~ II½. The Jacobson Radical

Def^n: For any ring A define the Jacobson radical
$$\mathrm{Jac}(A) = \bigcap_{\mathfrak{m} \subseteq A} \mathfrak{m},$$
where the intersection runs across all maximal ideals $\mathfrak{m} \subseteq A$.
Note that this is an ideal in A.

Example: Let k be an infinite field and consider $k[x]$. We have at each $\alpha \in k$ the corresponding max ideal $\mathfrak{m}_\alpha = (x-\alpha) \subseteq k[x]$. Thus
$$\mathrm{Jac}(k[x]) \subseteq \bigcap_{\alpha \in k} \mathfrak{m}_\alpha$$
and any $p(x) \in \mathrm{Jac}(k[x])$ has $p(\alpha) = 0$ at all $\alpha \in k$. (This follows since $\mathfrak{m}_\alpha =$ all functions which vanish at α.) Since k is infinite this forces $p(x) = 0$.
Hence $\mathrm{Jac}(k[x]) = 0$.

Exercise: Generalize this result to $k[x]$ at arbitrary field k.

Example: $\mathrm{Jac}\big(\overbrace{\prod_{i=1}^{n} M_{n_i}(D_i)}^{A}\big)$ we have the max ideals
$$\mathfrak{m}_i = \prod_{j \neq i} M_{n_j}(D_j) \subseteq A$$
$$= \ker(\pi_i : A \to M_{n_i}(D_i)).$$

For $a \in \mathrm{Jac}(A)$ then we have

$a \in \mathrm{Ker}(\pi_i)$ for all i

$\Rightarrow$ $a = (a_i : 1 \leq i \leq n)$ w/ all $a_i = 0$

$\Rightarrow$ $a = 0$.

So $\mathrm{Jac}\left(\prod_{i=1}^n M_{n_i}(D_i)\right) = 0$.

Proposition 1: Any semisimple algebra A has $\mathrm{Jac}(A) = 0$.

Proof: By Wedderburn $A \cong \prod_{i=1}^n M_{n_i}(D_i)$. $\blacksquare$

~ II. Jac(A) and nilpotence

Let's call an ideal $I \subseteq A$ nilpotent if $I^n = 0$ at some (large) n. Here we've taken

$$I^p = \mathrm{Span}_{\mathbb{Z}}\{x_1 \cdots x_p : x_i \in I\},$$

and we note that this is always an ideal in A.

Lemma 2: Given a maximal ideal $m \subseteq A$ and ideals I and J for which

$$I \cdot J = \{\textstyle\sum_i x_i y_i : x_i \in I, y_i \in J\} \subseteq m,$$

we have $I \subseteq m$ or $J \subseteq m$.

In particular, any nilpotent ideal I is contained in every maximal ideal.

Proof: If $\mathfrak{m} \supseteq I \cdot J$ and both $I, J \not\subseteq \mathfrak{m}$ then we have $I + \mathfrak{m} = J + \mathfrak{m} = A$, since $I + \mathfrak{m}$ and $J + \mathfrak{m}$ are ideals properly containing $\mathfrak{m}$. Hence $1 = x + a = y + b$ for some $x \in I$, $y \in J$, $a, b \in \mathfrak{m}$. This gives $1 = 1^2 = xy + xb + ay + ab \in \mathfrak{m}$, and hence $A = \mathfrak{m}$, a contradiction. ∎

Corollary 3: Any nilpotent ideal $I \subseteq A$ is contained in the Jacobson radical.

Now, for any pair of nilpotent ideals I_0 and $I_1 \subseteq A$ we have

$$(I_1 + I_2)^{m_1 \cdot m_2} = \sum_\ell \left(\sum_{r_1 + \cdots + r_\ell = m_1 \cdot m_2} I_1^{r_1} I_2^{r_2} I_1^{r_3} \cdots I_\ell^{r_\ell} \right)$$

$$\subseteq I_1^{m_1} + I_2^{m_2}.$$

Taking m_1 and m_2 large we find that $I_1 + I_2$ is nilpotent.

Lemma 4: If A is Noetherian then A contains a unique maximal nilpotent ideal. In particular, for $\{I_\lambda : \lambda \in \Lambda\}$ the collection of all nilpotent ideals the sum

$$\sum_{\lambda \in \Lambda} I_\lambda = \{\text{fin. sums of elem. in the } I_\lambda\}$$

is a nilpotent ideal in A.

Corollary 5: If A is a finite dimensional alg over a field k, then A contains a maximal nilpotent ideal and

$$\mathrm{MaxNil}(A) \subseteq \mathrm{Jac}(A)$$

Example: Let S_n act on $k[x_1, \ldots, x_n]/(x_i^n : 1 \le i \le n)$, by permuting the generators. Consider

$$I = (x_1, \ldots, x_n) \subseteq A = k[x_1, \ldots, x_n]/(x_i^n) \rtimes S_n.$$

and note that

$$I^{n \cdot n} = \mathrm{Span}_{\mathbb{Z}}\{\text{length } n \cdot n \text{ products of the } x_i\} \cdot A$$
$$= 0.$$

Hence I is nilpotent and thus $I \subseteq \mathrm{Jac}(A)$.

Suppose now that $\mathrm{char}\, k = 0$. Then

$$A/I \cong kS_n \text{ is semisimple}$$

and thus has $\mathrm{Rad}(kS_n) = 0$.

Lemma 4: For any surjective ring map $\pi : A \to B$, $\pi^{-1}(\mathrm{Jac}(B)) \supseteq \mathrm{Jac}(A)$.

Proof: Follows from the fact that $\pi^{-1}(m)$ is maximal in A whenever m is max in B. Then

$$\pi^{-1}(\mathrm{Jac}(B)) = \pi^{-1}\Big(\bigcap_m m\Big) = \bigcap_m \pi^{-1}(m)$$
$$\supseteq \bigcap_n n = \mathrm{Jac}(A). \qquad \blacksquare$$

Continuing our example, Prop 1 and Maschke now give
$$\mathrm{Jac}(A) \subseteq \pi^{-1}(\mathrm{Jac}(kS_n)) = \pi^{-1}(0) = I,$$
so that $I = \mathrm{Jac}(A)$.

Let's think about one more example in this nilpotent direction. For commutative A we have
$$\mathrm{Jac}(A) = \mathrm{Rad}(A)$$
since max submodules in A = max ideals in this case, and hence $A/\mathrm{Jac}(A)$ is a sum of simple A-modules. Further, each simple A-mod appears with positive mult. in $A/\mathrm{Jac}(A)$ since each simple L appears as a quotient $A \to L$.

Proposition 5: For any finite dimensional commutative algebra A over a field k,
$$\mathrm{Jac}(A) = \mathrm{MaxNil}(A).$$

Proof: We have $\mathrm{MaxNil}(A) \subseteq \mathrm{Jac}(A)$ by Corollary 5. Now for $x \in \mathrm{Jac}(A)$ we have $x \cdot L = 0$ at any simple A-module, since
$$A/\mathrm{Jac}(A) = A/\mathrm{Rad}(A) \cong \bigoplus_i m_i L_i$$
where $\{L_1, \dots, L_r\}$ runs over a complete list of simple A-modules (up to $\cong$) and each $m_i > 0$.

Considering a composition series

$$0 \subseteq M_1 \subseteq \cdots \subseteq M_\ell = A$$

for A we now find $x \cdot M_j \subseteq M_{j-1}$ at each j, for any $x \in \mathrm{Jac}(A)$, giving

$$x^\ell \cdot A = 0$$

and hence $x^\ell = 0$. So all $x \in \mathrm{Jac}(A)$ are nilpotent. Considering any finite basis $\{x_1, \ldots, x_r\} \subseteq \mathrm{Jac}(A)$ for $\mathrm{Jac}(A)$ over k, we find that $\mathrm{Jac}(A)$ itself is nilpotent. This gives

$$\mathrm{Jac}(A) \subseteq \mathrm{MaxNil}(A),$$

and thus $\mathrm{Jac}(A) = \mathrm{MaxNil}(A)$. ∎

Corollary 6: For any finite-dim commutative k-alg A,

$$A / \mathrm{Jac}(A) = \prod_{j=1}^{n} K_j,$$

where each K_j is a finite field extension of k.

Corollary 7: Let G be a finite group acting on a finite-dimensional commutative k-alg A, in char 0. Then

$$\mathrm{Jac}(A \rtimes G) = \mathrm{MaxNil}(A \rtimes G)$$

and $A \rtimes G / \mathrm{Jac}(A \rtimes G)$ is a semisimple algebra.

Proof: Since any alg automorphism preserves MaxNil we see that

$$g \cdot \mathrm{MaxNil}(A) = \mathrm{MaxNil}(A)$$

for all $g \in G$, and thus

$$\mathrm{MaxNil}(A) \rtimes G = \mathrm{MaxNil}(A)(A \rtimes G)$$

is a nilpotent ideal in $A \rtimes G$. Consequently,

$$G \curvearrowright A / \mathrm{MaxNil}(A) =: \bar{A} \left(\cong \prod_i K_i\right)$$

and we have the alg quotient

$$\pi : A \rtimes G \to \bar{A} \rtimes G$$

with nilpotent kernel $\mathrm{MaxNil}(A) \rtimes G$. One can argue that $\bar{A} \rtimes G$ is semisimple [Exercise] so that

$$\mathrm{Jac}(A \rtimes G) \subseteq \pi^{-1}(0) = \mathrm{MaxNil}(A) \rtimes G$$
$$\subseteq \mathrm{MaxNil}(A \rtimes G),$$

giving the proposed equality. ∎

~ I. The Main Point.

Let A be any finite-dim alg over a field, or more generally any Artinian ring. We consider the Jacobson radical $\mathrm{Jac}(A) \subseteq A$. We claim that $\mathrm{Jac}(A)$ is a magical person.

Main Results (Artinian Ring)

- $A/\mathrm{Jac}(A)$ is semisimple
- An A-module M is semisimple if and only if $\mathrm{Jac}(A) \subseteq \mathrm{Ann}_A(M)$.
- For any A-module M, $\mathrm{Jac}(A) \cdot M = \mathrm{Rad}(M)$.
- $\mathrm{Jac}(A) = \mathrm{MaxNil}(A)$.

Implication

- Any Artinian ring is Noetherian, and hence of finite length $\sim$ A is finite dimensional.

Our first (generic) ambition is to prove the above result, and hence gain quite a bit of traction in our study of finite-dimensional algebras.

Our process is first, study semisimple rings then second, lift our analysis of semisimplicity to the pairing $\{A, \mathrm{Jac}(A)\}$.

~ II. The double centralizer theorem

We consider a ring A and a semisimple A-mod M. Take $E = \mathrm{End}_A(M)$, and note that

a) M is naturally an E-module.

b) The left action of A on M provides a ring map $\phi_M : A \longrightarrow End_E(M)$. (*)

We claim that under advantageous circumstances that map (*) is surjective, or even an isomorphism.

Theorem (Double centralizer theorem): For any Artinian ring A, finitely generated semisimple A-mod M, and $E = End_A(M)$, the natural map
$$A \to End_E(M)$$
is surjective. Furthermore M is fin gen'd over E.

We approach the result in steps.

Lemma 8: Consider any semisimple A-module M, for general A, and take $E = End_A(M)$. For any $m \in M$ and $f \in End_E(M)$ there exists $a \in A$ for which $f(m) = a \cdot m$.

Proof: Take $M_0 = A \cdot m$ and complementary $M_1 \subseteq M$ so that $M = M_0 \oplus M_1$ [Thm 10, Ssmodls]. Hence for $e \in E$ the composite
$$e = \{ M \xrightarrow{\pi_0} M_0 = A \cdot m \xrightarrow{i_0} M$$
we have
$$f(m) = f(e \cdot m) = e \cdot f(m) \in A \cdot m.$$
We obtain the result. ∎

Lemma (Jacobson Density): Take any ring A, semisimple A-module M, and $E = \mathrm{End}_A(M)$. For any $f \in \mathrm{End}_E(M)$ and finite collection $m_1, \ldots, m_r \in M$, there exists $a \in A$ for which $f(m_i) = a \cdot m_i$ at all i.

Proof: Consider $M^{\oplus r}$ so that $\mathrm{End}_A(M^{\oplus r}) = M_r(E)$. We have for

$$f_r = \begin{bmatrix} f & & 0 \\ & \ddots & \\ 0 & & f \end{bmatrix} : M^{\oplus r} \to M^{\oplus r}$$

an element $a \in A$ with

$$(f(m_1), \ldots, f(m_r)) = f_r(m_1, \ldots, m_r) = a \cdot (m_1, \ldots, m_r)$$
$$= (a m_1, \ldots, a \cdot m_r),$$

by the previous Lemma. ∎

Sketch Proof of the double centralizer: Recall that in this setting E is semisimple [Proposition 6, Maschkaw]. Hence every E-module is projective (even co-generated) and any inclusion of a E-submodule $M_0 \leq M$ splits over E. This implies that the restriction maps

$$\mathrm{End}_E(M) \to \mathrm{Hom}_E(M_0, M)$$

is a surjective map of left $\mathrm{End}_E(M)$-mods, and those of left E-mods. Furthermore Jac. density tells us that the sequence

$$A \longrightarrow \mathrm{End}_E(M) \qquad (*)$$
$$\longrightarrow\!\!\!\!\rightarrow \mathrm{Hom}_E(M_0, M)$$

is a surjective map of left A-modules whenever M_0 is finitely gen'd over E.

This tells us, in particular, that

$$A \to \mathrm{End}_E(M)$$

is surjective whenever M is finite over E.

Supposing M is <u>not</u> finitely gen'd over E we can produce an ∞-ascending sequence

$$M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subseteq M$$

of finitely gen'd E-submodules which produce an ∞-descend sequence of left A-submods

$$A \supsetneq K_0 \supsetneq K_1 \supsetneq K_2 \supsetneq \cdots$$

for

$$K_i = \mathrm{Ker}(A \to \mathrm{End}_E(M) \to \mathrm{Hom}_E(M_i, M)).$$

This contradicts Artinian-ness of A, and hence never occurs. We conclude that M is in fact finite over E in this case. ∎

— III. Consequences of double centralizing

Def^{n}: Call a ring A <u>simple</u> if A has no ideals other than 0 and A itself.

Corollary 9: If A is simple and Artinian then A has a single simple module, up to isomorphism, and $A \cong M_n(D)$ for some division ring D.

Proof: Take any simple module L. Then E is a division algebra $L \cong E^n$ for some n by double centralizer thm, and we have the isom

$$\mathrm{End}_E(L) \cong M_n(\mathrm{End}_E(E)) = M_n(D)$$

where $D = E^{op}$.

Via simplicity of A the action map

$$A \to \mathrm{End}_E(L) \cong M_n(D).$$

is injective, and thus a ring isom by double centralizer. The fact that $M_n(D)$ has a unique simple module now tells us that A has a unique simple module. ∎

Corollary 10: For any maximal ideal $m \subseteq A$ in Artinian A, A/m is a semisimple ring and a semisimple A-module.

Corollary 11: For Artinian A and simple L, $\mathrm{Ann}_A(L) \subseteq A$ is a maximal ideal.

Proof: The Annihilator is the kernel of the action map $A \to \mathrm{End}_E(L) \cong M_n(D)$, giving
$$A / \mathrm{Ann}_A(L) \cong M_n(D).$$ ∎

Corollary 12: An Artinian ring A is semisimple if and only if A admits a faithful, finitely generated semisimple module M.

Proof: We have the injective map
$$A \to \mathrm{End}_E(M) \subseteq \mathrm{End}_{\mathbb{Z}}(M)$$
which is furthermore surjective, by Double Centralizer, and thus an isomorphism onto the semisimple alg $\mathrm{End}_E(M) \cong \prod_{i=1}^{t} M_{n_i}(D_i)$. ∎

IV. Radical v. Radical

Theorem 12: For any Artinian ring A, $A/\mathrm{Jac}(A)$ is semisimple, and in fact
$$\mathrm{Jac}(A) = \mathrm{Rad}(A).$$

Proof: By Artinian-ness there is a finite collection of maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_r \subseteq A$ for which

$$\mathrm{Jac}(A) = \mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_r,$$

so that $\mathrm{Jac}(A)$ is the kernel of the corresponding algebra / A-module map

$$A \to \prod_{i=1}^{r} A/\mathfrak{m}_i .$$

By Corollary 10, each $A/\mathfrak{m}_i$ is semisimple as a module over itself, and hence as a module over A as well. We therefore obtain $A/\mathrm{Jac}(A)$ as a submodule of the semisimple module $\prod_i A/\mathfrak{m}_i$, giving semisimplicity of $A/\mathrm{Jac}(A)$ by [Prop 9, Smoke].

By semisimplicity of $A/\mathrm{Jac}(A)$ the surjection $A \twoheadrightarrow A/\mathrm{Jac}(A)$ has $\mathrm{Rad}(A)$ in its kernel [Lemma 13, Smoke]. Thus

$$\mathrm{Rad}(A) \subseteq \mathrm{Jac}(A).$$

For the opposite inclusion, it suffices to show that every surjection onto a simple $\pi : A \to L$ has $\mathrm{Jac}(A) \subseteq \ker(\pi)$. However we have

$$\mathrm{Ann}_A(L) \subseteq \ker(\pi) \text{ with } \mathrm{Ann}_A(L) \text{ maximal}$$

by Corollary 11. Thus

$$\mathrm{Jac}(A) \subseteq \mathrm{Ann}_A(L) \subseteq \ker(\pi)$$ ∎

as desired, and we obtain an equality $\mathrm{Jac}(A) = \mathrm{Rad}(A)$.

Corollary 13: When A is Artinian, the module theoretic radical $\mathrm{Rad}(A)$ is a two sided ideal.

Corollary 14: For $\mathrm{Rad}'(A)$ = the intersection of all maximal right submodules, we have

$$\mathrm{Rad}(A) = \mathrm{Jac}(A) = \mathrm{Rad}'(A).$$

Proof: We have

$$\mathrm{Rad}'(A) = \mathrm{Rad}(A^{op}) = \mathrm{Jac}(A^{op}) = \mathrm{Jac}(A)$$
$$= \mathrm{Rad}(A). \quad \blacksquare$$

§V An alternate characterization of $\mathrm{Jac}(A)$

Theorem 15: For any ring A and $x \in A$ the following are equivalent

a) $x \in \mathrm{Jac}(A)$

b) For each $a, b \in A$, the element $1 + axb$ is a unit in A.

To help with the proof, we record a simple lemma

Lemma 16: For any fin gen'd A-module M, and proper submodule $M' \subseteq M$, there is a maximal submodule M_{max} with $M' \subseteq M_{max} \subseteq M$.

Proof: After specifying a min gen set $\{m_1, \ldots, m_r\}$

M, proper submodules $N \subseteq M$ are precisely those submodules with $m_i \notin N$ at some i. Hence any totally ordered sequence of proper submodules $\{M_\lambda : \lambda \in \Lambda\}$ with $M' \subseteq M_\lambda \subsetneq M$ at all λ we can find some common m_i with $m_i \notin M_\lambda$ at each λ. Hence $m_i \notin \bigcup_\lambda M_\lambda$ and we obtain an upper bound among the collection of proper submodules in M which contain M'. By Zorn it follows that this collection contains a maximal element

$$M' \subseteq M_{max} \subsetneq M. \quad \blacksquare$$

Proof of Thm 15: If $x \in Jac(A)$ then for each $a, b \in A$, $y = axb \in Jac(A)$ and we consider the endomorphism

$$-\cdot(1+y) : A \to A.$$

If $1+y$ has no left inverse then $M_y = A\cdot(1+y)$ is a proper left submodule in A and we can choose max left submod $M_y \subseteq M_{max} \subseteq A$ giving

$$\overline{(1+y)} = 0 \text{ in } A/M_{max} \Rightarrow \bar{y} = \bar{1}$$

in A/M_{max}. But $y \in M_{max}$ by the equality $Jac(A) = Rad(A)$, so that $\bar{1} = \bar{y} = 0$ in the quotient giving $A/M_{max} = 0$, a contradiction. So we see

$(1+axb)$ has a left inverse whenever x is in $\mathrm{Jac}(A)$. Repeating the argument with the right module map

$$(1+y)\cdot - : A \to A,$$

and employing the right radical $\mathrm{Rad}'(A)$, with Corollary 14, we see that $(1+y) = (1+axb)$ admits a right inverse as well.

Conversely, if $x \notin \mathrm{Jac}(A)$, take $M \subseteq A$ max sub-module with $x \notin M$. Then, since A/M is simple, we can find $a \in A$ with $1 - a\cdot x = 0 \mod M$. Then $1-ax$ admits <u>no</u> left inverse in A, since otherwise we can find u with

$$1 = u\cdot(1-ax) = 0 \mod M \Rightarrow M = A,$$

a contradiction. ∎

~ VI Nakayama!

Theorem (Nakayama Lemma): For any ring A, and finitely generated A-module M, we have

$$\mathrm{Jac}(A)\cdot M = M$$

if and only if $M = 0$.

Proof: Suppose $M \neq 0$ and let $m_1, \dots, m_r$ be a generating set of minimal size. If $M = \mathrm{Jac}(A) \cdot M$ then we have

$$\sum_{i=1}^{r} m_i = \sum_{i=1}^{r} x_i m_i \quad \text{for some } x_i \in \mathrm{Jac}(A),$$

giving

$$\sum_{i=1}^{r} (1 - x_i) m_i = 0.$$

By Theorem 15 we find now $u_1 \in A$ with $u \cdot (1 - x_1) = 1$ giving

$$0 = u \cdot \left(\sum_{i=1}^{r} (1 - x_i) m_i \right) = m_1 + \sum_{j \neq 1} (u - u x_j) m_j$$

$$\Rightarrow \quad m_1 = \sum_{j \neq 1} (u x_j - u) \cdot m_j,$$

contradicting minimality of our generating set.

We conclude that $\mathrm{Jac}(A) \cdot M \subsetneq M$ whenever M is finitely generated and nonzero. ∎

Corollary 17: For any map of finitely generated A-modules $f: M \to N$, f is surjective if and only if the induced map

$$\bar{f}: M / \mathrm{Jac}(A) \cdot M \to N / \mathrm{Jac}(A) \cdot N$$

is surjective.

Proof: Exercise. [HW] ∎

Anti-Example: Consider the ring of power series

$A = \mathbb{C}[\![x]\!]$ and Laurent series $\mathbb{C}((x))$,

$\mathbb{C}((x)) = \{ \sum'_{n \in \mathbb{Z}} c_n x^n : c_{-N} = 0$ at all suff large $N \}$.

One can show that

a) $\mathbb{C}[\![x]\!]$ has a unique max ideal $\mathfrak{m} = (x)$

b) $\mathbb{C}((x))$ is a field.

Thus $\mathrm{Jac}(\mathbb{C}[\![x]\!]) = \mathfrak{m}$.

We have the ring inclusion $\mathbb{C}[\![x]\!] \to \mathbb{C}((x))$ realizing $\mathbb{C}((x))$ as a (non-fin gen'd) module over $\mathbb{C}[\![x]\!]$. Since all elem in $\mathbb{C}[\![x]\!]$ become invertible in $\mathbb{C}((x))$ we have

$$\mathrm{Jac}(\mathbb{C}[\![x]\!]) \cdot \mathbb{C}((x)) = \mathbb{C}((x)),$$

even though $\mathbb{C}((x))$ is nonzero. So we see the fin. gen. hypothesis in Nakayama is <u>necessary</u> at general A.

~ VII Nilpotence of Jacobson radical

Theorem 18: For any Artinian ring A, the Jacobson radical $\mathrm{Jac}(A)$ is nilpotent. In particular, $\mathrm{Jac}(A)$ is the (unique) maximal nilpotent ideal in A.

Proof: By Nakayama, $Jac(A)^{n+1}$ is properly contained in $Jac(A)^n$ at all n, unless $Jac(A)^n=0$. However, by Artinianness there exists n at which the descending sequence of ideals (and thus submodules)

$$A \supseteq Jac(A) \supseteq Jac(A)^2 \supseteq \cdots$$

stabilizes. This forces $Jac(A)^n = 0$ at large n. ∎

This essentially immediately implies Noetherianity of A, after we note the following.

Lemma 19: A module M over a semisimple ring is Artinian if and only if, it is Noetherian, if and only if M is a finite sum of simple submodules, if and only if M is finitely gen'd.

Proof: If M is fin gen'd then it's both Artinian and Noetherian, since any semisimple ring is both Art. and Noeth. If M is not fin gen'd then, by a Zorn argument, M admits an embedding from an ∞-sum of simples $\bigoplus_{i \in \mathbb{Z}_{>0}} L_i \hookrightarrow M$. Such an ∞ sum is easily seen to neither be Artinian nor Noetherian, which implies M is neither Artinian nor Noetherian. [Prop 9, Friday]. ∎

Corollary 20: Any Artinian ring is also Noetherian, and hence a fin. length module over itself.

Proof: We have the sequence of submodules

$$A \supseteq \mathrm{Jac}(A) \supseteq \mathrm{Jac}(A)^2 \supseteq \cdots \supseteq \mathrm{Jac}(A)^{n+1} = 0$$

with each subquotient $\mathrm{Jac}(A)^m / \mathrm{Jac}(A)^{m+1}$ having A-action factoring through the surjection

$$A \to A/\mathrm{Jac}(A)$$

onto the semisimple alg $A/\mathrm{Jac}(A)$ (Theorem 12). Since each subquotient is Artinian, Lemma 19 tells us that each subquotient is Noetherian as well.

Take $M_r = \mathrm{Jac}(A)^r / \mathrm{Jac}(A)^{r-1}$. Use the sequence of extensions

$$0 \to \mathrm{Jac}(A)^n \to \mathrm{Jac}(A)^{n-1} \to M_{n-1} \to 0$$

$$0 \to \mathrm{Jac}(A)^{n-1} \to \mathrm{Jac}(A)^{n-2} \to M_{n-2} \to 0$$

$$\vdots$$

$$0 \to \mathrm{Jac}(A) \to A \to M_0 \to 0$$

Noetherianity of the M_r, and $M_n = \mathrm{Jac}(A)^n$, implies Noetherianity of A. ∎

HW

1. For any Artinian ring A, and arbitrary (non-finitely generated) M, prove that

$$\mathrm{Jac}(A)\cdot M = M$$

if and only if $M = 0$.

2. For any map of rings $A \to B$, prove that the base change functor $B \otimes_A - : A\text{-mod} \to B\text{-mod}$ is right exact. Rather, prove that any exact sequence of A-modules $0 \to M' \xrightarrow{i} N \to M \to 0$ induces an exact sequence

$$B \otimes_A M' \to B \otimes_A N \to B \otimes_A M \to 0.$$

Prove, by way of examples, that $\hat{B \otimes_A i} : B \otimes_A M' \to B \otimes N$ can be non-injective for injective $i : M' \to N$.

2. For a general ring A, and map between finitely generated A-modules $f : M \to N$, prove that f is surjective if and only if the induced map

$$\bar{f} : M / \mathrm{Jac}(A)\cdot M \to N / \mathrm{Jac}(A)\cdot N$$

is surjective.

4. Suppose A is Artinian, and consider any f.g. A-module M. Prove that M is semisimple if and only if $\mathrm{Jac}(A)\cdot M = 0$.

5. Prove that restricting along the projection map provides a fully faithful functor

$$A/\mathrm{Jac}(A)\text{-mod} \to A\text{-mod}$$

which is an equivalence onto the subcategory of all semisimple A-modules.

6. Prove that every simple S_3-rep over $\overline{\mathbb{F}_3}$ is of dimension 1. Classify all such simples.
[Hint: You know S_3 has at least two 1-dim simples, and you know something about $\dim \overline{\mathbb{F}_p} S_3$ Jac.]

7. Describe the Jacobson radical $\mathrm{Jac}(\overline{\mathbb{F}_3} S_3)$. Provide, at least, its dimension.