Representation Theory!

§1 Universal enveloping algebra

Let $\mathfrak{g}$ be a Lie alg. We define the universal enveloping algebra

$$U(\mathfrak{g}) = \mathbb{C}\langle \mathfrak{g} \rangle \big/ (xy - yx - [x,y] : x, y \in \mathfrak{g})$$

Note that the construction of $U(\mathfrak{g})$ is functorial in $\mathfrak{g}$, i.e. each Lie alg map $\varphi: \mathfrak{g} \to \mathfrak{g}'$ def an alg map $U\varphi: U\mathfrak{g} \to U\mathfrak{g}'$ def by

$$U\varphi(x_1 \cdots x_n) = \varphi(x_1) \cdots \varphi(x_n).$$

(By univ prop of free alg + pres of rels.)

By construction we have the inclusion of Lie alg

$$i_{\mathfrak{g}}: \mathfrak{g} \to U(\mathfrak{g})^{Lie}, \quad x \mapsto x, \qquad (*)$$

which defines a natural transformation $i: \mathrm{id} \to (-)^{Lie} \circ U$ between endofunctors on Lie.

Lemma 1.1: The transformation i defines an adjunction

$$\begin{tikzcd}
\mathrm{Hom}_{Alg}(U(\mathfrak{g}), A) \arrow[rr, "\cong"] \arrow[dr, "Lie"'] & & \mathrm{Hom}_{Lie}(\mathfrak{g}, A^{Lie}) \\
& \mathrm{Hom}_{Lie}(U(\mathfrak{g})^{Lie}, A^{Lie}) \arrow[ur, "i^*"'] &
\end{tikzcd}$$

Proof: For the inverse send a Lie alg map $\varphi: \mathfrak{g} \to A^{Lie}$ to the unique alg map $\tilde{\varphi}: U(\mathfrak{g}) \to A$ w/ $\tilde{\varphi}\big|_{i(\mathfrak{g})} = \varphi$. Specifically, we have a map

$$\mathrm{Free}\,\varphi: \mathbb{C}\langle \mathfrak{g} \rangle \longrightarrow A$$

w/ $\mathrm{Free}\,\varphi\big|_{\mathfrak{g}} = \varphi$ and the fact that φ is a Lie alg map gives

$$\mathrm{Free}\,\varphi(xy - yx) = [\varphi x, \varphi y] = \varphi([x,y]) = \mathrm{Free}\,\varphi([x,y])$$

So that Free ϕ vanishes on the rels

$$(xy - yx - [x,y] : x, y \in \mathfrak{g}).$$

Hence we get an induced map from the quotient

$\tilde{\phi} = \text{Free}\,\phi : \mathcal{U}(\mathfrak{g}) \to A$. By construction

$\tilde{\phi}\,|_{i(\mathfrak{g})} = \phi$. Direct verification tells us that

the assignments $f \mapsto f^{Lie} \circ i$ and

$\phi \mapsto \tilde{\phi}$ are mutually inverse. ∎

Now, any $\mathcal{U}\mathfrak{g}$-module is naturally a $\mathbb{C}$-vect space, via the action of the scalars $\mathbb{C} \hookrightarrow \mathcal{U}(\mathfrak{g})$, and via centrality of the scalars all elem $a \in \mathcal{U}(\mathfrak{g})$ act by $\mathbb{C}$-linear endos. Hence any $\mathcal{U}(\mathfrak{g})$-module M is determined by its underlying $\mathbb{C}$-vect space structure and action map

$$act_M : \mathcal{U}(\mathfrak{g}) \to \text{End}_{\mathbb{C}}(M).$$

This action map is an alg map and so induces a Lie alg map $act_M^{Lie} : \mathcal{U}(\mathfrak{g})^{Lie} \to \text{End}_{\mathbb{C}}(M)^{Lie} = \mathfrak{gl}(M)$.

Restricting along $i_{\mathfrak{g}} : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})^{Lie}$ now defines a $\mathfrak{g}$-rep

$$\rho_M := act_M^{Lie} \circ i_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{gl}(M).$$

Colloquially, ρ_M simply restricts the $\mathcal{U}(\mathfrak{g})$-action on M to the generators $\mathfrak{g} \to \mathcal{U}(\mathfrak{g})$. Further, since $\mathcal{U}\mathfrak{g}$ is generated by the image of $\mathfrak{g}$ under $i_{\mathfrak{g}} : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ a linear map $f : M \to N$ of $\mathcal{U}(\mathfrak{g})$-modules is $\mathcal{U}(\mathfrak{g})$-linear iff f is a map of $\mathfrak{g}$-reps. So we obtain a fully faithful functor

(*) $\quad res_i : \mathcal{U}(\mathfrak{g})\text{-mod} \to \boxed{\mathfrak{g}\text{-mod}}$ $\quad$ ∞-dim $\mathfrak{g}$-rep w/ no finiteness cond

$$(M, act_M) \longmapsto (M, \rho_M)$$

$$(f : M \to N) \longmapsto (f : M \to N)$$

to the category of generally ∞-dim $\mathfrak{g}$-reps.

Corollary 1.2: The functor (*) is an equivalence, and restricts to an equivalence

$$(res_i)_{fin} : \mathcal{U}(\mathfrak{g})\text{-mod}_{fin} \to rep(\mathfrak{g})$$

from the cat of fin dim $\mathcal{U}(\mathfrak{g})$-modules to the cat of $\mathfrak{g}$-reps.

Proof: Since we've already observed fully faith, need only est. essential surjectivity. But, by Lemma 1.1, alg maps $act_M : \mathcal{U}(\mathfrak{g}) \to End_{\mathbb{C}}(M)$ are precisely the same thing as Lie alg maps

$$\rho_M : \mathfrak{g} \to \mathfrak{gl}(M).$$

So we see that res; is in fact a bijection on objects. ∎

From the proof (Lemma 1.1) we see that the cats $\mathcal{U}(\mathfrak{g})$-mod and $\mathfrak{g}$-mod are essentially "the same thing", and we'll treat them as interchangeable.

Rem: $rep(\mathfrak{g})$ is not the same thing as $\mathcal{U}(\mathfrak{g})$-mod!

– $1\frac{1}{2}$ Aside: Alg. group reps

For semisimple $\mathfrak{g}$ define

$Rep(\mathfrak{g})^{Int} := \{$ The cat of $\mathfrak{g}$-modules V w/ $V = \bigcup_\alpha V_\alpha$ where $V_\alpha \leq V$ runs over all subreps in V.

integrable $\mathfrak{g}$-reps

We have $Rep(\mathfrak{g}) \underset{full}{\hookrightarrow} \mathcal{U}(\mathfrak{g})$-mod.

Let $\mathfrak{g}$ be semisimple $/\mathbb{C}$.

Thm: There is a unique semisimple, simply-conn, alg group G w/ Lie $G = \mathfrak{g}$. For this group we have $Rep(G) = Rep(\mathfrak{g})^{Int}$.

This is where rep theory happens [not $\mathcal{U}(\mathfrak{g})$-mod].

– 2 PBW Theorem

Theorem (Ado) Every fin-dim Lie alg over an alg closed field of char 0) admits a faithful rep V.

Thm (PBW): Let $\mathfrak{g}$ be an arbitrary Lie alg.

i) The natural map $i_{\mathfrak{g}} : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ is injective.

ii) For any choice of ordered basis $\{x_1, \ldots, x_n\}$ in $\mathfrak{g}$, the collection of ordered monomials $\{x_1^{m_1} \cdots x_n^{m_n} : m_i \geq 0\}$ provides a basis for $\mathcal{U}(\mathfrak{g})$.

Proof: (i) For any faithful $\mathfrak{g}$-rep V we have the inj Lie alg map

$$\phi : \mathfrak{g} \to \mathfrak{gl}(V) = \mathrm{End}_{\mathbb{C}}(V)^{Lie}$$

which then provides an alg map

$$\mathcal{U}_\phi : \mathcal{U}(\mathfrak{g}) \longrightarrow \mathrm{End}_{\mathbb{C}}(V)$$

w/ $\phi = \mathcal{U}_\phi \circ i_{\mathfrak{g}}$, by Lemma 1.1. Since ϕ is injective, we conclude $i_{\mathfrak{g}}$ is injective.

(ii) Take $\mathcal{B} = \{x_1, \dots, x_n\}$. Define

$$\mathcal{U}(\mathfrak{g})_n = \mathrm{Span}_{\mathbb{C}}\{y_1 \cdots y_m : m \leq n,\ y_i \in \mathfrak{g}\} + \mathbb{C}\cdot 1.$$

and $\deg(a) = \min\{n : a \in \mathcal{U}(\mathfrak{g})_n\}$ for any $a \in \mathcal{U}(\mathfrak{g})$.

By extending linearly we see

$$\mathcal{U}(\mathfrak{g})_n = \mathrm{Span}_{\mathbb{C}}\{x_{i_1} \cdots x_{i_m} : m \leq n,\ x_{i_j} \in \mathcal{B}\} + \mathbb{C}\cdot 1.$$

We have $\mathcal{U}(\mathfrak{g})_0 = \mathbb{C}$ and suppose now

$$\mathcal{U}(\mathfrak{g})_{n'} = \mathrm{Span}_{\mathbb{C}}\{x_1^{m_1} \cdots x_n^{m_n} : \textstyle\sum m_i \leq n'\}$$

for each $n' < n$. Then for an arbitrary deg n monomial $a = x_t x_{i_2} \cdots x_{i_n}$ we have

$$a = x_t \cdot a' = x_t \cdot \left(\sum_{\vec{r}} c_{\vec{r}} \cdot x_1^{r_1} \cdots x_n^{r_n}\right)$$
$$= \sum_{\vec{r}} c_{\vec{r}}\, x_t x_1^{r_1} \cdots x_n^{r_n}$$

w/ the $\sum r_i \leq n-1$. So suffices to show

$$x_t x_1^{r_1} \cdots x_t^{r_t} \cdots x_n^{r_n}, \quad \sum r_i = n-1$$

is in the span of the $x_1^{m_1} \cdots x_n^{m_n}$, $\sum m_i \leq n$.

We have

$$x_t x_1^{r_1} \cdots x_n^{r_n} = [x_t, x_1^{r_1} \cdots x_{t-1}^{r_{t-1}}]\, x_t^{r_t} \cdots x_n + x_1^{r_1} \cdots x_t^{r_t+1} \cdots x_n^{r_n}$$
$$= x_1^{r_1} \cdots x_t^{r_t+1} \cdots x_n^{r_n} + \sum_j \sum_{l=1}^{r_j} x_1^{r_1} \cdots [x_t, x_j] \cdots x_n^{r_n}$$
$$\in x_1^{r_1} \cdots x_t^{r_t+1} \cdots x_n^{r_n} + \mathcal{U}(\mathfrak{g})_{n-1} \subseteq \text{desired span}.$$

So we conclude by induction

$$\mathcal{U}(\mathfrak{g}) = \mathrm{Span}\{\text{ordered monomials}\}.$$

The proof that ordered monomials are linearly independent is hard, and I know of no intuitive argument. [Hf, Section 17.4]. ∎

— 3 Triangular decomposition and (dominant) wts

Let $\mathfrak{g}$ be a semisimple Lie alg, w/ chosen Cartan $\mathfrak{h} \subseteq \mathfrak{g}$. Suppose we've fixed a base $\Delta \subseteq \Phi$, or rather that we've fixed a splitting

$$\Phi = \Phi^+ \sqcup \Phi^-.$$

This is our setting for studying rep theory for $\mathfrak{g}$.

Def^n: In the setting above, take

$$\mathfrak{n}^+ = \bigoplus_{\gamma \in \Phi^+} \mathfrak{g}_\gamma \quad \text{and} \quad \mathfrak{n}^- = \bigoplus_{\gamma \in \Phi^+} \mathfrak{g}_{-\gamma}.$$

The triangular decomposition for $\mathfrak{g}$ is the linear decomp

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

$\mathfrak{b}^+ = \mathfrak{n}^+ \oplus \mathfrak{h}$ ← Borels

Example: For

$$\mathfrak{sp}_{2n}(\mathbb{C}) = \left\{ \begin{array}{l} \text{lin endos of } W = \mathbb{C}^{2n} \text{ which preserve} \\ \text{an anti-symm non-degen form} \end{array} \right\}.$$

Choose specific form $\quad (v_i, v_j) = \begin{cases} 1 & \text{if } j = i+n \\ -1 & \text{if } i = j+n \\ 0 & \text{else} \end{cases}$

$x \in \mathfrak{sp}_{2n}$ satisfy $(x \cdot v, w) = -(v, x \cdot w)$ at all $v, w \in \mathbb{C}^n$.

We have the Cartan $\mathfrak{h} \subseteq \mathfrak{sp}_{2n}(\mathbb{C})$,

$$\mathfrak{h} = \operatorname{Span}\{E_{ii} - E_{n+i\, n+i} : 1 \leq i \leq n\}.$$

Humph; Ch 8 Ex 1]. For

$$\left.\begin{array}{l} e_i = E_{i\, i+1} - E_{n+i+1\, n+i} \\ f_i = E_{i+1\, i} - E_{n+i\, n+i+1} \end{array}\right\} \quad i = 1, \dots, n-1$$

$$h_i = E_{ii} - E_{i+1\, i+1} - E_{n+i\, n+i} + E_{n+i+1\, n+i+1}$$

$$e_n = E_{n\, 2n}, \quad f_n = E_{2n\, n}$$

$$h_n = E_{nn} - E_{2n\, 2n}$$

$$[h_n, e_i] = \begin{cases} 0 & \text{if } i < n-1 \\ -e_i & \text{if } i = n-1 \end{cases}$$

$$[h_i, e_n] = \begin{cases} 0 & \text{if } i < n-1 \\ -2e_n & \text{if } i = n-1 \end{cases}$$

$\Delta = \{\alpha_1, \dots, \alpha_n : \alpha_i = \text{root for } e_i\}$

$\Phi^+ =$ roots for upper Δ mat in $\mathfrak{sp}_{2n}(\mathbb{C})$

$\Phi^- =$ roots for lower Δ mat in $\mathfrak{sp}_{2n}(\mathbb{C})$.

— $\mathfrak{n}^+ =$ upper Δ'r mat $\quad \mathfrak{n}^- =$ lower Δ'r mat.

Theorem 3.1 (Triangular decomp): Let $\mathfrak{g}$ be a semisimple Lie-alg w/ Δ decomp $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$.

i) Multiplication provides an iso of $(\mathcal{U}(\mathfrak{n}^-), \mathcal{U}(\mathfrak{n}^+))$-bimodules
$$\mathcal{U}(\mathfrak{n}^-) \otimes \mathcal{U}(\mathfrak{h}) \otimes \mathcal{U}(\mathfrak{n}^+) \xrightarrow{\cong} \mathcal{U}(\mathfrak{g}).$$

ii) Mult provides an $\cong$ of $(\mathcal{U}(\mathfrak{n}^-), \mathcal{U}(\mathfrak{b}^+))$-bimodules
$$\mathcal{U}(\mathfrak{n}^-) \otimes \mathcal{U}(\mathfrak{b}^+) \xrightarrow{\cong} \mathcal{U}(\mathfrak{g}).$$

Proof: First note that both mult maps are maps of bimodules. For any orderings
$$\Phi = \{\gamma_1, \dots, \gamma_t\}, \quad \Delta = \{\alpha_1, \dots, \alpha_n\}$$
the ordered basis
$$\mathcal{B} = \{e_{\gamma_1}, \dots, e_{\gamma_t}, h_{\alpha_1}, \dots, h_{\alpha_n}, f_{\gamma_1}, \dots, f_{\gamma_t}\}$$
for $\mathfrak{g}$. Then the results (i) and (ii) follow by PBW Thm. ∎

4 (Dominant) Weights

Def^n: Let $\mathfrak{g}$ be semisimple w/ chosen Cartan. A function $\lambda : \mathfrak{h} \to \mathbb{C}$ is a weight if
$$\lambda(\gamma) \in \mathbb{Z} \quad \text{for all } \gamma \in \Phi.$$
Having chosen a base $\Delta \subseteq \Phi$, we call a weight λ a dominant weight if
$$\langle \lambda, \alpha \rangle \geq 0 \text{ for all } \alpha \in \Delta \quad (\text{equiv } \lambda(\gamma) \geq 0 \text{ for all } \gamma \in \Phi^+).$$
Given $\alpha \in \Delta$, the fundamental weight ω_α is the unique wt w/ $\omega_\alpha(\beta) = \delta_{\alpha\beta}$ at all $\beta \in \Delta$.

Def^n: The weight lattice $P \subseteq \mathfrak{h}^*$ is the lattice of weights. The root lattice $Q \subseteq P$ is the sublattice $Q = \mathbb{Z} \cdot \Phi$ spanned by the roots.

Theorem: Q and P are free $\mathbb{Z}$-modules w/ respective bases Δ and $\{\omega_\alpha : \alpha \in \Delta\}$. In particular $\operatorname{rank}_{\mathbb{Z}} Q = \operatorname{rank}_{\mathbb{Z}} P = |\Delta|$. Hence Q is of finite index in P.

§5 Verma modules and highest wt repr

Def$^{\underline{n}}$: Given any $\mathfrak{g}$-rep V, w/ chosen cartan $\mathfrak{h} \subseteq \mathfrak{g}$, and $\lambda : \mathfrak{h} \to \mathbb{C}$, we have
$$V_\lambda := \{ v \in V : h.v = \lambda(h).v \ \forall h \in \mathfrak{h} \}.$$
Having chosen ... we call a weight vector $v \in V$ a highest wt. vector if $v \neq 0$ and $e_\gamma . v = 0$ at all $\gamma \in \Phi^+$.
(Equiv. $e_\alpha . v = 0$ at all $\alpha \in \Delta$, by Prop 8.4.)

By reducing to the sl_2-case, by res along the root subalgs
$$\psi : sl_2(\mathbb{C}) \to \mathfrak{g},$$
one finds the following.

Prop 5.1: If $V_\lambda \neq 0$ then $\lambda \in P$. If $v \in V$ is a highest wt. vector of weight λ then λ is a dominant weight.

For each function $\lambda : \mathfrak{h} \to \mathbb{C}$ we have the 1-dim $\mathfrak{h}$-rep $\mathbb{C}(\lambda)$ on which $\mathfrak{h}$ acts via λ, $h.1 = \lambda(h).1$, and we restrict along the projection
$$\mathfrak{g} \supseteq \mathfrak{b}^+ \to \mathfrak{b}^+/\mathfrak{n}^+ \cong \mathfrak{h}$$
to obtain a 1-dim simple $\mathfrak{b}^+$-rep $\mathbb{C}(\lambda)$ on which $\mathfrak{n}^+$ acts as 0.

Def$^{\underline{n}}$: For any fn $\lambda \in \mathfrak{h}^*$ we define
$$M(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b}^+)} \mathbb{C}(\lambda).$$
This vect space is a $\mathcal{U}(\mathfrak{g})$-module, and hence a $\mathfrak{g}$-module, w/ $\mathcal{U}(\mathfrak{g})$ acting via the left action on $\mathcal{U}(\mathfrak{g})$,
$$\mathcal{U}(\mathfrak{g}) \circlearrowright \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b}^+)} \mathbb{C}(\lambda).$$

Lemma 5.2: As a $\mathcal{U}(\mathfrak{n}^-)$-module,
$$M(\lambda) = \mathcal{U}(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathbb{C}(\lambda).$$

Proof: We have

$$U(\mathfrak{g}) = U(\mathfrak{n}^-) \otimes U(\mathfrak{b})$$

by the triangular decomposition, as $(U(\mathfrak{n}^-), U(\mathfrak{b}^+))$-bimod, so that

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}(\lambda)$$

$$= U(\mathfrak{n}^-) \otimes_{\mathbb{C}} \left(U(\mathfrak{b}) \otimes_{U(\mathfrak{b})} \mathbb{C}(\lambda) \right)$$

$$= U(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathbb{C}(\lambda),$$

via associativity ~~of~~ of the $\otimes$-product functor. ∎

Corollary 5.3: Let $\{\gamma_1, \dots, \gamma_t\} = \Phi^+$ be any ordering of the positive roots. Then the set

$$\{ f_{\gamma_1}^{m_1} \cdots f_{\gamma_t}^{m_t} \cdot 1_\lambda : m_i \geq 0 \} \subseteq M(\lambda)$$

provides a linear basis for $M(\lambda)$.

Lemma 5.4: $M(\lambda)$ is a semisimple $\mathfrak{h}$-module, and for any collection of positive roots $\{\gamma_{i_1}, \dots, \gamma_{i_s}\}$ the vector

$$f_{\gamma_{i_1}} \cdots f_{\gamma_{i_s}} \cdot 1_\lambda \in M(\lambda)$$

is of weight $\lambda - \sum_{j=1}^{s} \gamma_{i_j}$.

Corollary 5.5: $\mathfrak{h}$ acts semisimply on any $\mathfrak{g}$-submodule $M' \subseteq M(\lambda)$. Furthermore,

$M' = M(\lambda)$ if and only if $1_\lambda \in M(\lambda)$ iff $(M')_\lambda \neq 0$.

Proof: For semisimplicity over $\mathfrak{h}$ [Exercise]. Since $M(\lambda)_\lambda = \mathbb{C} \cdot 1_\lambda$ by Corollary 5.3 and Lemma 5.4 we have $1_\lambda \in M'$ iff $(M')_\lambda \neq 0$. If $1_\lambda \in M'$ then all spanning vectors

$$f_{\gamma_{i_1}} \cdots f_{\gamma_{i_s}} \cdot 1_\lambda \in M'.$$

Hence $M(\lambda) \subseteq M'$. ∎

Corollary 5.6: For each function λ, $M(\lambda)$ contains a unique maximal proper $\mathfrak{g}$-submodule,

$M_{max}(\lambda) \subseteq M(\lambda)$. Proof: [Exercise]. ∎

Def: For any $\lambda \in \mathfrak{h}^*$ define the $\mathfrak{g}$-module

$$L(\lambda) = M(\lambda) \big/ M_{max}(\lambda).$$

Note that $L(\lambda)$ is simple, as we have a bijection

$$\{\text{submods } V \subseteq L(\lambda)\} \leftrightarrow \left\{ \begin{array}{l} \text{submods } N \subseteq M(\lambda) \\ \text{w/ } M_{max}(\lambda) \subseteq N \end{array} \right\}.$$

Proposition 5.2: If $L(\lambda)$ is finite-dimensional then λ is dominant integral.

Proof: By restricting along each root subalg

$$\iota_\gamma : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g}, \quad \iota_\gamma(e) = e_\gamma,\ \iota_\gamma(f) = f_\gamma$$
$$\iota_\gamma(h) = h_\gamma$$

we have, in this case, $L(\lambda)$ is a fin. dim $\mathfrak{sl}_2$-rep w/ $v_\lambda = \bar{1}_\lambda$ a highest wt. vector. Then by $\mathfrak{sl}_2$-theory

$$\lambda(h_\gamma) \in \mathbb{Z}_{\geq 0} \text{ necessarily}$$

at each $\gamma \in \Phi^+$. $\blacksquare$

- 6 Universal property of the Vermas

Theorem 6.1: For any $\mathfrak{g}$-rep V, evaluating at the vector $1_\lambda \in M(\lambda)$ provides a bijection

$$(*) \quad \mathrm{Hom}_{\mathfrak{g}}(M(\lambda), V) \xrightarrow{\cong} \mathrm{Span}_{\mathbb{C}} \left\{ \begin{array}{l} \text{highest wt vectors} \\ v \in V_\lambda \end{array} \right\}.$$

Proof: We have for $\mathfrak{b} = \mathfrak{b}^+$,

$$\mathrm{Hom}_{\mathfrak{b}}(\mathbb{C}(\lambda), V) \xrightarrow{\cong} \mathbb{C}\cdot\{\text{highest wt. vects } v \in V_\lambda\}$$
$$f \longmapsto f(1_\lambda),$$

and by $\otimes$-Hom adjunction

$$\mathrm{Hom}_{\mathfrak{g}}(M(\lambda), V) = \mathrm{Hom}_{\mathfrak{g}}(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}(\lambda), V)$$
$$\xrightarrow{\cong} \mathrm{Hom}_{\mathfrak{b}}(\mathbb{C}(\lambda), V)$$
$$\xi \longmapsto \xi|_{\mathbb{C}(\lambda)}.$$

So that in total we obtain the claimed bijection $(*)$. $\blacksquare$

Def: Consider the partial ordering on P def by $\mu \leq \lambda$ if $\lambda - \mu \in \mathbb{Z}_{\geq 0}\Delta$.

By Lemma 5.4 we have $M(\lambda)_\mu \neq 0$ if and only if $\mu \leq \lambda$.

Hence for any $\mathfrak{g}$-linear map
$$f : M(\lambda) \to V$$
we have $\operatorname{im} M(\lambda) \subseteq \sum_{\mu \leq \lambda} V_\mu$.

Prop 6.2: For any simple $\mathfrak{g}$-rep L,

a) L has a unique highest wt vector $v \in L_\lambda$, up to scaling.

b) If $L_\mu \neq 0$ then $\mu \leq \lambda = \mathrm{wt}(v)$.

c) $\dim(L_\lambda) = 1$.

Proof: Let λ be a maximal element in the (finite) set $\{\mu : L_\mu \neq 0\}$.

Then any nonzero vector $v \in L_\lambda$ is a highest wt vector. Suppose $w \in L$ is another highest wt vector, of weight $\mu = \mathrm{wt}(w)$. Then we have the map from the verma module
$$f_w : M(\mu) \to L$$
with $f_w(1_\mu) = w$. Take $L' = \operatorname{im}(f_w)$.

Note $L' \neq 0$ by construction and
$$L'_\mu = \mathbb{C} \cdot f(1_\mu) = \mathbb{C} \cdot w.$$
So $\dim L'_\mu = 1$. Furthermore $L'_\tau \neq 0$ implies $\tau \leq \mu$.

By simplicity we have $L' = L$, and thus $v \in L'$. So $\lambda \leq \mu$. Arguing similarly with $f_v : M(\lambda) \to L$ we find $\mu \leq \lambda$. Hence $\mu = \lambda$, $w \in \mathbb{C}^\times \cdot v$, and conclusions (a) – (c) follow. ∎

Theorem 6.3: For L a simple $\mathfrak{g}$-rep of highest wt λ, any choice of highest wt. vector v specifies a unique isomorphism
$$f : L(\lambda) \to L \text{ w/ } f(1_\lambda) = v.$$

Furthermore, for any $\mu \in P^+ - \{\lambda\}$ there exist no nonzero morphisms $L(\mu) \to L$.

Proof: By univ prop of Verma there exists a nonzero map $\tilde{f} : M(\lambda) \to L$ w/ $\tilde{f}(1_\lambda) = v$.

By simplicity $\mathrm{im}(\tilde{f}) = L$ so that, for $K = \ker(\tilde{f})$, $\tilde{f}$ induces an iso

$$f : M(\lambda)/K \xrightarrow{\sim} L.$$

As $M(\lambda)/K$ is now a simple quotient of $M(\lambda)$, we have K maximal in $M(\lambda)$, $K = M_{max}(\lambda)$ by Corollary 5.6, and hence $M(\lambda)/K = L(\lambda)$. We thus obtain the claimed isomorphism

$$f : L(\lambda) \xrightarrow{\sim} L.$$

If $\mu \neq \lambda$, then for any map $\xi : L(\mu) \to L$ we have $\xi(1_\mu) = 0$ or a highest wt vector of wt μ. By Prop 6.2 the latter cannot occur, so that $\xi(1_\mu) = 0$. Since $L(\mu)$ is generated by 1_μ it follows that $\xi = 0$, as claimed. ∎

Corollary 6.4: For any semisimple Lie alg $\mathfrak{g}$, w/ chosen Cartan, base etc., simple $\mathfrak{g}$-reps are determined up to isomorphism by their highest wt. We thus obtain an injective map of sets

$$\mathrm{Irrep}(\mathfrak{g}) \longrightarrow P^+$$
$$L \longmapsto \max \lambda \in P^+ \text{ w/ } L_\lambda \neq 0.$$

— 7 Existence

Theorem 7.1 [H, 21.2]: For semisimple $\mathfrak{g}$, w/ chosen Cartan etc., and dominant weight $\lambda \in P^+$, the simple $\mathfrak{g}$-module $L(\lambda)$ is finite-dimensional.

Theorem 7.2 [H, 21.2] For

$$\Pi(\lambda) := \{\mu \in P : L(\lambda)_\mu \neq 0\},$$

$\Pi(\lambda)$ is a finite subset in P which is stable under the action of W.

In a just world, one proves 7.1 by explicitly, elem. constructing a tensor generator, however, we're stuck w/ a technical proof from [H]. We'll only prove the structure Thm 7.2.

Note that on any fin-dim $\mathfrak{g}$-rep V we have each e_α and f_α acting nilpotently. Hence the ops $\exp(c \cdot e_\alpha), \exp(c \cdot f_\alpha) : V \to V$ are well-defined linear automorphisms, at each $c \in \mathbb{C}$.

Lemma 7.3: The linear automorphism
$$A_\alpha := \exp(e_\alpha)\exp(-f_\alpha)\exp(e_\alpha) : V \to V$$
satisfies $A_\alpha(V_\mu) = V_{\sigma_\alpha(\mu)}$ at each weight μ.

Proof: For each $h \in \mathfrak{h}$, $v \in V$, we have
$$h A_\alpha \cdot v = A_\alpha (A_\alpha^{-1} \cdot h \cdot A_\alpha) \cdot v,$$
and
$$A_\alpha^{-1} h A_\alpha = \exp(-\mathrm{ad}_{e_\alpha})\exp(\mathrm{ad}_{f_\alpha})\exp(-\mathrm{ad}_{e_\alpha}) h$$
$$= \exp(-\mathrm{ad}_{e_\alpha})\exp(\mathrm{ad}_{f_\alpha})(h + \alpha(h) \cdot e_\alpha)$$
$$= \exp(-\mathrm{ad}_{e_\alpha})(h + \alpha(h) f_\alpha + \alpha(h) e_\alpha - \alpha(h) h_\alpha - \alpha(h) f_\alpha)$$
$$= \exp(-\mathrm{ad}_{e_\alpha})(h + \alpha(h) e_\alpha - \alpha(h) h_\alpha)$$
$$= h + 2\alpha(h) e_\alpha - \alpha(h) h_\alpha - 2\alpha(h) h_\alpha$$
$$= h - \alpha(h) h_\alpha.$$

Thus for $\gamma \in \Phi^+$ and $v \in V_\mu$
$$h_\gamma \cdot A_\alpha \cdot v = A_\alpha (h_\gamma - \langle \alpha, \gamma \rangle h_\alpha) \cdot v$$
$$= (\mu(h_\gamma) - \langle \alpha, \gamma \rangle \mu(h_\alpha)) \cdot A_\alpha \cdot v$$
$$= (\mu(h_\gamma) - \langle \alpha, \gamma \rangle \cdot \langle \mu, \alpha \rangle) \cdot A_\alpha \cdot v$$
$$= (\mu - \langle \mu, \alpha \rangle \alpha)(h_\gamma) A_\alpha \cdot v.$$
$$= \sigma_\alpha(\mu)(h_\gamma) \cdot A_\alpha \cdot v.$$

So $A_\alpha \cdot v$ is of weight $\sigma_\alpha(\mu)$, and since A_α is an automorphism we must have
$$A_\alpha(V_\mu) = V_{\sigma(\mu)}.$$ ∎

Corollary 7.4: For any finite-dim rep V, the set

$$\Pi(V) = \{\lambda \in P : V_\lambda \neq 0\}$$

is stable under the action of the Weyl group.

Proof of Theorem 7.2: Apply Corollary 7.4 to the case $V = L(\lambda)$. ∎