– Reps of algebraic groups

– 1. Algebraic groups

Def^{n}: A group scheme is a scheme G over a base S equipped w/ an assoc mult map

$$m : G \times_S G \to G$$

unit map $\quad u : S \to G$

and inverse morphism $\quad i : G \to G$

which satisfy the expected axioms.

e.g. We have a diagram

$$\begin{tikzcd} & G \times G \arrow[r, "i \times id"] & G \times_S G \arrow[dr, "m"] & \\ G \arrow[ur, "\Delta"] \arrow[r] & S \arrow[rr, "u"] & & G \end{tikzcd}$$

We generally assume $S = \mathrm{Spec}(k)$ for an alg closed field $k = \bar{k}$, and G of finite type.

E.g. Elliptic curves are projective varieties which are naturally equipped w/ a group scheme structure

$$E \hookrightarrow \mathbb{P}^N.$$

Def^{n}: An algebraic group is a finite type group scheme over an algebraically closed field $S = \mathrm{Spec}(k)$.

– 2. Over an algebraically closed field

Thm 2.2 (Zariski): Let k be alg closed. For finite type k-schemes X and Y, and map $f : X \to Y$, the underlying map on topological spaces is recoverable from the map on k-points

$$f(k) : X(k) \to Y(k).$$

In the case where $k = \mathbb{C}$ and X, Y smooth...

Corollary 2.3: If X and Y are smooth over $\mathbb{C}$, a map $f : X \to Y$ is determined, as a map of topological spaces, by the underlying map of complex manifolds $\quad f(\mathbb{C}) : X(\mathbb{C}) \to Y(\mathbb{C})$

where we remember, additionally, the Zariski topologies on the two spaces.

So we can think when $k = \mathbb{C}$ of a map of a map of finite type smooth $\mathbb{C}$-schemes as a map of complex manifolds

$$X(\mathbb{C}) \to Y(\mathbb{C})$$

which preserve algebraic functions on affine opens

$$\begin{tikzcd}
U(\mathbb{C}) \arrow[r] \arrow[dr, "\text{aff op}"] \arrow[ddr, dashed, "\text{is alg}"'] & V(\mathbb{C}) \arrow[dr, "\text{aff op}"] & \\
& X(\mathbb{C}) \arrow[r] \arrow[d, "\text{alg fun}"] & Y(\mathbb{C}) \\
& \mathbb{C} &
\end{tikzcd}$$

— 3 Affine k-schemes

Take k = any field [redacted] (We think $k = \bar{k}$, and eventually $k = \mathbb{C}$.)

Def^{n}: An affine k-scheme is a scheme X over $\mathrm{Spec}(k)$ which admits a closed embedding

$$X \hookrightarrow \mathbb{A}^n_k. \quad (*)$$

Rem: The embedding $(*)$ is not part of the data.

Thm 3.1: Taking global section provides an equivalence of categories

$$\Gamma : \mathrm{Aff}_k \underset{\mathrm{Spec}}{\rightleftarrows} k\text{-}\mathrm{Alg}_{f.g.}, \quad X \mapsto \mathcal{O}(X).$$

— 4 Affine group schemes and Hopf alg's

Fix $k = \bar{k}$.

Def^{n}: An affine alg group over k is an algebraic group over $\mathrm{Spec}(k)$ which is simultaneously affine over k.

E.g.

$$SL_n = \mathrm{Spec}\left(\mathbb{C}[x_{ij} : 1 \le i, j \le n] / (\det[x_{ij}] - 1)\right)$$

$$GL_n = \mathrm{Spec}\left(\mathbb{C}[x_{ij} : 1 \le i, j \le n]_{\det}\right)$$

$SL_n(\mathbb{C}) = \{ \text{Matrix } A = [a_{ij}] \text{ w/ } \det A = 1 \}$

$GL_n(\mathbb{C}) = \{ \text{Mat } A = [a_{ij}] \text{ w/ } \det A \neq 0 \}$.

(For the symplectic group we consider the closed subvariety)

$$Sp_{2n} \hookrightarrow \overbrace{M_n \times \cdots \times M_n}^{4 \text{ times}}$$
$$= \left\{ \begin{bmatrix} A & C \\ D & B \end{bmatrix} : A, B, C, D \text{ in } M_n \right\}$$

which is the locus of all points $\nparallel$ satisfying

$$\left.\begin{array}{l} -C^t A + A^t C = 0 \\ -C^t B + A^t D = I_n \\ -D^t B + B^t D = 0 \end{array}\right\} \text{a collection of } 3n^2 \text{ quadratic equations in } 4n^2 \text{ variables}$$

Alternatively, for $\Omega = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ we have the algebraic map

$$F : GL_{2n} \to GL_{2n}, \quad M \mapsto M^t \Omega M$$

and Sp_{2n} is the fiber over the point Ω,

$$\begin{tikzcd} Sp_{2n} \arrow[r] \arrow[d, "\text{fiber}"] & GL_{2n} \arrow[d, "F"] \\ \operatorname{Spec}(\mathbb{C}) \arrow[r, "\Omega"] & GL_{2n} \end{tikzcd}$$

For affine schemes X, we have

$$X \times_{\operatorname{Spec}(k)} X = \operatorname{Spec}(\mathcal{O}(X) \otimes_k \mathcal{O}(X))$$

so that, under the equiv of Thm 3.1, for $G = \operatorname{Spec}(\mathcal{O})$ a group scheme structure on G is equiv to the following data

coprod) An alg map $\Delta : \mathcal{O} \to \mathcal{O} \otimes_k \mathcal{O}$

w/ $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$. (*)

counit) A specified alg map $\epsilon : \mathcal{O} \to k$.

antipode) An alg iso $S : \mathcal{O} \to \mathcal{O}$ w/

$$\Delta S = (S \otimes S)\Delta^{op}$$

($\Delta^{op} = \text{swap factors} \circ \Delta$) and satisfying

$$\text{mult} \circ (S \otimes 1)\Delta = \text{mult} \circ (1 \otimes S)\Delta$$
$$= \text{unit} \circ \epsilon.$$

Def: A commutative Hopf alg / k is a comm alg $\mathcal{O}$ w/ Δ, ϵ, S satisfying the conditions (*).

Corollary 4.1: The equivalence
$$\Gamma : \mathrm{Aff}_k \to k\text{-}\mathrm{Alg}_{fingen}$$
restricts to provide an equivalence
$$\Gamma : \mathrm{AffGrp}_k \underset{\mathrm{Spec}}{\rightleftarrows} k\text{-}\mathrm{CommHopf}_{fingen}.$$

Ex: For GL_n, $\mathcal{O} = \mathcal{O}(GL_n)$ the coproduct Δ is det on the gens $x_{ij} \in \mathcal{O}$, and given by the formula $\Delta(x_{ij}) = \sum_{k=1}^{n} x_{ik} \otimes x_{kj}$. (*)

Since the inclusion $SL_n \to GL_n$ and $Sp_{2n} \to GL_{2n}$ are group maps, the maps on funcs
$$\mathcal{O}(GL_n) \to \mathcal{O}(SL_n)$$
and
$$\mathcal{O}(GL_{2n}) \to \mathcal{O}(Sp_{2n})$$
are Hopf alg maps, So the formula Δ det coproducts on $\mathcal{O}(SL_n)$ and $\mathcal{O}(Sp_{2n})$ as well.

The antipode ϵ is given by $\epsilon(x_{ij}) = \delta_{ij}$, and the antipode given by
$$S(x_{ij}) = \det([x_{kl}])^{-1} (-1)^{ij} \det M_{ij}$$
where M_{ij} = Matrix $[x_{kl}]$ w/ ith row and jth col deleted.

$= ij$th entry of $[x_{ij}]^{-1}$.

-5 The universal coaction

For fin dim $\mathbb{C}$-vect space V we have the gen's $V^* \otimes V \hookrightarrow \mathcal{O}(GL(V))$, and linear map
$$V \xrightarrow{coev \otimes 1} V \otimes V^* \otimes V, \quad v_j \longmapsto \sum_{k=1}^{n} v_i \otimes x_{ij},$$
which composes to a map
$$\rho_{univ} : V \to V \otimes V^* \otimes V \longrightarrow V \otimes \mathcal{O}(GL(V)).$$

In this coord free expression the coprod. on $\mathcal{O}(GL)$ is det by the map on the gens

we have $(\rho \otimes 1)\rho = (1 \otimes \Delta)\rho$.

$$V^* \otimes V \xrightarrow{1 \otimes coev \otimes 1} V^* \otimes V \otimes V^* \otimes V \subseteq \mathcal{O} \otimes \mathcal{O}.$$

We note that for any group scheme G the R-points $G(R)$ have a natural group structure so that the functor of points is naturally group valued

$$G : k\text{-Alg} \longrightarrow \text{Group}.$$

Lemma 5.1: The functor

$$\text{Grp Sch}_k \longrightarrow \text{Fun}(k\text{-Alg}, \text{Grp})$$
$$G \longmapsto (R \mapsto G(R))$$

is fully faithful.

Proof: Follows by Yoneda and Zariski descent. ∎

Below, for a k-vect space V we consider the fun

$$\text{Aut}_V : k\text{-Alg} \to \text{Group}, \quad R \mapsto \text{Aut}_R(V \otimes R).$$

Theorem 5.1: For any R-point $\xi : \mathcal{O}(GL_V) \to R$ the corresponding R-linear endo $A_\xi : V \otimes R \to V \otimes R$ def on the gen by

$$V \xrightarrow{\rho} V \otimes \mathcal{O} \xrightarrow{1 \otimes \xi} V \otimes R$$

is an automorphism. Furthermore, for each k-alg R the assignment

$$GL_V(R) \to \text{Aut}_R(V \otimes R)$$

is an isomorphism of groups. These isos define a natural isom $GL_V \cong \text{Aut}_V$.

Proof: Consider a basis $\{v_1, \ldots, v_n\}$ for V. We recover ξ on the generators from A_ξ on the generators by composing

$$V^* \otimes V \to V^* \otimes V \otimes R \xrightarrow{ev \otimes 1} R$$

and the values $a_{ij} = f_\xi(v^i \otimes v_j) = f_\xi(x_{ij})$ determine A_ξ as a matrix

$$A_\xi = [a_{ij}].$$ Now we have

A_ξ is an R-linear isom iff $\det(A_\xi) \in A^\times$. But this holds by def^n of $\mathcal{O}(GL_V)$, since

$$\det(A_\xi) = f_\xi(\det[x_{ij}]).$$

Conversely, given an automorphism
$A \in GL_R(V \otimes R)$
A def a fun $V^* \otimes V \to R$, $x_{ij} \mapsto a_{ij}$,
w/ $\det([a_{ij}]) \in A^\times$. Hence the induced map
fr $\mathrm{Sym}(V^* \otimes V) \to R$
factors through the localizatn to define
$$\xi^A : \mathcal{O}(GL_V) \to R.$$
The assignments $\xi \mapsto A_\xi$, $A \mapsto \xi^A$, are seen to be mutually inverse, and the resulting $\cong$'s
$$GL_V(R) \xrightarrow{\sim} \mathrm{Aut}_V(R)$$
are seen to be group maps which are natural in R. ∎

– 6 Coactions and G-representations

Def$^{\underline{n}}$: Let G be an affine alg group/k. A G-representation is the information of a fin dim'l vector space V and a map of group schemes
$$\phi_V : G \to GL(V)$$
over $\mathrm{Spec}(k)$.

Now, for any such map $\phi_V : G \to GL(V)$ we have the corresp. maps
$$\rho_V : V \to V \otimes \mathcal{O}(G)$$
def by
$$\rho_V = (1 \otimes \phi_V^*)\rho_{univ} : V \to V \otimes \mathcal{O}(GL_V) \to V \otimes \mathcal{O}(G).$$

Lemma 6.1: ρ_V satisfies
$$(\rho_V \otimes 1)\rho_V = (1 \otimes \Delta)\rho_V, \qquad (**)$$
and $id_V = (1 \otimes \epsilon)\rho_V$.

Proof: Inherited from ρ_{univ}, since ϕ_V^* is a Hopf map. ∎

Def$^{\underline{n}}$: A corepresentation for a Hopf alg $\mathcal{O}$ is a vect space V equipped w/ a coaction $\rho_V : V \to V \otimes \mathcal{O}$ satisfying $(**)$.

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Theorem 6.2: For any affine group scheme over k, the functor

$$\mathrm{rep}(G) \to \mathrm{corep}(\mathcal{O}), \quad (V, \rho_V) \mapsto (V, \rho_V),$$

is an equivalence of categories.

Proof: Given any corep (V, ρ) we obtain a map between the fun of points

$$G(-) \to \mathrm{Aut}_V(-) \cong GL_V(-)$$

def. by taking an R-point $x: \mathcal{O}(G) \to R$ to the linear endo ξ_x def by

$$V \otimes R \xrightarrow{\rho \otimes 1} V \otimes \mathcal{O} \otimes R \xrightarrow{1 \otimes m(x \otimes 1)} V \otimes R.$$

One sees that ξ_x is invertible w/ inverse $\xi_{x^{-1}}$.

We then obtain, via Yoneda, a Hopf alg map

$$\rho_V^*: \mathcal{O}(GL_V) \to \mathcal{O}(G)$$

which is explicitly defined on the generators by

$$V^* \otimes V \xrightarrow{1 \otimes \rho} V^* \otimes V \otimes \mathcal{O}(G) \xrightarrow{ev \otimes 1} \mathcal{O}(G).$$

One sees that the corresp. functor

$$\mathrm{corep}(\mathcal{O}(G)) \to \mathrm{rep}(G)$$

is inverse to the given functor. ∎

7 Representations of Tori.

Take $\mathbb{G}_m$ the multiplicative group scheme over k, w/ fun of pts

$$\mathbb{G}_m(R) = R^\times.$$

For $k[x, x^{-1}]$ w/ Hopf structure

$$\Delta(x^n) = x^n \otimes x^n, \quad S(x^n) = x^{-n}, \quad \epsilon(x^n) = 1,$$

we have $\mathbb{G}_m = \mathrm{Spec}(k[x, x^{-1}])$. A split torus is a group scheme T w/ an isomorphism

$$T \cong \mathbb{G}_m^{\times r} \quad \text{for some } r.$$

Ex: In SL_n we have the torus of diag matrices $T \hookrightarrow SL_n$, $\mathcal{O}(T) = \mathbb{C}[x_1, \dots, x_n]/(x_1 \cdots x_n - 1)$

$$\cong \mathbb{C}[x_1^{\pm 1}, \dots, x_{n-1}^{\pm 1}].$$

For fixed $\mathcal{O}(T)$ on any torus we have the associated lattice

$$X_T := \{\lambda \in \mathcal{O}(T) : \Delta(\lambda) = \lambda \otimes \lambda\}$$
$$\cong \mathrm{Hom}_{\mathrm{GrpSch}}(T, \mathbb{G}_m).$$

If $\mathcal{O}(T) \cong k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ then

$$X_T \cong \langle x_1, \ldots x_n \rangle \cong \mathbb{Z}^{\oplus n}.$$

For each T-rep V and $\lambda \in X_T$ define

$$V_\lambda := \{v \in V : \rho(v) = v \otimes \lambda\}.$$
$$= \{v \in V : t \cdot v = \lambda(t) v \text{ at all } t \in T\}.$$

Proposition 7.1: For each T-rep V,

$$V = \bigoplus_{\lambda \in X_T} V_\lambda.$$

Proof: For arbitrary $v \in V$ we have

$$\rho(v) = \sum_\lambda{}' v_\lambda \otimes \lambda \quad \text{w/ fin many } v_\lambda \text{ nonzero}$$

and via coassociativity $\rho(v_\lambda) = v_\lambda \otimes \lambda$. Further via counitality we find

$$v = (1 \otimes \epsilon)\rho(v) = \sum_\lambda{}' v_\lambda.$$

Then $V = \bigoplus_{\lambda \in X_T} V_\lambda$. ∎

Ex: For SL_n take $P = X_T$ for T = diag matrices. By restricting along the inclusion $T \to SL_n$ we see that any SL_n-representation decomposes into eigenspaces for the action of the torus

$$V = \bigoplus_{\lambda \in P} V_\lambda.$$

— 8 Additive groups

Take $\mathbb{G}_a$ w/ $\mathbb{G}_a(R) = (R, +)$, the underlying additive group. We have

$$\mathbb{G}_a = \mathrm{Spec}(k[x]) \text{ w/ } \begin{aligned} \Delta(x) &= x \otimes 1 + 1 \otimes x \\ S(x) &= -x \\ \epsilon(x) &= 0. \end{aligned}$$

Ex: For SL_2 we have the two additive subgroups

$E : \mathbb{G}_a \to SL_2, \quad c \mapsto \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$

$F : \mathbb{G}_a \to SL_2, \quad c \mapsto \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}.$

From any $\mathbb{G}_a$-rep V we extract an endomorphism

$y_V: V \to V, \quad v \mapsto (1 \otimes x^*)\rho(v)$

providing a functor supposing char $= 0$

$\mathrm{inf}: \mathrm{rep}(\mathbb{G}_a) \to \mathrm{rep}(\mathfrak{g}_a), \quad \mathfrak{g}_a = \mathbb{C}\cdot y.$

Theorem 8.1: The functor inf is an equivalence onto the subcategory $\mathrm{rep}(\mathfrak{g}_a)_{nilp}$ of $\mathfrak{g}_a$-rep's on which the generator y acts nilpotently.

Proof: We have for any $v \in V$,

$\rho(v) = v + v_1 \otimes x^1 + \cdots + v_n \otimes x^n$

and for $\rho^m(v) = \overbrace{(\rho \otimes 1 \cdots 1)(\rho \otimes 1 \cdots 1) \cdots \rho(v)}^{m \text{ times}}$

we have $y^m = (1 \otimes x^* \otimes \cdots x^*)\rho^m$ so that

$y^m \cdot v = m!\, v_m.$

In particular $y^{n+1} \cdot v = 0$. So y acts locally nilpotently and thus nilpotently on all of V. Hence inf has image in $\mathrm{rep}(\mathfrak{g}_a)_{nilp}$.

For the inverse functor

$\exp: \mathrm{rep}(\mathfrak{g}_a)_{nilp} \to \mathrm{rep}(\mathbb{G}_a)$

take V w/ nilpotent endo $y: V \to V$ to V w/ coaction $\rho: V \to V \otimes k[x]$ def. by

$\rho(v) := \sum_{i \geq 0} (i!)^{-1} y^i \cdot v \otimes x^i.$

One checks directly that ρ is in fact a coaction, and one also checks directly that the two composites

$\mathrm{inf}|_{nilp} \circ \exp = id$ and $\exp \circ \mathrm{inf}|_{nilp} = id.$ ∎

9. Representations of SL_2

Given a complex SL_2-rep V take

$e, f: V \to V$

the infinitesimal action maps obtained by the root subgroups

$E, F: \mathbb{G}_a \to SL_2$

respectively.

For the torus $T \to SL_2$ take the cocharacter

$$\mathbb{Z} \to X_T, \quad m \mapsto x_{11}^m \in \mathcal{O}(T),$$

so that any SL_2-rep is naturally $\mathbb{Z}$-graded.

Lemma 9.1: Consider an SL_2-rep V w/ weight decomposition $V = \bigoplus_{\lambda \in \mathbb{Z}} V_\lambda$. The infinitesimal endos $e, f : V \to V$ satisfy

a) $e \cdot V_\lambda \subseteq V_{\lambda+2}$

b) $f \cdot V_\lambda \subseteq V_{\lambda-2}$

c) $[e,f] \cdot v = \lambda \cdot v$ when $v \in V_\lambda$.

Rather, the three endos $e, f, h = [e,f] : V \to V$ define an $sl_2(\mathbb{C})$-rep structure on V.

Proof: Exercise. ∎

Theorem 9.2: The functor

$$\mathrm{inf} : \mathrm{rep}(SL_2) \to \mathrm{rep}(sl_2(\mathbb{C}))$$

$$(V, \rho) \longmapsto (V, e_V, f_V, h_V)$$

is an equivalence of categories.

Proof: Exercise. ∎