Given a ring map $\phi : R \to S$ and T mult. subset in R, note that the module localization $S[T^{-1}]$ is naturally a ring under expected $+$ and mult,

$$f_1/t_1 + f_2/t_2 = (f_1 t_2 + f_2 t_1)/t_1 t_2$$

and $(f_1/t_1)\cdot(f_2/t_2) = f_1 f_2 / t_1 t_2$.

Observe that the ring map $\phi : R \to S$ localizes to provide a corresp. ring map $\phi_T : R[T^{-1}] \to S[T^{-1}]$. Indeed $S[T^{-1}] = S[\phi(T)^{-1}]$.

1. By a finite extension of commutative rings we mean an injective ring map $\phi : R \to S$ with S a finitely gen'd module over R. For any such fin extension, and prime $\mathfrak{p} \subseteq R$, prove that $S_{\mathfrak{p}}$ is a nonzero ring, and the natural map $R_{\mathfrak{p}} \to S_{\mathfrak{p}}$ is also a finite extension.

2. Given a finite extension $\phi R \to S$ and a prime ideal $\mathfrak{p}$ in R, show that there is a prime $\tilde{\mathfrak{p}}$ in S with $\phi^{-1}\tilde{\mathfrak{p}} = \mathfrak{p}$. Show that the number of such $\tilde{\mathfrak{p}}$ is finite.

3. Given a finite extension $\phi : R \to S$ prove that

the induced map on spectra $\phi^*: \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective with finite fibers. Prove further that there is an integer N with $|\phi^{*-1}(\mathfrak{p})| \le N$ at each prime $\mathfrak{p}$ in $\mathrm{Spec}(R)$.

4. Suppose $\phi: R \to S$ is a finite extension. Prove that the induced map on spectra $\phi^*: \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is a closed map.

5. Construct a map of rings $\phi: R \to S$ for which the induced map on spectra is not closed.