Algs via gen's and rel's 1

I Free algebras and presentations

Given a finite set $\{x_1, \dots, x_n\} = X$ and a comm ring k, we let

$$k\langle x_1, \dots, x_n\rangle$$

denote the free k-module spanned by words in the alphabet X

$$k\langle x_1, \dots, x_n\rangle = \bigoplus_{w \in \mathcal{W}_X} k\cdot w,$$

where $\mathcal{W}_X = \left\{ \begin{array}{l} \text{the set of (non commuting)} \\ \text{words } x_{i_1} x_{i_2} \cdots x_{i_r} \text{ in} \\ \text{the } x_i \end{array} \right\}$

$$= \coprod_{r \geq 0} \mathrm{Hom}_{Set}(\{1, \dots, r\}, X).$$

We endow this k-module with the assoc. product provided by concatenation of words

$$\left(\sum_{\vec{i}}{}' c_{\vec{i}}\, x_{i_1} \cdots x_{i_r}\right) \cdot \left(\sum_{\vec{j}}{}' c_{\vec{j}}\, x_{j_1} \cdots x_{j_t}\right)$$

$$= \sum_{\vec{i}, \vec{j}}{}' c_{\vec{i}} \cdot c_{\vec{j}}\, x_{i_1} \cdots x_{i_r} x_{j_1} \cdots x_{j_t}.$$

The inclusion

$$k \to k\langle x_1, \dots, x_n\rangle, \quad c \longmapsto c \cdot \underbrace{\text{empty word}}_{1}$$

gives $k\langle x_1, \dots, x_n\rangle$ the structure of a k-algebra.

We furthermore have the inclusion of sets

$$i_X : X \to k\langle x_1, \dots, x_n\rangle, \quad x_i \longmapsto x_i$$

Theorem 1: For any k-alg A, restricting along the inclusion i_X provides a bijection

$$i_X^*: \mathrm{Hom}_{\mathrm{Alg}_k}(k\langle x_1,\dots,x_n\rangle, A) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Set}}(X, A).$$

Proof: Since $k\langle x_1,\dots,x_n\rangle$ is gen'd by the x_i, as an algebra, i_X^* is injective. For surjectivity, any set map $\bar{\varphi}: X \to A$ extends to an alg. map $\varphi: k\langle x_1,\dots,x_n\rangle \to A$ defined by)

$$\varphi(c\, 1_{k\langle x_1,\dots,x_n\rangle}) = c\, 1_A$$

$$\varphi\Big(\sum_{\vec{i}} c_{\vec{i}}\, x_{i_1}\cdots x_{i_r}\Big) := \sum_{\vec{i}} c_{\vec{i}}\, \bar{\varphi}(x_{i_1})\cdots\bar{\varphi}(x_{i_r})$$

We note that φ is a k-module map, via freeness of $k\langle x_1,\dots,x_n\rangle$ as a k-module, and φ is mult just by distributivity, and k-bilinearity, of the product on A. ∎

Corollary 2: For any set of "relations" (any subset) $R \subseteq k\langle x_1,\dots,x_n\rangle$,

$$R = \{ r_\lambda(x_1,\dots,x_n) : \lambda \in \Lambda \},$$

and ideal (R) in $k\langle x_1,\dots,x_n\rangle$ gen'd by R, restricting along the map $i_X: X \to k\langle x_1,\dots,x_n\rangle/(R)$ provides a bijection

$$\eta_X^*: \cancel{\mathrm{Hom}_{\mathrm{Alg}_k}(k\langle x_1,\dots,x_n\rangle/(R), A)} \xrightarrow{\sim} \left\{\begin{array}{l}\text{choices of elem}\\ a_1,\dots,a_n \text{ in } A\\ \text{which satisfy}\\ r_\lambda(a_1,\dots,a_n)=0 \text{ in } A\\ \text{for all } \lambda\in\Lambda.\end{array}\right\}$$

Def^n: A presentation of a k-algebra A is a choice of elements $a_1, \dots, a_n \in A$ and subset $\{r_\lambda(x_1,\dots,x_n) : \lambda\in\Lambda\} \subseteq k\langle x_1,\dots,x_n\rangle$ for which

a) $r_\lambda(a_1,\dots,a_n)=0$ for all $\lambda\in\Lambda$.

b) The induced alg map

$$\frac{k\langle x_1,\dots,x_n\rangle}{(r_\lambda(x_1,\dots,x_n) : \lambda\in\Lambda)} \to A, \quad x_i \mapsto a_i,$$

is an isomorphism.

Example: $\cancel{\varphi}: $

$$k\langle x,y\rangle \Big/ (x^3-1, y^2-1, yxy-x^2) \xrightarrow{\sim} kS_3.$$

$x \mapsto (123)$, $y \mapsto (12)$.

Indeed $(123)^3 = (12)^2 = 1$, and

$$(12)(123)(12) = (213) = (123)^2,$$

and $(123)^3 = (12)^2 = 1$. So we have such an alg map, and since S_3 is gen^d by (123), (12) as a group kS_3 is gen^d by these elem as a k-alg. Thus φ is surjective.

For injectivity, it suffices to prove

$\dim_k k\langle x,y\rangle/\text{rels} \le \dim_k kS_3 = 6.$

We have in $k\langle x,y\rangle/\text{rels}$ $y^2=1$ $x^3=1$

$\Rightarrow y^{-1}=y$ and $x^{-1}=x^2$ and

$xy = yx^2$

$\Rightarrow$ all noncommuting words $x^{m_1}y^{m_2}\cdots x^{m_{t-1}}y^{m_t}$

are identified w/ the ordered word

$$x^{\sum_i m_{2i+1}}\, y^{\sum_i m_{2(i+1)}} = x^r y^s$$

w/ $r \le 2$ and $s \le 1$. Thus

$k\langle x,y\rangle/\text{rels}$ has a spanning set

$\underbrace{1,\ x,\ x^2,\ y,\ xy,\ x^2y}_{6}$

$\Rightarrow \dim_k k\langle x,y\rangle/\text{rels} \le 6 \Rightarrow \phi$ iso.

Example: Polynomial ring

$k\langle x_1,\dots,x_n\rangle/(x_ix_j - x_jx_i : 1\le i,j\le n) \xrightarrow{\sim} k[x_1,\dots,x_n].$

Example: Exterior alg ($2^{-1}\in k$)

$k\langle x_1,\dots,x_n\rangle/(x_ix_j + x_jx_i : 1\le i,j\le n) \xrightarrow{\sim} \Lambda(x_1,\dots,x_n)$

comm rel $x_ix_j = -x_jx_i$, $x_i^2=0$, $\dim 2^n$.

Example (Skew poly) $q_{ij} \in k^\times$, $q_{ji} = q_{ij}^{-1}$

$$k\langle x_1, \ldots, x_n\rangle / (x_i x_j - q_{ij} x_j x_i : 1 \le i < j \le n) = k_q[x_1, \ldots, x_n].$$

Has comm rel, $x_i x_j = q_{ij} x_j x_i$, basis of ordered monom. Is f.g. mod over its center if and only if all q_{ij} are roots of unity.

~ # Basic free construction (k a field)

For k a field and V a vect space over k, def

$$T(V) = \bigoplus_{n \ge 0} V^{\otimes_k n}$$ w/ unique distributive mult prov. by the univ. bilinear maps

$$(V^{\otimes n_1}) \times (V^{\otimes n_2}) \xrightarrow{\sim} V^{\otimes n_1 + n_2},$$

$$(v_1 \otimes \cdots \otimes v_{n_1}, w_1 \otimes \cdots \otimes w_{n_2}) \longmapsto v_1 \otimes \cdots \otimes v_{n_1} \otimes w_1 \otimes \cdots \otimes w_{n_2}.$$

For any k-alg A and linear map $\bar{\phi} : V \to A$ we obtain the uniquely def alg map

$$\phi : T(V) \to A \quad \text{w/} \quad \phi(v) = \bar{\phi}(v)$$

for all $v \in V$. Under shorthand

$$v_1 \cdots v_n := v_1 \otimes \cdots \otimes v_n, \quad \phi(v_1 \cdots v_n) = \bar{\phi}(v_1) \cdots \bar{\phi}(v_n).$$

So we derive a bijection univ. res. to the gens

$$\operatorname{Hom}_{Alg_k}(T(V), A) \xrightarrow{\sim} \operatorname{Hom}_k(V, A)$$

Lemma 3: For any choice of basis $x_1, \ldots, x_n$ for fin dim V, we have an alg $\cong$

$$\varphi : k\langle x_1, \ldots, x_n\rangle \xrightarrow{\cong} T(V), \quad x_i \longmapsto x_i.$$

Proof: You have such an alg map via univ prop for $k\langle x_1, \ldots, x_n\rangle$ and inv. provided by the alg map $T(V) \to k\langle x_1, \ldots, x_n\rangle$ specified by the inclusion $V \cong k\{x_1, \ldots, x_n\} \xrightarrow{\mathrm{incl}} k\{x_1, \ldots, x_n\} \subseteq k\langle x_1, \ldots, x_n\rangle$. ∎

Example: Basic free polynomials

$$\mathrm{Sym}(V) := T(V)/(vw - wv : v, w \in V)$$

Example: Basic free ext alg ($2^{-1} \in k$)

$$\Lambda(V) = T(V)/(vw + wv : v, w \in V)$$

Example: For $sl_n \subseteq gl_n(V) = \mathrm{End}_k(V)$ the subspace of traceless matrices, have

$[a, b] \in sl_n$ whenever $a, b \in sl_n$ (or gl_n).

Def: $\mathcal{U}(sl_n) := T(sl_n)/(xy - yx - [x,y]_{sl_n} : x, y \in sl_n)$

Example: $\mathcal{U}_q(sl_2) = k\langle E, F, K, K^{-1}\rangle / \mathrm{rels}$, $q \in k^\times$

$$\mathrm{rels} = \begin{cases} KK^{-1} - 1, & KEK^{-1} - q^2 E \\ KFK^{-1} - q^{-2}F, & EF - FE - \dfrac{K - K^{-1}}{q - q^{-1}}. \end{cases}$$

When q is finite order, $l = \mathrm{ord}(q^2)$ have "small" quantum group

$$\underbrace{u_q(sl_2)}_{\text{fin dim}} = U_q(sl_2) \Big/ (E^l, F^l, K^l - 1).$$

(Used for link invariants, tqft, CFT.)

Example (Symplectic Ref. alg) $/\mathbb{C}$ $G \circlearrowright V$

$\Rightarrow G \circlearrowright T(V)$ via the unique alg auts ϕ_g w/ $\phi_g(v) = g \cdot v$ on V.

For a G-inv. symplectic form $\beta : V \otimes_{\mathbb{C}} V \to \mathbb{C}$, $t \in \mathbb{C}$ and c: some mystery function take

$$H_{t,c}(G) := T(V^*) \rtimes G \Big/ \left(vw - wv - t\beta(v,w) - 2\sum_{s \in S} c(s)\, \beta_s(v,w) \cdot s : v, w \in V^* \right)$$

Has to do w/ resolving the singularities in the quotient V/G.

Other examples: Hecke Algebras (# theory), Path/Quiver algs (AG), Diagrammatic Algs e.g. Temperley Lieb (integrality), Operator Algs (CFT), differential operator algs (formal diff eqs in AG), etc.

HW

1. Let G be a finite group with a finite presentation

$$G \cong \langle g_1, \ldots, g_n \mid r_1(g_1, \ldots, g_n) = r_1'(g_1, \ldots, g_n), \ldots, r_t(g_1, \ldots, g_n) = r_t'(g_1, \ldots, g_n) \rangle.$$

Prove that the group ring kG admits a presentation

$$k\langle x_1, \ldots, x_n \rangle / \text{rels}, \quad \text{rels} = \begin{cases} x_i^{\operatorname{ord}(g_i)} - 1 & i = 1, \ldots, n \\ r_j(x_1, \ldots, x_n) - r_j'(x_1, \ldots, x_n) & j = 1, \ldots, t. \end{cases}$$