$
Hence we calculate
$$T_1 Sp_{2n} = sp_{2n}(\mathbb{C}) \subseteq gl_{2n}(\mathbb{C}).$$

Similarly $T_1 SO_n = so_n(\mathbb{C}) \subseteq gl_n(\mathbb{C})$

– XIII The Lie structure on the tangent space

Def^n: For functions $\alpha, \beta : \mathcal{O}(G) \to \mathbb{C}$ (here I just mean linear functions) the convolution product $\alpha * \beta : \mathcal{O}(G) \to \mathbb{C}$ is the new function defined by

$$\alpha * \beta(\xi) = \sum_i \alpha(\xi_{i_1})\beta(\xi_{i_2}),$$

where

$$\Delta(\xi) = \sum_i \xi_{i_1} \otimes \xi_{i_2}.$$

Note now that we have the inclusion

$$\mathrm{incl} : T_1 G \hookrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}(G), \mathbb{C})$$

$$\left(\tilde{\alpha} = 1_G + \alpha\varepsilon : \mathcal{O}(G) \to \mathbb{C}[\varepsilon]/\varepsilon^2\right) \longmapsto (\alpha : \mathcal{O}(G) \to \mathbb{C}).$$

This inclusion identifies $T_1 G$ with functions $\alpha : \mathcal{O}(G) \to \mathbb{C}$ which satisfy

T1) $\alpha|_{\mathfrak{m}_G^2} = 0$

T2) $\alpha|_{\mathbb{C}1_{\mathcal{O}}} = 0$.

Example: For $G = GL_n$, the inclusion incl sends A in $\mathfrak{gl}_n(\mathbb{C}) \cong T_1 GL_n$ to the unique linear function

$$\text{"}A\text{"} : \mathcal{O}(GL_n) = \mathbb{C}[x_{ij} : 1 \le i,j \le n][\det^{-1}] \to \mathbb{C}$$

which satisfies T1 and T2 and has

$$\text{"}A\text{"}(x_{ij}) = a_{ij}.$$

Theorem 7: Considering $\mathrm{Hom}_{\mathbb{C}}(\mathcal{O}(G), \mathbb{C})$ as an assoc. alg under convolution, the inclusion

$$\mathrm{incl}: T_1 G \hookrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}(G), \mathbb{C})$$

identifies $T_1 G$ with a Lie subalg in

$$\underbrace{\mathrm{Hom}_{\mathbb{C}}(\mathcal{O}(G), \mathbb{C})^{Lie}}_{\text{Lie alg under convolution commutator } \alpha * \beta - \beta * \alpha.}$$

O(G) = C \oplus m
Thus O(G)\ot O(G) =
C\ot C + m\ot C + C\ot m + m\ot m

Proof: Take any two such functions α, β which satisfy (T1) and (T2). As $\Delta(1) = 1 \otimes 1$

$$\alpha * \beta - \beta * \alpha \big|_{\mathbb{C} \cdot 1_0} = 0,$$

and for $x \in m$ we have

$$\Delta(x) = 1 \otimes x + x \otimes 1 \mod (m \otimes m)$$

since $(\epsilon \otimes 1)\Delta(x) = (1 \otimes \epsilon)\Delta(x) = x$. Thus for $x \cdot y$ with $x, y \in m$

$$\begin{aligned} \Delta(xy) &= \Delta(x) \cdot \Delta(y) \\ &= xy \otimes 1 + 1 \otimes xy + x \otimes y + y \otimes x \mod (m^2 \otimes m^2) \\ &= x \otimes y + y \otimes x \mod (m^2 \otimes m^2) \end{aligned}$$

so that

$$\begin{aligned} \alpha * \beta(x \cdot y) &= \alpha(x)\beta(y) + \alpha(y)\beta(x) \\ &= \beta * \alpha(xy). \end{aligned}$$

Hence $[\alpha, \beta](xy) = 0$. Since

$$m^2 = \mathrm{Span}_{\mathbb{C}}(xy : x, y \in m)$$

we have $[\alpha, \beta]\big|_{m^2} = 0$. ∎

Def$^{\underline{n}}$: For G an affine algebraic group

$$\mathrm{Lie}\, G := \{ T_1 G \text{ w/ convolution bracket} \}$$

– XIV. Functoriality of Lie(–)

: AffAlgGrp_C -> Lie_C

For $\phi : H \to G$ a map of algebraic groups the corresp map on functions

$$\phi^* : \mathcal{O}(G) \to \mathcal{O}(H)$$

is a Hopf map. Thus restricting along ϕ^* gives a map of (Lie) algebras

$$\phi_* : \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}(H), \mathbb{C})^{\mathrm{Lie}} \to \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}(G), \mathbb{C})^{\mathrm{Lie}}$$

which furthermore fits into a diagram

$$\begin{tikzcd} T_1 H \arrow[r, "T_1\phi"] \arrow[d, hook] & T_1 G \arrow[d, hook] \\ \mathrm{Hom}(\mathcal{O}(H), \mathbb{C})^{\mathrm{Lie}} \arrow[r, "\phi_*"'] & \mathrm{Hom}(\mathcal{O}(G), \mathbb{C})^{\mathrm{Lie}} \end{tikzcd}.$$

(Just follows by $(\phi^*)^{-1}(m_H) = m_G$.) Thus

$T_1\phi : T_1 H \to T_1 G$ is a map of Lie algebras.

$$\mathrm{Lie}\,\phi : \mathrm{Lie}\, H \to \mathrm{Lie}\, G.$$

– XV. Examples

Theorem 8: The linear identification
$$\mathfrak{gl}_n(\mathbb{C}) \xrightarrow{\sim} \mathrm{Lie}\, GL_n, \quad A \longmapsto I_n + A\varepsilon,$$
is an isomorphism of Lie algebras.

Proof: The above isomorphism identifies, further, a matrix A with the function "A" $: \mathcal{O} \to \mathbb{C}$ w/ "A"$(x_{ij}) = a_{ij}$ as before. Then for matrices A, B we have
$$["A", "B"](x_{ij}) = \sum_k A(x_{ik}) B(x_{kj}) - \sum_k B(x_{ik}) A(x_{kj})$$
$$= \sum_k a_{ik} \cdot b_{kj} - b_{ik} \cdot a_{kj}$$
$$= \text{"}[A,B]\text{"}(x_{ij}).$$
We're done. ∎

Corollary 9: The identifications
- $\mathfrak{sl}_n(\mathbb{C}) \xrightarrow{\sim} \mathrm{Lie}\, SL_n$
- $\mathfrak{sp}_{2n}(\mathbb{C}) \xrightarrow{\sim} \mathrm{Lie}\, Sp_{2n}$
- $\mathfrak{so}_n(\mathbb{C}) \xrightarrow{\sim} \mathrm{Lie}\, SO_n$

are all isomorphisms of Lie algebras.

Proof: For $G = SL_n, Sp_{2n}, SO_n$ and $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}), \mathfrak{sp}_{2n}(\mathbb{C}), \mathfrak{so}_n(\mathbb{C})$ we've established diagrams of linear inclusions

$$\begin{tikzcd} \mathfrak{g} \arrow[r] \arrow[d, "\wr"'] & \mathfrak{gl}_m(\mathbb{C}) \arrow[d, "\wr"] \\ \mathrm{Lie}\, G \arrow[r] & \mathrm{Lie}\, GL_m \end{tikzcd}$$

where each map but $\mathfrak{g} \xrightarrow{\sim} \mathrm{Lie}\, G$ is now understood to be a map of Lie algebras. But now the diagram forces this final map to be an isomorphism of Lie algebras. ∎

XVI. Infinitesimal action for G-reps

Let's fix G algebraic and $\mathfrak{g} = \mathrm{Lie}(G)$. Fix also $\mathcal{O} = \mathcal{O}(G)$. Of our many options, we consider now $\mathrm{Rep}\, G$ via $\mathrm{Corep}(\mathcal{O})$ and consider

$$\mathfrak{g} \subseteq \{\text{Linear functs } \mathcal{O} \to \mathbb{C}\}$$

consisting of all maps which

T1) Vanish on $\mathfrak{m}_G^2$

T2) Vanish on $1 \in \mathcal{O}$.

algebra under convolution

Proposition 10: Take $H_G = \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}, \mathbb{C})$.

a) For any G-rep V, considered as $\mathcal{O}$-corep, the formula

$$\alpha \cdot v = (1 \otimes \alpha)\rho_V(v) = \textstyle\sum_i \alpha(v_{i_1}) \cdot v_{i_0} \in V$$

defines an assoc. action of H_G on V.

b) Restricting to elements A in $\mathfrak{g}$, the action $A \cdot v =$ as in (a) defines an action of $\mathfrak{g}$ on V, i.e. gives it the structure of a $\mathfrak{g}$-representation.

Proof: a) Recall the product on $\mathcal{H}_G$,

$$\alpha * \beta(\xi) := (\alpha * \beta)\underline{\Delta}(\xi)$$

at $\xi \in \mathcal{O}$. Then for $v \in V$,

$$
\begin{aligned}
(\alpha * \beta)(v) &= (1 \otimes \alpha \otimes \beta)(1 \otimes \Delta)\rho_V(v) \\
&= (1 \otimes \alpha \otimes \beta)(\rho_V \otimes 1)\rho_V(v) \quad \text{(Coassoc.)} \\
&= (1 \otimes \alpha \otimes 1)(\rho_V \otimes \beta)\rho_V(v) \\
&= (1 \otimes \alpha \otimes 1)(\rho_V \otimes 1)(1 \otimes \beta)\rho_V(v)
\end{aligned}
$$

$$
\begin{tikzcd}
V \arrow[r, "\rho_V"] \arrow[drr, "\beta\cdot -"'] & V \otimes \mathcal{O} \arrow[r, "1 \otimes \beta"] & V \otimes \mathbb{C} \arrow[r, "\rho_V \otimes 1"] \arrow[d, "\cong"] & V \otimes \mathcal{O} \otimes \mathbb{C} \arrow[r, "1 \otimes \alpha \otimes 1"] \arrow[d, "\cong"] & V \otimes \mathbb{C} \otimes \mathbb{C}. \arrow[d, "\cong"] \\
& & V \arrow[r] \arrow[rr, bend right, "\alpha\cdot -"'] & V \otimes \mathcal{O} \arrow[r] & V
\end{tikzcd}
$$

$$= \alpha \cdot (\beta \cdot v).$$

We thus obtain associativity.

b) Any $\mathcal{H}_G$-module becomes a $\mathcal{H}_G^{Lie}$-rep, under the same action. Hence restricting to the subalgebra

$$\mathfrak{g} \hookrightarrow \mathcal{H}_G^{Lie}$$

produces a $\mathfrak{g}$-rep structure on any $\mathcal{O}$-corepresentation V. ∎

Def^{n}: For a G-representation V we call the corresponding action of $\mathfrak{g} = \mathrm{Lie}\, G$ on V, as in Proposition 10, the *infinitesimal action* of $\mathfrak{g}$ on V.

Clearly the infinitesimal action is natural, in the sense that a map $f: V \to W$ of G-representations is also a map of $\mathfrak{g}$-reps, under the infinitesimal action. So we get a functor (!)

$$\mathrm{inf} = \mathrm{inf}_G : \mathrm{Rep}(G) \to \mathrm{Rep}(\mathfrak{g}).$$

Proposition 11: Let $\phi: H \to G$ be a map of algebraic groups and $\mathrm{Lie}\,\phi : \mathrm{Lie}\, H \to \mathrm{Lie}\, G$ be the induced map on Lie algebras. Then for any G-rep V with restriction $V|_H$ to H

$$\mathrm{inf}_H (V|_H) = \mathrm{inf}_G (V) \Big|_{\mathrm{Lie}\, H} \qquad (*)$$

Proof: $\phi : H \to G \Rightarrow \phi^* : \mathcal{O}(G) \to \mathcal{O}(H)$
$\Rightarrow \phi_* : \mathcal{H}_H \to \mathcal{H}_G$
which fits into a diagram

$$\begin{array}{ccc} \mathcal{H}_H & \xrightarrow{\phi_*} & \mathcal{H}_G \\ \uparrow & & \uparrow \\ \mathfrak{h} & \xrightarrow[\mathrm{Lie}\,\phi]{} & \mathfrak{g} \end{array}.$$

This is sufficient for (*). ∎

Example (The universal example):
We have the standard representation V over $GL(V)$ with corresponding universal coaction

$$\rho_V^{univ}(v) = \sum_{i=1}^{n} v_i \otimes v^i \otimes v \in V \otimes \mathrm{End}(V)^* \subseteq V \otimes \mathcal{O}.$$

In coordinates, A in $gl(V) = \mathrm{Lie}\, GL(V)$ is given by

$$A = [a_{ij}] = \sum_{i,j} a_{ij}\, v_i \otimes v^j .$$

Then for the expression

$$v = \sum_j c_j v_j$$

we get for the inf action of $gl(V)$ on V,

$$A \cdot v = \sum_i A(v^i \otimes v) \cdot v_i$$

$$= \sum c_j a_{ij} \cdot v_i = A(v).$$

This is to say, the standard rep for $GL(V)$ becomes the standard rep for $gl(V) = \mathrm{Lie}\, GL(V)$ under the infinitesimal action.

As a corollary to this Example and Proposition 11 we find an alternate definition of the infinitesimal action.

Theorem 12: For a G-representation $\phi_V : G \to GL(V)$ the infitesimal action of $\mathfrak{g} = \mathrm{Lie}\, G$ on V is determined by the map $\mathrm{Lie}\,\phi_V : \mathfrak{g} \to gl(V)$, and the standard action of $gl(V)$ on V.

Remark: The description of the infinitesimal action as in Proposition 10 is good because it makes it clear that we have a *functor*

$$\mathrm{inf}: \mathrm{Rep}(G) \to \mathrm{Rep}(\mathfrak{g}).$$

On the other hand Theorem 12 makes it clear that the infinitesimal action of $\mathfrak{g}$ on V is defined in the expected way: An infinitesimal path through the identity

$$\mathbb{C} \supseteq D \xrightarrow{t} G$$

composes to give an infinitesimal path in $GL(V)$

$$\begin{tikzcd} & G \arrow[dr, "\phi"] & \\ D \arrow[ur, "t"] \arrow[rr] & & GL(V) \end{tikzcd}$$

$$\varepsilon \longmapsto I_n + A\varepsilon + O(\varepsilon^2),$$

and the element $[t]$ in $\mathfrak{g}$ acts "infinitesimally" on V via the matrix

$$\partial \phi \circ t / \partial z \Big|_{z=0} = A \in gl(V).$$

– XVII. Symmetric monoidality of inf_G

For G-representations V and W we have the apparent $G(\mathbb{C})$-action on $V \otimes W$ given by the expected formula $g.(v \otimes w) = (g.v) \otimes (g.w)$.

This "diagonal" action of $G(\mathbb{C})$ extends uniquely to a G-representation structure on the tensor product. In terms of comodules,

$$\rho_{V\otimes W} = (1\otimes 1\otimes \mathrm{mult}_G)(1\otimes \mathrm{swap}\otimes 1)(\rho_V\otimes\rho_W):$$

$$V\otimes W \xrightarrow{\rho's} (V\otimes W)\otimes(\mathcal{O}\otimes\mathcal{O}) \xrightarrow{1\otimes \mathrm{mult}} (V\otimes W)\otimes\mathcal{O}.$$

The standard vector space swap

$$\tau_{VW}: V\otimes W \longrightarrow W\otimes V, \quad v\otimes w \mapsto w\otimes v,$$

is an isomorphism of G-representations. Thus we have a symmetric tensor structure on $\mathrm{Rep}\, G$.

Theorem 13: The functor

$$\mathrm{inf}: \mathrm{Rep}(G) \longrightarrow \mathrm{Rep}(\mathfrak{g})$$

is symmetric monoidal.

Proof: Exercise. ∎

XVIII. Full faithfulness of inf_G

Theorem 14: For any algebraic group G with Lie algebra $\mathfrak{g} = \mathrm{Lie}\, G$, the functor

$$\mathrm{inf}_G: \mathrm{Rep}(G) \to \mathrm{Rep}(\mathfrak{g})$$

is <u>fully faithful</u> provided G is connected.

However one approaches this, there are two basic principles we need to deal with. We sketch some details

Principle I (Actions are determined in an infinitesimal neighborhood of the identity)

By restricting along the embedding $\hat{G}_1 \to G$ from the formal neighborhood around the identity we obtain a functor

res: $\mathrm{Rep}(G) \to \mathrm{Rep}(\hat{G}_1)$.

Now, rep $\hat{G}_1 = \mathrm{Corep}(\hat{\mathcal{O}})$ where

$$\hat{\mathcal{O}} = \varprojlim_n \mathcal{O}/m_G^n \quad , \quad m_G = \ker(1_G).$$

complete local ring at $1_G \in G(\mathbb{C})$.

The Hopf structure on $\mathcal{O}$ becomes a topological Hopf structure on $\hat{\mathcal{O}}$, $\quad \hat{\Delta} : \hat{\mathcal{O}} \to \hat{\mathcal{O}} \hat{\otimes} \hat{\mathcal{O}} = \varprojlim_n \mathcal{O}/m^n \otimes \mathcal{O}/m^n$

so we can consider

$$\mathrm{Rep}(\hat{G}_1) := \mathrm{Corep}(\hat{\mathcal{O}})$$

and the restriction functor is just given by "corestricting"

res: $\mathrm{Corep}(\mathcal{O}) \to \mathrm{Corep}(\hat{\mathcal{O}})$.

Since G is smooth [a fact] and connected the canonical map

$$\mathcal{O} \to \hat{\mathcal{O}}$$

is injective, hence res is fully faithful. This is to say, any representation $\phi : G \to GL(V)$ is determined by its restriction to the "infinitesimal nbd" $\phi|_{\hat{G}_1}$, and

and any linear map $f: V \to W$ is a map of G-repr if and only if f is a map of $\hat{G}_1$ repr.

Rem: Any nbd (Zariski or analytic) $1 \in \mathcal{U} \subseteq G$ contains $\hat{G}_1 \subseteq \mathcal{U} \subseteq G$. So any algebraic rep V has G-action def on any nbd $\mathcal{U} \subseteq G$, and G-linearity of any map holds iff
$$f(x \cdot v) = x \cdot f(v) \text{ for all } x \in \mathcal{U}(\mathbb{C}).$$

Principle II (The Lie algebra $\mathfrak{g}$ knows everything about the formal nbd $\hat{G}_1$)

Consider the topological dual
$$\mathcal{U}_G := \varinjlim_n (\mathcal{O}/m_G^n)^* \subseteq \mathcal{H}_G.$$
This is a subalgebra with
$$\mathfrak{g} \overset{\text{Lie}}{\hookrightarrow} \mathcal{U}_G \subseteq \mathcal{H}_G^{\text{Lie}},$$
since all $\text{ev}(x)$ for x in $\mathfrak{g}$ vanish on m^2. Now, we have
$$\mathcal{U}_G^* = \left(\varinjlim_n (\mathcal{O}/m_G^n)^*\right)^* = \varprojlim (\mathcal{O}/m^n)^{**}$$
$$= \varprojlim \mathcal{O}/m^n = \hat{\mathcal{O}},$$
so that we have an identification of categories
$$\mathrm{Rep}(\hat{G}_1) = \mathcal{U}_G\text{-mod}_{\text{findim}},$$
(Cf.
$$\mathrm{Hom}_{\mathbb{C}}(V \otimes \mathcal{U}_G, V) = \mathrm{Hom}_{\mathbb{C}}(V, \mathrm{Hom}_{\mathbb{C}}(\mathcal{U}_G, V)))$$

$= \mathrm{Hom}_{\mathbb{C}}(V, V \otimes \mathcal{U}_G^*)$

$\qquad = \mathrm{Hom}_{\mathbb{C}}(V, V \otimes \hat{\mathcal{O}}).$)

Now, the Lie embedding

$$\mathfrak{g} \hookrightarrow \mathcal{U}_G^{Lie}$$

determines an algebra map

$$\mathrm{can}: \mathcal{U}(\mathfrak{g}) \to \mathcal{U}_G^{Lie}.$$

Using smoothness of G, one can prove that in fact this map can is an isomorphism, giving

$$\begin{tikzcd} \mathrm{Rep}(G) \arrow[r, "\mathrm{inf}"] \arrow[d, "\text{full faith}"'] & \mathrm{Rep}(\mathfrak{g}) \\ \mathrm{Rep}(\hat{G}_1) = \mathrm{Corep}(\hat{\mathcal{O}}) \arrow[r, "\cong"] & \mathcal{U}(\mathfrak{g})\text{-mod}_{\text{findim}} \arrow[u, "="'] \end{tikzcd}.$$

Thus inf is fully faithful. ∎

– XIX. Integrable representations

Def^{n}: For $\mathfrak{g} = \mathrm{Lie}\, G$, we say a $\mathfrak{g}$-representation V is integrable (for G) if there exists V' in $\mathrm{Rep}(G)$ with $\mathrm{inf}(V') = V$.

In this case note that $V' = V$, and that the G-action on V is actually fixed, due to full faithfulness of $\mathrm{inf}: \mathrm{Rep}(G) \to \mathrm{Rep}(\mathfrak{g})$.

So we are simply asking if we can "integrate" the $\mathfrak{g}$-action on V to an action of the group G.

Another way of saying Thm 14 is as follows.

Theorem 15: The functor

$$\inf_G : \mathrm{Rep}(G) \longrightarrow \mathrm{Rep}(\mathfrak{g})$$

is a symmetric monoidal equivalence (in fact isomorphism) onto the full subcategory of integrable $\mathfrak{g}$-reps in $\mathrm{Rep}(\mathfrak{g})$,

$$\inf_G : \mathrm{Rep}(G) \xrightarrow{\sim} \mathrm{Rep}(\mathfrak{g})_{\mathrm{Int}} .$$

The following is also clear from the proof of Thm 14.

Corollary (to proof) 16: The subcategory of integrable reps in $\mathrm{Rep}(\mathfrak{g})$ is closed under taking subobjects and quotients.

~~XX~~. Examples

Example 1: For discrete (finite) G we have $\mathrm{Lie}(G) = 0$ so that $\mathrm{Rep}(\mathfrak{g}) = \mathrm{Vect}$, and the infinitesimal action functor

is just the forgetful functor
$$\mathrm{inf}_G : \mathrm{Rep}(G) \to \mathrm{Vect}.$$

This is faithful, but not full, and clearly not an equivalence when G is nontrivial.

Example 2: Consider $B \leq GL_n$ consisting of non-strictly upper Δ inv. matrices. Then an element
$$\tilde{A}(\varepsilon) = I_n + A\varepsilon \in \mathrm{Lie}\, GL_n$$
lies in $\mathrm{Lie}\, B \leq \mathrm{Lie}\, GL_n$ if and only if $\tilde{A}(\varepsilon)$ is upper Δ. This occurs if and only if A is upper Δ'r (with arbitrary entries in a_{ij}, $i<j$).
Hence
$$\mathrm{Lie}\, B = \mathfrak{b} \leq \mathfrak{gl}_n(\mathbb{C}).$$
Take $T \leq GL_n$ the inv. diagonal matrices. We have $\mathrm{Lie}\, T = \mathfrak{h} = \mathrm{diags} \leq \mathfrak{gl}_n(\mathbb{C})$, and there is a diagram
$$\begin{tikzcd} \mathrm{Rep}(B) \arrow[r, "\mathrm{inf}_B"] \arrow[d, "\mathrm{res}"'] & \mathrm{Rep}(\mathfrak{b}) \arrow[d, "\mathrm{res}"] \\ \mathrm{Rep}(T) \arrow[r, "\mathrm{inf}_T"'] & \mathrm{Rep}(\mathfrak{h}) \end{tikzcd}$$

by Prop 11. Since $T = \mathbb{G}_m^{\times n}$ we see an $\mathfrak{h}$-rep V is integrable if and only if $\mathfrak{h}$ acts semisimply on V and all the $h_i = e_{ii}$ act as integers on V.

Consequently, any integrable $\mathfrak{b}$-rep V must have $\mathfrak{h} \subseteq \mathfrak{g}$ acting semisimply with integral eigenvalues. One can show that in fact this integral Cartan condition characterizes integrable $\mathfrak{b}$-reps,

$$\mathrm{inf}_B : \mathrm{Rep}(B) \xrightarrow{\sim} \left\{ \begin{array}{c} \mathfrak{b}\text{-reps } V \text{ on} \\ \text{which } \mathfrak{h} \subseteq \mathfrak{b} \\ \text{acts semisimply w/} \\ \text{the } h_i = e_{ii} \text{ acting} \\ \text{as integers} \end{array} \right\} \subseteq \mathrm{Rep}(\mathfrak{b}).$$

As a general consequence of Thm 14 we have the following.

Corollary 17: Suppose G is connected w/ Lie alg $\mathfrak{g}$. If $\mathrm{Rep}(\mathfrak{g})$ is semisimple then $\mathrm{Rep}(G)$ is also semisimple.

So we see, for example that $\mathrm{Rep}(SL_n)$ is semisimple. Also for any finite quotient

$$\pi : SL_n \to SL_n/\mu = G$$

$\mathrm{Rep}\, G$ is semisimple, as $\mathrm{Lie}\, G = \mathrm{Lie}\, SL_n = \mathfrak{sl}_n(\mathbb{C})$ in this case.

Corollary 18: $\mathrm{Rep}(PGL_n)$ is semisimple.

Example 3: For $\mathrm{Rep}(SL_n)$, we've already seen that the standard representation $\mathbb{V}$ for $\mathfrak{sl}_n(\mathbb{C})$ integrates to the standard rep for SL_n, using Theorem 12.

Since $\inf : \mathrm{Rep}(SL_n) \to \mathrm{Rep}(\mathfrak{sl}_n(\mathbb{C}))$ is symmetric monoidal and has image stable under subquotients, we see

$$\{\text{Symm } \otimes\text{-subcat gen'd by } \mathbb{V} \text{ in } \mathrm{Rep}(\mathfrak{sl}_n)\}$$

$$\subseteq \text{ image of } \mathrm{Rep}(SL_n).$$

But, this $\otimes$-subcat is all of $\mathrm{Rep}(\mathfrak{sl}_n)$, so that $\inf_{SL_n}$ is <u>essentially surjective</u>, and thus an equivalence

$$\inf_{SL_n} : \mathrm{Rep}(SL_n) \xrightarrow{\sim} \mathrm{Rep}(\mathfrak{sl}_n(\mathbb{C})).$$

Example 4: We have

$$PGL_n = SL_n / \mu_n$$

where

$$\mu_n = \left\{ \begin{bmatrix} \zeta & & 0 \\ & \ddots & \\ 0 & & \zeta \end{bmatrix} : \zeta \text{ an } n\text{-th root of } 1 \right\}.$$

Thus $\mathrm{Rep}\, PGL_n \subseteq \mathrm{Rep}\, SL_n$ identifies w/ the full subcat of reps on which $\mu_n \curvearrowright V$ trivially.

Now, $H(\zeta) = \mathrm{diag}\{\zeta, \dots \zeta\} = H_1(\zeta) H_2(\zeta^2) \cdots H_{n-1}(\zeta^{n-1})$

for the root subgroups

$$H_i : \mathbb{C}^\times \to SL_n(\mathbb{C}), \; z \mapsto \begin{bmatrix} 1 & & & & & & 0 \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & z & & & \\ & & & & z^{-1} & & \\ & & & & & 1 & \\ & & & & & & \ddots \\ 0 & & & & & & 1 \end{bmatrix}.$$

i-th pos

Thus for a weight $\lambda = c_1\omega_1 \cdots + c_{n-1}\omega_{n-1} \in P$
$H(\zeta) \cdot v = \zeta^{(\sum_k^i k c_k)} \cdot v$ for $v \in V_\lambda$,
and we find V is in $\mathrm{Rep}(PGL_n) \subseteq \mathrm{Rep}(SL_n)$
if and only if V is graded by the sublattice

$$X_{PGL_n} := \{ \sum_k^i c_k \cdot \omega_k : \sum_k^i k \cdot c_k \text{ is } n\cdot\mathbb{Z} \} \subseteq P.$$

Now for the weights $\alpha_i = -\omega_{i-1} + 2\omega_i - \omega_{i+1}$ we have $-(i-1) + 2i - (i+1) = 0$. Hence we have a seq. of inclusions

$$\mathbb{Z}\cdot\Phi = \text{Root lattice} \subseteq X_{PGL_n} \subseteq P$$

Since P has a $\mathbb{Z}$-basis $\{\omega_n, \alpha_2, \ldots, \alpha_{n-1}\}$
with

$$\omega_n = \frac{1}{n}(\alpha_1 + 2\alpha_2 + \cdots + (n-1)\alpha_{n-1})$$

and Root lattice has basis

$$\{\alpha_1 + 2\alpha_2 + \cdots + (n-1)\alpha_{n-1}, \alpha_2, \ldots, \alpha_{n-1}\}$$

we have $P/\text{Root Lattice} \cong \mathbb{Z}/n\mathbb{Z}$ and

By construction X_{PGL_n} is the kernel of the map

$$[1\ 2\ \cdots\ (n-1)] : P \to \mathbb{Z}/n\mathbb{Z}$$

so that

$$P/X_{PGL_n} \cong \mathbb{Z}/n\mathbb{Z} \text{ as well.}$$

Since $|P/X_{PGL_n}| = |P/\text{RootLattice}|$ we find the inclusion

$$\text{RootLattice} \subseteq X_{PGL_n}$$

is an equality.

Proposition 19: For $\mathfrak{pgl}_n(\mathbb{C}) = \mathfrak{sl}_n(\mathbb{C}) = \mathrm{Lie}\, PGL_n$, the infinitesimal action functor

$$\mathrm{inf}_{PGL_n} : \mathrm{Rep}(PGL_n) \to \mathrm{Rep}(\mathfrak{pgl}_n(\mathbb{C}))$$

restricts to an equivalence

$$\mathrm{inf}_{PGL_n} : \mathrm{Rep}(PGL_n) \xrightarrow{\sim} \left\{ \begin{array}{l} \mathfrak{pgl}_n(\mathbb{C}) = \mathfrak{sl}_n(\mathbb{C})\text{-reps} \\ \text{whose } P\text{-grading is} \\ \text{supported on the sublattice} \\ \text{spanned by the roots} \end{array} \right\}.$$

This is the $\otimes$-subcat gen'd by the adjoint representation.

Other classical examples

$$\mathrm{inf}_G : \mathrm{Rep}(G) \xrightarrow{\sim} \mathrm{Rep}(\mathfrak{g})$$

for $G = Sp_{2n}$, SO_{2n} and

$$\mathrm{inf}_{GL_n} : \mathrm{Rep}(GL_n) \xrightarrow{\sim} \left\{ \begin{array}{l} \mathfrak{gl}_n(\mathbb{C})\text{-reps on which} \\ \text{the identity matrix } I_n \text{ acts} \\ \text{semisimply w/ integral eigen-} \\ \text{values} \end{array} \right\}.$$

Since $\mathfrak{gl}_n(\mathbb{C}) = \mathfrak{sl}_n(\mathbb{C}) \oplus \mathbb{C} \cdot I_n$ we observe now that $\mathrm{Rep}(GL_n)$ is semisimple.