- Example [Abelian Lie alg's]

For any vector space V we can endow V w/ the trivial bracket $[\,,\,]_{triv} : V \otimes V \to V$ def by $[v,w] = 0$ at all $v, w \in V$. The Jacobi identity holds trivially $(0=0)$ so that the pairing

$$(V, [\,,\,]_{triv}) \text{ forms a Lie algebra.}$$

Def^n: A Lie algebra $\mathfrak{h}$ is called abelian if the bracket operation on $\mathfrak{h}$ is identically 0, i.e. if

$$\mathfrak{h} = (V, [\,,\,]_{triv})$$

for a vector space V.

Sub-example: The Lie alg A^{Lie} assoc to an alg A is abelian iff A is commutative.

- Example [$\mathfrak{gl}(V)$].

For any vector space V we have the algebra of linear endomorphisms $End_{\mathbb{C}}(V)$.

Def^n: The general linear Lie alg for V is

$$\mathfrak{gl}(V) := End(V)^{Lie}$$

$$= \left\{ \begin{matrix} \text{linear endos } A: V \to V \text{ w/ commutator} \\ \text{bracket } [A,B] = AB - BA \end{matrix} \right\}.$$

In the particular case $V = \mathbb{C}^n$ we write

$$\mathfrak{gl}_n(\mathbb{C}) := \mathfrak{gl}(\mathbb{C}^n) = M_n(\mathbb{C})^{Lie}.$$

- Lie subalgebras and ideals

Def^n: A Lie subalgebra is a Lie alg $\mathfrak{g}$ is a vector subspace $\mathfrak{f} \subseteq \mathfrak{g}$ for which

$$[x,y] \in \mathfrak{f} \text{ whenever } x, y \in \mathfrak{f}.$$

An ideal is ~~$\mathfrak{g}$~~ is a subspace $I \subseteq \mathfrak{g}$ which satisfies $[x,z] \in I$ whenever one of x or z is in I.

Def$^{\underline{n}}$: A homomorphism of Lie alg's $\phi: \mathfrak{g} \to \mathfrak{g}'$ is a linear map which satisfies

$$\phi([x,y]) = [\phi(x), \phi(y)]' \quad \text{at all } x,y \in \mathfrak{g}.$$
[also iso...]

Lemma 4: a) Any Lie subalg $\mathfrak{f} \subseteq \mathfrak{g}$ is itself a Lie alg, w/ bracket inherited from that of $\mathfrak{g}$.

b) For any ideal $I \subseteq \mathfrak{g}$, I is a Lie subalg and the quotient $\mathfrak{g}/I$ inherits a unique Lie alg structure so that the quotient map $\pi: \mathfrak{g} \to \mathfrak{g}/I$ is a Lie alg homomorphism.

Proof: Exercise. ∎

Lemma 5: The kernel $\ker\phi \subseteq \mathfrak{g}$ of any Lie alg homomorphism $\phi: \mathfrak{g} \to \mathfrak{g}'$ is an ideal in $\mathfrak{g}$.

Example [$\mathfrak{sl}_n(\mathbb{C})$]: Let $\mathbb{C}$ = 1-dim abelian Lie alg. Then the trace function

$$tr: \mathfrak{gl}_n(\mathbb{C}) \to \mathbb{C}, \quad A \mapsto tr(A)$$

satisfies $\quad tr([A,B]) = 0 = [trA, trB]$.

Hence the trace function is a Lie alg homomorphism, and the kernel

$$\mathfrak{sl}_n(\mathbb{C}) := \ker(tr) = \left\{ \begin{array}{c} n\times n \text{ traceless matrices} \\ \text{w/ commutator} \\ \text{bracket} \end{array} \right\}.$$

We have
$$\dim \mathfrak{gl}_n(\mathbb{C}) = n^2$$
$$\dim \mathfrak{sl}_n(\mathbb{C}) = n^2 - 1.$$

In the particular case $n=2$, $\dim \mathfrak{sl}_2(\mathbb{C}) = 3$, and we have the spanning set

$$\mathfrak{sl}_2(\mathbb{C}) = \mathrm{span}_{\mathbb{C}}\left\{ e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}.$$

The Lie bracket is specified by the formulas:

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h.$$

$\mathfrak{sl}_2(\mathbb{C})$ is a very special individual.

- Lie alg in low dim.

Dim 1: In dim 1, the only Lie alg $\mathfrak{h} = \mathbb{C}\cdot x$ is the abelian one. This follows from antisymmetry

$$[a\cdot x, b\cdot x] = a\cdot b\,[x,x] = 0.$$

Dim 2: In dim 2, have $\mathfrak{h} = \mathbb{C}x \oplus \mathbb{C}y$

$$[x,x] = [y,y] = 0, \qquad [x,y] = ax + by.$$

If $a \neq 0$ then replace x w/ $x + \frac{b}{a}y$ to get alt expression $[x,y] = a\cdot x$. Then

$$[y,[x,y]] = -a^2x$$
$$= [[y,x],y] + [x,[y,y]] = a^2x,$$

giving $0 = 2a^2x$, a contradiction.

Conclusion: The only 2-dim Lie alg, up to isomorphism, is the abelian one.

Dim 3: In dim 3 we have the non-abelian Lie alg

$$\mathfrak{n}_3 := \left\{ \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{bmatrix} : a_{ij} \in \mathbb{C} \right\} \leq \mathfrak{gl}_3(\mathbb{C}).$$

Precisely $\mathfrak{n}_3 = A^{Lie}$ for the (non-unital) alg of strictly upper Δ matrices.

Exercise: Prove that any 3-dimensional Lie alg $\mathfrak{g}$ is either abelian, or isomorphic to $\mathfrak{n}_3$.

- Representations of Lie algebras

Def: A representation of a Lie alg $\mathfrak{g}$ is a vector space V equipped w/ an linear map

$$\cdot : \mathfrak{g} \otimes V \to V$$

satisfying $[x,y]\cdot v = x\cdot(y\cdot v) - y\cdot(x\cdot v)$.

Lemma: For any $\mathfrak{g}$-rep V, the map

$$\rho_V : \mathfrak{g} \to \mathfrak{gl}(V), \quad x \mapsto (v \mapsto x \cdot v),$$

is a Lie alg homomorphism, and any Lie alg hom $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ defines a $\mathfrak{g}$-rep structure on V by $x \cdot v := \rho(x) \cdot v$.

Proof: Exercise. ∎

Example [Adjoint rep] For any Lie alg $\mathfrak{g}$, the adjoint action $x \cdot y := [x, y]$ gives $\mathfrak{g}$ the structure of a $\mathfrak{g}$-representation. Indeed, the Jacobi identity is equiv to the requisite formula $[x, y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z)$. This is the adjoint representation.

Example [The standard rep] For any vector space V, $\mathfrak{gl}(V)$ acts on V "tautologically",

$$x \cdot v = x(v) \leftarrow \text{viewed as linear endo.}$$

This gives V the structure of a $\mathfrak{gl}(V)$-representation, and we call it the "standard representation".

Recall, we have some examples of Lie algs

$\mathfrak{gl}(V)$, $\mathfrak{gl}_n(\mathbb{C})$

$\mathfrak{sl}(V)$, $\mathfrak{sl}_n(\mathbb{C})$, $\mathfrak{sl}_2(\mathbb{C}) = \left(\begin{array}{l} \text{span}\{e, f, h\} \\ [h, e] = 2e \\ [h, f] = -2f \\ [e, f] = h. \end{array} \right.$

A $\mathfrak{g}$-representation is a vect space V equipped w/ an "action" of $\mathfrak{g}$, $\cdot : \mathfrak{g} \otimes V \to V$, which satisfies

$$[x, y] \cdot v = x \cdot y \cdot v - y \cdot x \cdot v.$$

Any rep specifies, and is specified by, its corresponding map to $\mathfrak{gl}(V)$, $\rho_V : \mathfrak{g} \to \mathfrak{gl}(V)$, $\rho_V(x) = x \cdot -$.

Ex [Adjoint rep] Any Lie alg $\mathfrak{g}$ acts on itself via the adjoint action $\cdot_{adj} : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$.

$$x \cdot_{adj} y = [x, y].$$

The requisite eq

$$[x, y] z = x \cdot y \cdot z - y \cdot x \cdot z$$

$$[[x, y] z] = [x [y, z]] - [y [x, z]]$$

is equiv to the Jacobi identity

$$[x [y, z]] = [[x, y] z] + [y [x, z]],$$

so that the adj rep $(\mathfrak{g}, \cdot_{adj})$ is seen to be a $\mathfrak{g}$-representation.

Re-Def: $\mathfrak{g}$ is simple if $\mathfrak{g}$ has no proper nonzero ideals, and $\mathfrak{g}$ is not the 1-dim abelian Lie alg.

Observation 1: If $\mathfrak{g}$ is simple, then the adj rep map $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is an inj Lie alg hom.

Proof: We already know it's a Lie alg hom. Simplicity of $\mathfrak{g}$ then says $\ker(ad) = 0$ or $\ker(ad) = \mathfrak{g}$. The latter case occurs iff $\mathfrak{g}$ is abelian, which contradicts simplicity of $\mathfrak{g}$. Hence $\ker = 0$. ∎

Aug 28 (2)

Short term plan:

- Provide complete analysis of rep(sl_2). (3-4 classes)
- Discuss sl_n.
- Begin w/ general theory from Humphreys.

- Some category stuff

Def^n: For any finite-dimensional Lie alg $\mathfrak{g}$ we let rep($\mathfrak{g}$) denote the category of $\mathfrak{g}$-representations. The objects are $\mathfrak{g}$-reps, and morphisms are homomorphisms of $\mathfrak{g}$-representations, i.e. linear maps $\phi: V \to W$ which satisfy

$$\phi(x \cdot v) = x \cdot \phi(v) \text{ for all } x \in \mathfrak{g}, v \in V.$$

A subrepresentation $V' \subseteq V$ is a linear subspace which is stable under the action of $\mathfrak{g}$.

Note that V' inherits a $\mathfrak{g}$-action, or $\mathfrak{g}$-rep structure, in this case. Call a $\mathfrak{g}$-rep <u>simple</u> if it has no proper, nonzero subrepresentations.

Example: The $\mathfrak{g}$-subreps in the adj rep are precisely the ideals $I \subseteq \mathfrak{g}$. (Hence $\mathfrak{g}$ is simple if and only if $\mathfrak{g}$ nonabelian w/ simple adjoint representation.

Lemma 2: If $\phi: V \to W$ is a homomorph of $\mathfrak{g}$-reps then

a) The kernel $\ker(\phi) \subseteq V$ is a subrepresentation in V.

b) The image $\phi(V) \subseteq W$ is a subrep in W.

c) The quotient $W/\phi(V)$ inherits a unique $\mathfrak{g}$-rep structure so that the quotient map $\pi: W \to W/\phi(V)$ is a map of $\mathfrak{g}$-reps.

d) ϕ is an isomorphism iff $\ker(\phi) = 0$ and $\phi(V) = W$

(3)

Proof: The proof just follows by standard observations. For example, (a) if $v \in \ker(\phi)$ then $\phi(x \cdot v) = x \cdot \phi(v) = x \cdot 0 = 0$. Hence the kernel is stable under the action of $\mathfrak{g}$, and thus a $\mathfrak{g}$-subrep. For (c) we have the right exact seq of vect spaces $V \to W \to W' \to 0$ for $W' = W/\phi(U)$ and apply the right exact fun $\mathfrak{g} \otimes -$ to get

$$\mathfrak{g} \otimes V \to \mathfrak{g} \otimes W \to \mathfrak{g} \otimes W' \to 0$$

and by univ prop of cokernel ∃ unique action $\mathfrak{g} \otimes W' \to W'$ which completes the diag

$$\begin{tikzcd} \mathfrak{g} \otimes V \arrow[r] \arrow[d, "\cdot"] & \mathfrak{g} \otimes W \arrow[r] \arrow[d, "\cdot"] & \mathfrak{g} \otimes W' \arrow[r] \arrow[d, dashed, "\exists !"] & 0 \\ V \arrow[r] & W \arrow[r] & W' \arrow[r] & 0 \end{tikzcd}$$

This action is given on classes by $x \cdot \bar{w} := \overline{x \cdot w}$, and inherits the identities $[x, y] \cdot \bar{w} = x \cdot y \cdot \bar{w} - y \cdot x \cdot \bar{w}$ from the corresp. id on W.

(d) For the linear inverse ϕ^{-1} we have

$$\phi^{-1}(x \cdot w) = \phi^{-1}(\phi(x \cdot \phi^{-1}(w))) = x \cdot \phi^{-1}(v),$$

so that ϕ^{-1} seen to be a map of $\mathfrak{g}$-reps. ∎

Also easy to check the following:

- Any $\mathbb{C}$-scaling $c \cdot \phi$ of a $\mathfrak{g}$-rep hom $\phi: V \to W$ is again a map of $\mathfrak{g}$-rep, as is any sum $\phi + \phi'$ of $\mathfrak{g}$-rep homs. Hence

$$\mathrm{Hom}_{\mathfrak{g}}(V, W) := \mathrm{Hom}_{\mathrm{rep}(\mathfrak{g})}(V, W)$$

is a vector subspace in $\mathrm{Hom}_{\mathbb{C}}(V, W)$.

- The sum $V_1 \oplus V_2$ inherits a unique $\mathfrak{g}$-rep structure so that the two inclusions $V_i \to V_1 \oplus V_2$ are maps of $\mathfrak{g}$-reps. Furthermore, this sum is both a product and coproduct in $\mathrm{rep}(\mathfrak{g})$ (look it up).

Taken together we conclude that

rep$(\mathfrak{g})$ is a $\mathbb{C}$-linear abelian category.

can take linear combos of morphisms — has kernels and cokernels

Def^n: Call an abelian cat $\mathcal{C}$ Artinian if every seq of subobjects $V_0 \supseteq V_1 \supseteq V_2 \supseteq \cdots$ stabilizes.

Call $\mathcal{C}$ semisimple if every exact sequence

$$0 \to V \xrightarrow{\phi} W \xrightarrow{\phi'} V' \to 0$$

splits, i.e. if there exists $\psi: W \to V$ satisfying $\psi\phi = \mathrm{id}_V$ or $\psi': V' \to W$ w/ $\phi'\psi' = \mathrm{id}_{V'}$.

Observe that $\mathcal{C} = \mathrm{rep}(\mathfrak{g})$ is Artinian.

Indeed, since each obj is fin.-dim / $\mathbb{C}$ and desc. seq of subreps must stabilize ~~for~~ dim reasons.

- Aside: Lengths and JH series.

Goal: rep$(\mathfrak{sl}_2)$ is semisimple.

Let $\mathcal{C}$ be an Artinian cat, and V be an object. A Jordan-Hölder series for V is a seq of proper submodules

$$0 = V_n \subsetneq V_{n-1} \subsetneq \cdots \subsetneq V_0 = V \quad (*)$$

for which each quotient V_i / V_{i+1} is a nonzero simple object in $\mathcal{C}$. (Here simple means cont. no proper nonzero subobj.) The length of such a series $(*)$ is n.

Theorem 3 (JH series) For any two JH series

$$0 = V'_m \subsetneq V'_{m-1} \subsetneq \cdots \subsetneq V'_0 = V$$

$$0 = V_n \subsetneq V_{n-1} \subsetneq \cdots \subsetneq V_0 = V$$

we have $n = m$, and for some permutation $\sigma \in S_n$ there are isoms

$$V_i / V_{i+1} \cong V'_{\sigma(i)} / V'_{\sigma(i)+1}$$

in $\mathcal{C}$.

Proof: Exercise. ∎

Def^n: For any object V in an Artinian cat $\mathcal{C}$, the length of V is the length n of any JH series

$$0 = V_n \leq V_{n-1} \leq \cdots \leq V_0 = V.$$

The composition factors are, up to isomorphism, the simples which appear in the collection $\{ V_i / V_{i+1} : 0 \leq i \leq n-1 \}$.

Proposition 4: For an Artinian category $\mathcal{C}$ the following are equivalent.

a) $\mathcal{C}$ is semisimple.

c) Every obj V decomp. as a sum of simples $V = \bigoplus_{i=1}^m L_i$.

b) Any extension $0 \to V \to W \to V' \to 0$ in which V and V' are simple splits.

Sketch Proof: (a) $\Rightarrow$ (b) is trivial. Assume now that (a) holds. Then any seq

$$0 \to V \to W \to V' \to 0 \qquad (*)$$

in which

$$\text{length}(W) = \text{length}(V) + \text{length}(V') \leq 2$$

is split. Suppose now that a seq $(*)$ is s.t. $\text{length}(W) = n+1$ and that all seq w/ middle term of length $\leq n$ split. We can assume $n > 2$, so that one of $\text{length}(V)$ or $\text{length}(V') > 1$. Assume first that $\text{length}(V') > 1$, and consider an exact sequence

$$0 \to V_1' \to V' \to V_0' \to 0$$

with V_0' simple.

By taking fiber products we obtain an exact seq

$$0 \to V \to W_1 = W \times_{V'} V_1' \to V_1' \to 0,$$

which is split since $\text{length}(W_1) = \text{length}(V) + \text{length}(V_1') = n$.

So we have a splitting

$$W_1 \cong V \oplus V_1'.$$

Take now $W_0 = W / \mathrm{im}\, V_1'$, via the splitting map $V_1' \xrightarrow{\sim} W_1 \hookrightarrow W$, and note the exact seq $0 \to V \to W_0 \to V_0' \to 0$ and a diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
V \arrow[r, "="] \arrow[d] & V \arrow[d] \\
W \arrow[r] \arrow[d] & W_0 \arrow[d] \\
V' \arrow[r] \arrow[d] & V_0' \arrow[d] \\
0 & 0
\end{tikzcd}$$

and the induced map to the fiber product

$$W \to V' \times_{V_0} W_0$$

is an isomorphism. So we see that the projection $W \to V'$ is split if the projection $W_0 \to V_0'$ is split. However, the latter splitting occurs by our induction hypothesis, so that the seq $0 \to V \to W \to V' \to 0$ is in fact split.

The argument in the case $\mathrm{length}(V) > 1$ is similar. ∎

Erratta to Aug 28:

In order for all obj. in an abelian cat $\mathbb{C}$ to have comp. series $\mathbb{C}$ must be both Artinian and Noetherian. So, in the statement of [Prop 4, Aug 28], which char semisimpleness via ext. of simples, we should replace "Let $\mathbb{C}$ be Artinian" with "Let $\mathbb{C}$ be Artinian and Noetherian". It is in the Art + Noeth setting that all objects have specified length and composition factors. We note that all familiar Art. cats are already Noeth. as well:

Ex 1: The cat Vect of finite-dim vect spaces is both Art. and Noeth.

Ex 2: The cat $A\text{-mod}_{fin}$ of fin-dim modules over any $\mathbb{C}$-alg A is both Art and Noeth.

Ex 3: The cat $R\text{-mod}_{fg}$ of fin. gen'd mods over an Artinian ring R is both Art. and Noeth.

Ex 4: The cat $\text{rep}(\mathfrak{g})$ of fin-dim $\mathfrak{g}$-rep's for any Lie alg $\mathfrak{g}$ is both Art. and Noeth.

Anti-Ex 5: The opposite cat $(\mathbb{C}[x]\text{-mod}_{fg})^{op}$ is Artinian but not Noeth.

These fin. constraints hold for Examples 1, 2, 4 for simple dimension reasons.

- sl_2-reps pt I: weights

$[h,e]=2e$
$[h,f]=-2f$
$[e,f]=h$.

Let V be a fin. dim sl_2-rep, V decomposes into generalized wt. spaces for the actn. of h

$$V=\bigoplus_{i=1}^{n} V_{\lambda_i}^{gen},$$

where each λ_i is a complex scalar and

$$V_{\lambda_i}=\ker\left((h-\lambda_i\cdot id_V)^{>>0}:V\to V\right).$$

This is clear from a consideration of the Jord. norm. form. of the endo

$$h|_V=\begin{bmatrix} \underbrace{\begin{matrix}\lambda_1 & * & * \\ 0 & \ddots & * \\ & & \lambda_1\end{matrix}}_{V_{\lambda_1}} & & 0 \\ & \ddots & \\ 0 & & \underbrace{\begin{matrix}\lambda_n & * & * \\ 0 & \ddots & * \\ & & \lambda_n\end{matrix}}_{V_{\lambda_n}}\end{bmatrix}$$

Def^n: A wt vector in V is an eigenvector $v\in V$ for the actn. of h, and the assoc. wt. $\lambda=wt(v)$ is the unique scalar so that $h\cdot v=\lambda\cdot v$.

We say a wt vector $v\in V$ is a highest wt vector if $e\cdot v=0$. Define lowest wt. vectors... $f\cdot v=0$.

Given a scalar $\lambda\in\mathbb{C}$, the assoc. wt. space in V is the subspace $V_\lambda\leq V$ of all λ-eigen vectors in V, for the actn. of h.

We say V is <u>weight graded</u> if $V=\bigoplus_{i=1}^{n}V_{\lambda_i}$ for scalars λ_i.

Ex. For the adjoint rep $V=sl_2$ we have

$$V=V_{-2}\oplus V_0\oplus V_2$$

w/ each V_i of dim 1, and $e\in V$ is the unique highest wt vector, up to scaling.

Ex: For $sl_2(\mathbb{C})=sl_2(\mathbb{C}^2)$ we have the standard representation

$\mathbb{V} = \mathbb{C}^2$ w/ e, f, h acting via

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ resp. Then

$\mathbb{V} = \mathbb{V}_{-1} \oplus \mathbb{V}_1$ w/ highest

wt vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathbb{V}$.

[Rem: For $\mathfrak{sl}_n(\mathbb{C})$ we always have the standard

rep $\mathbb{V} = \mathbb{C}^n$ w/ nat. action of $\mathfrak{sl}_n(\mathbb{C}) \subseteq \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$]

- Existence of highest wt vectors ← Trivial rep

Lemma 1: If V is a $\mathfrak{sl}_2$-rep and

v is a wt vector of wt λ, then the following hold:

a) $e \cdot v \in V_{\lambda+2}$.

b) $h \cdot v \in V_\lambda$.

c) $f \cdot v \in V_{\lambda-2}$.

Proof: We have directly

a) $h \cdot (e \cdot v) \overset{\text{rel Jacobi}}{=} [h, e] v + e \cdot (h \cdot v)$

$= 2 e \cdot v + \lambda e \cdot v = (\lambda+2) e \cdot v$

b) $h \cdot v = \lambda \cdot v \in V_\lambda$

c) $h \cdot (f \cdot v) = [h, f] \cdot v + f \cdot (h \cdot v)$

$= -2 f \cdot v + \lambda f \cdot v = (\lambda - 2) f \cdot v$.

We're done. ∎

Proposition 2: Any nonzero $\mathfrak{sl}_2$-rep V contains

a highest wt vector $v \in V$, and the subspace

$$\bigoplus_{i=1}^n V_{\lambda_i} \subseteq V$$

spanned by wt vectors in V is a nontrivial $\mathfrak{sl}_2$-subrep.

Proof: Let $V' = \bigoplus_{i=1}^m V_{\lambda_i}$ be

the span of the wt vectors in V. By considering

the Jord. norm. form for h it is clear that V

has some nonzero wt. vector $v \in V'$, since $\dim V > 0$ by

hypothesis. Take λ w/ $v \in V_\lambda$.

By Lemma 1 $e^n \cdot v \in V_{\lambda+2n}$, and by fin.-dim'ness $V_{\lambda+2n} = 0$ for n large n. So there exists some minimal n_0 w/ $e^{n_0} \cdot v \neq 0$ and $e^{n_0+1} \cdot v = 0$. $e^{n_0} \cdot v$ is therefore a highest wt. vector in V.

The latter claim, that $V' \subseteq V$ is a nontrivial subrep follows by Lemma 1. ∎

Corollary 3: Every simple $\mathfrak{sl}_2$-representation V is weight graded,

$$V = \bigoplus_{i=1}^{n} V_{\lambda_i}.$$

— Constraining weights

Structure Thm for $\mathfrak{sl}_2$-reps: Let V be a simple $\mathfrak{sl}_2$-rep. Then

a) V has a unique highest wt. vector v, up to scaling.

b) The highest wt vector v has non-neg integral (!) wt $\lambda \in \mathbb{Z}_{\geq 0}$.

c) The nonzero wt spaces in V are precisely

$$V_{\lambda-2m} \text{ for } 0 \leq m \leq \lambda.$$

d) For each $0 \leq m \leq \lambda$, $V_{\lambda-2m}$ is 1-dim and spanned by f^m.

We decompose the proof into a seq. of Lemmas and their consequences.

Lemma 4: If $v \in V$ is a highest weight vector of weight λ then, for each $m \geq 0$,

$$e \cdot (f^m \cdot v) = m(\lambda - m + 1) f^{m-1} \cdot v$$

and

$$e^m \cdot (f^m \cdot v) = \left[\prod_{k=1}^{m} k(\lambda - k + 1)\right] \cdot v.$$

—

Proof: We have

$$
\begin{aligned}
e \cdot f^m \cdot v &= [e, f^m] \cdot v \\
&= \sum_{i=0}^{m-1} f^{m-i-1} [e,f] f^i \cdot v \\
&= \sum_{i=0}^{m-1} f^{m-i-1} h \cdot f^i \cdot v \\
&= \sum_{i=0}^{m-1} (\lambda - 2i) \cdot f^{m-1} \cdot v \\
&= \left(m\lambda - 2\left(\sum_{i=0}^{m-1} i\right)\right) \cdot f^{m-1} \cdot v \\
&= m(\lambda - m + 1) \cdot f^{m-1} \cdot v.
\end{aligned}
$$

The second eq is an immediate consequence of the first. ∎

Corollary 5: If V is a nonzero, fin-dim $\mathfrak{sl}_2$-rep, then for any highest wt. vector $v \in V$, $wt(v) \in \mathbb{Z}_{\geq 0}$.

Proof: Take $\lambda = wt(v)$. Since V is fin dim $V_{\lambda - 2m}$ vanishes for $m \gg 0$. Hence $f^m \cdot v = 0$ for $m \gg 0$, and by Lemma 4 we see

$$k \cdot (\lambda - k + 1) = 0 \quad \text{for some } k > 0.$$

$\Rightarrow \lambda = k - 1$ for some $k > 0$

$\Rightarrow \lambda$ is nonnegative integral.

Proposition 6: If V is a fin-dim $\mathfrak{sl}_2$-rep. w/ highest wt vector v of wt $\lambda \geq 0$. Then

$$f^m \cdot v = 0 \text{ if and only if } m > \lambda,$$

and the vectors $\{v, f \cdot v, \ldots, f^\lambda \cdot v\}$ span a simple $\mathfrak{sl}_2$ subrep $L(\lambda) \subseteq V$ which has unique highest wt. vector v, up to scaling, and is of dim $\dim_{\mathbb{C}} L(\lambda) = \lambda + 1$.

(5)

Proof: By the formula from Lemma 4,

$e^{\lambda} f^{\lambda} \cdot v = \kappa \cdot v$ for a nonzero scalar κ,

so that $f^m \cdot v \neq 0$ whenever $m \leq \lambda$.

At $\lambda + 1$ we have

$$e \cdot f^{\lambda+1} \cdot v = (\lambda+1)(\lambda-\lambda) \cdot f^{\lambda} \cdot v = 0$$

so that either $f^{\lambda+1} \cdot v = 0$ or $f^{\lambda+1} \cdot v$ is a highest wt vector of wt

$$\lambda - 2(\lambda+1) = -\lambda - 2 < 0.$$

By Corollary 5 $\nexists$ highest wt vectors of negative wt in V, so that $f^{\lambda+1} \cdot v = 0$ and all $f^m \cdot v = 0$ when $m > \lambda$.

The fact that $L(\lambda)$ is a subrep, i.e. is closed under the actions of e, f, and h, is immediate from Lemma 4. For simplicity, any nonzero subrep $L \leq L(\lambda)$ has a highest wt vector $w \in L$, which is therefore a highest wt vector in $L(\lambda)$. But the only highest wt vector in $L(\lambda)$ is v, up to scaling, so that

$v = c \cdot w$ for some scalar c, $v \in L$, and hence $L(\lambda) = \mathrm{Span}\{f^m \cdot v\} \leq L$, so that $L = L(\lambda)$. $\blacksquare$

Corollary 7: Any simple $\mathfrak{sl}_2$-rep L has a unique highest wt vector v, up to scaling, $\lambda = \mathrm{wt}(v)$ is a non-negative integer,

$$L = \mathrm{Span}_{\mathbb{C}}\{f^m \cdot v : 0 \leq m \leq \lambda\},$$

and

$$\dim L = \lambda + 1.$$

Def^n: For any simple $\mathfrak{sl}_2$-rep L, w/ highest wt vector of wt $\lambda \geq 0$, we say L is a simple of highest wt λ.

- Uniqueness of highest wt simples.

Proposition 8: For simple $\mathfrak{sl}_2$-reps L and L' with highest wt vectors v and v' of

$$wt(v) = \lambda = wt(v'),$$

there exists a unique isomorphism of $\mathfrak{sl}_2$-reps

$$\phi : L \longrightarrow L'$$

with $\phi(v) = v'$.

On the other hand, if L and L' have distinct highest wts then L and L' are not isom as $\mathfrak{sl}_2$-reps.

Suppose $\lambda = wt(v) = wt(v')$.

Proof: We have $L = \mathrm{Span}\{v, fv, \ldots, f^{\lambda} v\}$ and $L' = \mathrm{Span}\{v', f v', \ldots, f^{\lambda} \cdot v'\}$ so that, by the action formulas of Lemma 4, the unique linear map

$$\phi : L \to L' \text{ w/ } \phi(f^{n} \cdot v) = f^{n} \cdot v'$$

provides the desired isomorphism. ∎

Example (adj rep): The adjoint rep

$$V_{adj} = V_{-2} \oplus V_0 \oplus V_2$$

is the unique simple of highest wt 2, $V_{adj} = L(2)$

Example (standard rep): The standard rep

$$\mathbb{V} = \mathbb{V}_{-1} \oplus \mathbb{V}_1$$

is the unique simple of highest wt 1, $\mathbb{V} = L(1)$.

Example (trivial rep): The trivial rep $\mathbb{C}$ is the unique simple of highest wt. 0, $\mathbb{C} = L(0)$.

- Aside: Tensor products of $\mathfrak{g}$-reps.

-

Lemma 9: Let $\mathfrak{g}$ be an arbitrary Lie algebra. For any two $\mathfrak{g}$-reps V and W the tensor product $V\otimes W = V\otimes_{\mathbb{C}} W$ admits a unique $\mathfrak{g}$-rep structure under the action

$$x\cdot(v\otimes w) := (x\cdot v)\otimes w + v\otimes(x\cdot w).$$

Proof: For each $x\in\mathfrak{g}$ we have the endos $x_V : V\to V$ and $x_W : W\to W$ so that we have the assoc. linear endo

$$x_V\otimes id_W + id_V\otimes x_W : V\otimes_{\mathbb{C}} W \to V\otimes_{\mathbb{C}} W,$$

via naturality of the tensor product. We claim that the assoc. linear map

$$\phi_{V\otimes W}: \mathfrak{g}\to \mathfrak{gl}(V\otimes_{\mathbb{C}} W),\ \phi_{V\otimes W} = \phi_V\otimes id + id\otimes\phi_W,$$

defines a $\mathfrak{g}$-rep structure on the tensor product.

We check relative Jacobi directly on monomials in $V\otimes_{\mathbb{C}} W$,

$$[x,y]\cdot(v\otimes w) = [x,y]\cdot v\otimes w + v\otimes[x,y]\cdot w$$
$$= xyv\otimes w + v\otimes xyw - yx\cdot v\otimes w - v\otimes yxw$$
$$= xy\,v\otimes w + xv\otimes yw + y\cdot v\otimes x\cdot w + v\otimes xyw$$
$$-yxv\otimes w - yv\otimes xw - xv\otimes yw - v\otimes yxw$$
$$= x\cdot y\cdot(v\otimes w) - yx(v\otimes w).$$

∎

Example: Let L and L' be simple $\mathfrak{sl}_2$-reps of highest wts λ and λ' resp. Let $v\in L$ and $v'\in L'$ be highest wt vectors. Then $v\otimes v'$ is a highest wt vector in $L\otimes L'$ and

$$h\cdot(v\otimes v') = hv\otimes v' + v\otimes h\cdot v'$$
$$= (\lambda+\lambda')(v\otimes v').$$

So $L\otimes L'$ contains a highest wt vector of wt $\lambda+\lambda'$.

- Existence and uniqueness for simple $\mathfrak{sl}_2$-reps

Theorem 10: For each $\lambda\geq 0$, there exists a unique simple $\mathfrak{sl}_2$-representation $L(\lambda)$ of highest wt. λ. Furthermore, for any highest wt v, we have $f^m\cdot v\neq 0$ for all $m\leq\lambda$ and

$$L(\lambda) = \operatorname{span}_{\mathbb{C}}\{v, f\cdot v, \ldots, f^{\lambda}v\}. \quad (*)$$

Proof: Uniqueness was covered in Proposition 8, and the structure $(*)$ follows by Corollary 7. So we need only establish existence.

At low wts we have

$\mathcal{L}(0)$ = trivial rep, $\mathcal{L}(1)$ = standard rep

$\mathcal{L}(2)$ = adjoint rep.

Now for all $\lambda \geq 1$ we note that

$$\mathcal{L}(1)^{\otimes \lambda}$$

has a highest wt vector $v = \begin{bmatrix}1\\0\end{bmatrix} \otimes \cdots \otimes \begin{bmatrix}1\\0\end{bmatrix}$ of wt. λ, so that $\mathcal{L}(1)^{\otimes\lambda}$ contains a, and hence the, simple sl_2-rep $\mathcal{L}(\lambda) \subseteq \mathcal{L}(1)^{\otimes \lambda}$ of highest wt. λ by Proposition 6. ∎

We've now classified simple sl_2-representations:

$$\begin{aligned} \mathbb{Z}_{\geq 0} &\xrightarrow{\ \cong\ } \{\text{simple } sl_2\text{-reps}\}/\cong \\ \lambda &\longmapsto \mathcal{L}(\lambda). \end{aligned}$$

$$\mathcal{L}(\lambda) = \begin{array}{c} v_\lambda \\ v_{\lambda-2} \\ \vdots \\ v_{-\lambda+2} \\ v_{-\lambda} \end{array}$$

– Next: semisimplicity of $\mathrm{rep}(sl_2(\mathbb{C}))$.

We know, from [Prop 4, Aug 28], that $\mathrm{rep}(\mathfrak{g})$ is semisimple iff each extension of simples

$$0 \to \mathcal{L}(\mu) \to V \to \mathcal{L}(\lambda) \to 0$$

split. Since we know so much about simples, one can observe such splittings directly. However, let us take an approach which mirrors the higher rank setting.

– The Casimir element

For each $sl_2(\mathbb{C})$-rep V define $\Omega_V : V \to V$ as:

$$\Omega_V := \tfrac{1}{2} h^2 + ef + fe \in \mathrm{End}_{\mathbb{C}}(V)$$

Lemma 11: a) For each map $\xi : V \to W$ of $\mathfrak{sl}_2$-reps, the diagram

$$\begin{tikzcd} V \arrow[r, "\xi"] \arrow[d, "\Omega_V"'] & W \arrow[d, "\Omega_W"] \\ V \arrow[r, "\xi"'] & W \end{tikzcd}$$

commutes.

b) Each linear endo Ω_V is in fact an $\mathfrak{sl}_2$-linear endo of V.

c) For each simple rep $L(\lambda)$, $\lambda \in \mathbb{Z}_{\geq 0}$,
$\Omega_{L(\lambda)} = \frac{1}{2}\lambda(\lambda+2)\cdot \mathrm{id}_{L(\lambda)}$.

Proof: a) is clear as at each $v \in V$ we have

$$\xi\big((\tfrac{1}{2}h.h + e.f + f.e)\cdot v\big) = (\tfrac{1}{2}h^2 + e.f + f.e)\cdot \xi(v),$$

via $\mathfrak{sl}_2$-linearity of ξ. (b) We want to show $x \cdot \Omega_V = \Omega_V x$ for each $x \in \mathfrak{sl}_2$, i.e. $[x, \Omega_V] = 0$ in $\mathfrak{gl}(V) = \mathrm{End}(V)^{\mathrm{Lie}}$.

However this follows by the calculations

$$[h, \tfrac{1}{2}h^2 + ef + fe] = \underbrace{2ef + (-2)ef + (-2)fe + 2fe}_{=0}$$

$$[e, \tfrac{1}{2}h^2 + ef + fe] = \underbrace{-eh - he + eh + he}_{=0}$$

$$[f, \tfrac{1}{2}h^2 + ef + fe] = \underbrace{fh + hf - hf - fh}_{=0}.$$

c) By Schur's Lemma $\mathrm{End}_{\mathfrak{sl}_2}(L(\lambda)) = \mathbb{C}$, so that $\Omega_{L(\lambda)} = c \cdot \mathrm{id}$ for some scalar c.

We can find the scalar c by evaluating on the highest wt. vector $v \in L(\lambda)_\lambda$. We have

$$\begin{aligned}(\tfrac{1}{2}h^2 + ef + fe)\cdot v &= \tfrac{1}{2}\lambda^2 v + ef\cdot v \\ &= \tfrac{1}{2}\lambda^2 v + [e,f]\cdot v \\ &= \tfrac{1}{2}\lambda^2 v + \lambda v \\ &= \tfrac{1}{2}\lambda(\lambda+2)\cdot v.\end{aligned}$$

∎

Remark: Ω_V is the action of the element $\Omega = \frac{1}{2}h^2 + ef + fe$ in $\mathcal{U}(sl_2)$ on the given $sl_2(\mathbb{C})$-rep V. This element

$$\Omega \in \mathcal{U}(sl_2)$$

is central, by (☆). It is called the <u>Casimir element</u>.

— Splitting extensions

Proposition 12: Any extension of simple sl_2-reps

$$0 \to L(\mu) \to V \to L(\lambda) \to 0 \qquad (*)$$

is split.

Proof: If $\lambda = \mu$ then $V(\lambda) = \mathbb{C}w \oplus \mathbb{C}w'$ where w is the image of the highest wt. vector $v \in L(\lambda)$ under the given inclusion and w' maps to v under the projection $V \to L(\lambda)$. By Proposition 6 we have two simple subreps

$$L, L' \subseteq V, \quad L, L' \cong L(\lambda),$$

with highest wt. vectors w and w' respectively.

The map $L(\lambda) \to V$ is therefore an $\cong$ onto L and the map $V \to L(\lambda)$ restricts to an isomorphism $L \to V \to L(\lambda)$. The inverse morphism $L(\lambda) \to L \hookrightarrow V$ provides the desired splitting.

If $\mu \neq \lambda$ then $\frac{1}{2}\mu(\mu+2) \neq \frac{1}{2}\lambda(\lambda+2)$. By Lemma 11 the operator $\Omega_V : V \to V$ has eigenvalues $\frac{1}{2}\mu(\mu+1)$ and $\frac{1}{2}\lambda(\lambda+1)$ and the resp generalized eigenspaces $V(\mu)$ and $V(\lambda)$ are nonvanishing subreps in V w/

$$V(\mu) \oplus V(\lambda) = V.$$

Since $\mathrm{Length}(V)=2$ we have

$$V(\mu) = \operatorname{im} L(\lambda)$$

and the composite $V(\lambda) \to V \to L(\lambda)$ is an isomorphism of sl_2-repr. The inverse

$$L(\lambda) \xrightarrow{\cong} V(\lambda) \hookrightarrow V$$

then provides the required splitting. ∎

Theorem [semisimplicity of rep(sl_2)]:

a) The category $\mathrm{rep}(sl_2(\mathbb{C}))$ is semisimple.

b) The simples in $\mathrm{rep}(sl_2(\mathbb{C}))$ are classified by their highest wts,

$$\mathbb{Z}_{\geq 0} \xrightarrow{\cong} \{\text{simple } sl_2(\mathbb{C})\text{-reps}\}/\cong.$$

c) Every fin-dim $sl_2(\mathbb{C})$-rep V decomposes uniquely into a sum

$$V = \bigoplus_{i=1}^{n} m(\lambda_i) \cdot L(\lambda_i)$$

with $m(\lambda_i) = \dim \mathrm{Hom}_{sl_2(\mathbb{C})}(L(\lambda_i), V)$.

Proof: Immediate from Prop 12 and [Prop 4, Aug 28]. ∎