KERODON REMIX PART II: COCARTESIAN FIBRATIONS, LIMITS AND COLIMITS, AND YONEDA EMBEDDING [IN PREPARATION]

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Abstract. These are notes on ∞-categories which are (mostly) adapted from Lurie's digital text Kerodon [5]. Following Part I, which presents the basic foundations for studies of ∞-categories, we discuss cartesian and cocartesian fibrations, transport functors (i.e. Grothendeick straightening and unstraightening), limits and colimits, and Yoneda embedding. Specific topics include Hom functions for simplicial and dg categories, and calculations of limits and colimits of Kan complexes and ∞-categories.

In comparing with Part I, we omit more details in our treatment. This is especially the case when it comes to our discussions of transport and our arguments for completeness/cocompleteness of $\mathscr{C}at_\infty$. In our subsequent installment, Part III, we discuss the derived category as a stable and presentable ∞-category.

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1. Introduction to cartesian and cocartesian fibrations

[This document is not edited, nor is it complete, though many sections exist in a quasi-complete state. I've only made it publicly available at this point as the text is referenced (with frequency) in “Kerodon remix part III”, which concerns derived ∞-categories. According to the authors’ calculations, a functional version of the text should be available in Fall 2025.]

1.1. **Definitions.**

Definition 1.1. Consider a map of simplicial sets $q : X \to S$. We call a 1-simplex $\alpha : x \to y$ in X a q-cartesian morphism if any lifting problem

$$\begin{tikzcd} \Lambda^n_n \arrow[r, "\bar{\sigma}"] \arrow[d] & X \arrow[d, "q"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & S \end{tikzcd} \tag{1}$$ eq:121

with $n \geq 2$ and $\bar{\sigma}|\Delta^{\{n-1,n\}} = \alpha$ admits a solution. We say $\alpha : x \to y$ is q-cocartesian if any lifting problem

$$\begin{tikzcd} \Lambda^n_0 \arrow[r, "\bar{\tau}"] \arrow[d] & X \arrow[d, "q"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & S \end{tikzcd} \tag{2}$$ eq:128

with $n \geq 2$ and $\bar{\tau}|\Delta^{\{0,1\}} = \alpha$ admits a solution.

Though at some specific moments we will consider a case where X and S are not ∞-categories, we are primarily invested in the ∞-categorical setting.

Definition 1.2. We call a map of ∞-categories $q : \mathscr{C} \to \mathscr{D}$ a cartesian fibration if q is an inner fibration and, for any map $\bar{\alpha} : \bar{x} \to \bar{y}$ in $\mathscr{D}$ and y in $\mathscr{C}$ with $q(y) = \bar{y}$, there is a q-cartesian map $\alpha : x \to y$ in $\mathscr{C}$ with $q(\alpha) = \bar{\alpha}$.

Similarly, we call q a cocartesian fibration if it is an inner fibration and, for any map $\bar{\beta} : \bar{x} \to \bar{y}$ in $\mathscr{D}$ and x with $q(x) = \bar{x}$, there is a q-cocartesian fibration $\beta : x \to y$ with $q(\beta) = \bar{\beta}$.

The following is obvious. Recall our definitions of right and left fibrations from Definition I-5.23.

Proposition 1.3. *If $q : \mathscr{C} \to \mathscr{D}$ is a right fibration (resp. left fibration) then q is a cartesian (resp. cocartesian).*

Proof. In this case any lifting problem of the form (1), or (2) respectively, admits a solution simply by the defintion. □

Obviously when $q : \mathscr{C} \to \mathscr{D}$ is a Kan fibration it is both cartesian and cocartesian.

Example 1.4. Consider a diagram $p : K \to \mathscr{C}$, with K some simplicial set. The we have the overcategory $\mathscr{C}_{/p}$ and the undercategory $\mathscr{C}_{p/}$. The two forgetful functors

$$\mathscr{C}_{/p} \to \mathscr{C} \quad \text{and} \quad \mathscr{C}_{p/} \to \mathscr{C}$$

are, respectively, a right and left fibration Proposition I-5.25. Hence these maps are respectively a cartesian and cocartesian fibration.

In the case where K is a point $x : * \to \mathscr{C}$ we recall that the fibers of the fibration $\mathscr{C}_{/x} \to \mathscr{C}$ and $\mathscr{C}_{x/} \to \mathscr{C}$ are the right and left pinched mapping spaces $\mathrm{Hom}^{\mathrm{R}}_{\mathscr{C}}(w, x)$ and $\mathrm{Hom}^{\mathrm{L}}_{\mathscr{C}}(x, y)$.

Example 1.5 ([5, 01T8])**.** Consider an ∞-category $q : \mathscr{C} \to *$. A morphism $\alpha : x \to y$ is q-cartesian if and only if α is an isomorphism. To see this consider fillings of the horns

$$\begin{tikzcd} & x \arrow[dr, "\alpha"] & \\ y \arrow[ur, dotted, "\exists\beta"] \arrow[rr, "id"'] & & y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & y \arrow[dr, "\beta"] \arrow[dd, "id"] & \\ x \arrow[ur, "\alpha"] \arrow[rr, "id"] \arrow[dr, "\alpha"'] & & x. \arrow[dl, "\alpha"] \\ & y & \end{tikzcd}$$

One similarly finds that a morphism is q-cocartesian if and only if it is an isomorphism.

Via the existence of identity morphisms the structure map $q : \mathscr{C} \to *$ is always a cartesian and cocartesian fibration. Note that this map is not a left or right fibration unless $\mathscr{C}$ is a Kan fibration.

1.2. **Imaginings: Cartesian fibrations as lax moduli.** Give a cartesian fibration $q : \mathscr{C} \to \mathscr{D}$ one might think of $\mathscr{C}$ as a lax moduli of "stuff" varying over the objects in $\mathscr{D}$. The cartesian lifts of morphisms in $\mathscr{D}$ provide transition functions between these fibers, i.e. the stuff we are parametrizing, over $\mathscr{D}$. In the case of the cartesian fibration $\mathscr{C}_{/x} \to \mathscr{C}$ the category $\mathscr{C}_{/x}$ is, in an obvious sense, the "moduli of maps to x". Let us leave the latter point about lifting maps for now, and try to make some comment on the moduli point.

Let us just consider how one classically constructs a moduli space. Here we consider the base space $\mathscr{D} = \mathrm{Sch}_k$ of schemes over k, which we can endow with some Grothendieck topology if we like, though we don't care at the moment. Then a pre-stack is a choice of a functor of plain categories $q : \mathbb{M} \to \mathrm{Sch}_k$ which makes $\mathbb{M}$ into a category fibered in groupoids over Sch_k [6, 003S]. One simply compares definitions to see that

$$\left\{ \begin{array}{c} \mathbb{M} \text{ is fibered in} \\ \text{groupoids over } \mathrm{Sch}_k \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} q \text{ is a cartesian fibration} \\ \text{in which all maps in } \mathbb{M} \\ \text{are } q\text{-cartesian} \end{array} \right\}.$$

In this familiar setting one can now "invert" this functor q to produce an associated 2-functor

$$q^\vee : (\mathrm{Sch}_k)^{\mathrm{op}} \to \mathrm{Groupoids} \subseteq \mathrm{Cat}, \quad Y \mapsto \mathbb{M}_Y.$$

One establishes this functor via an abuse of the axion of choice.

To elaborate a bit more, for any map of schemes $\alpha : X \to Y$ we take a lift $\alpha^* y \to y$ in $\mathbb{M}$. This lift is unique up to unique isomorphism, and via unique filling defines a functor between the fibers

$$\alpha^* : \mathbb{M}_Y \to \mathbb{M}_X, \quad y \mapsto \alpha^* y.$$

On morphisms $\xi : y_1 \to y_2$ in the fiber $\mathbb{M}_Y$, we note that the cartesian property for maps in $\mathbb{M}$ asserts the existence of a unique map $\alpha^*\xi : \alpha^* y_1 \to \alpha^* y_2$ completing the diagram

$$\begin{tikzcd} \alpha^* y_1 \arrow[r] \arrow[d, dashed, "\exists !"'] & y_1 \arrow[d, "\xi"] \\ \alpha^* y_2 \arrow[r] & y_2, \end{tikzcd}$$

where we note that uniqueness comes from filling the appropriate 3-simplex in $\mathbb{M}$. Hence α^* is well-defined on morphisms via the assignment $\xi \mapsto \alpha^*\xi$.

We note that this inversion of $q : \mathbb{M} \to \mathrm{Sch}_k$ into a functor $q^\vee : (\mathrm{Sch}_k)^{\mathrm{op}} \to \mathrm{Cat}$ does not require all maps in $\mathbb{M}$ to be cartesian. This is simply a consequence of q being a cartesian fibration between plain categories.

In the general ∞-context, we again have this inversion property for (co)cartesian fibrations. Here a cartesian fibration $q : \mathscr{C} \to \mathscr{D}$ will specify, and be specified by, a functor

$$q^\vee : \mathscr{D}^{\mathrm{op}} \to \mathscr{C}at_\infty$$

whose values over objects y in $\mathscr{D}$ are the fibers $\mathscr{C}_y$. The functors between fibers $\alpha^* : \mathscr{C}_y \to \mathscr{C}_x$ are what we've referred to as *transport* along α (following Kerodon [5]).

While such fibrations play an essentially non-existent role in plain category theory, from the perspective of the working mathematician, they play an extraordinarily important role in the development of ∞-category theory. The main point is that they tame choices in the ∞-categorical setting. While in the plain category setting we can simply make a choice, and if that choice is not unique we can simply say it's unique up to a unique isomorphism, and then if I make two of the same types of choices then any ambiguities will vanish due to sufficient uniqueness, etc. etc., such a laissez faire attitude will lead to immediate intractable problems in the ∞-context. So one generally bundles all choices of a certain "type" into a cartesian or cocartesian fibrations, and manipulates these bundles in order to make global movements between choices of different types.

1.3. **Discussion: Classifying functors etc.**

2. Cartesian and cocartesian fibrations

2.1. **Cartesian morphisms via overcategories.**

prop:232

Proposition 2.1 ([5, 01TF]). *Let $q : \mathscr{C} \to \mathscr{D}$ be a map between ∞-categories. A morphism $\alpha : x \to y$ in $\mathscr{C}$ is q-cartesian if and only if the natural map*

$$\mathscr{C}_{/\alpha} \to \mathscr{C}_{/y} \times_{\mathscr{D}_{/q(y)}} \mathscr{D}_{/q(\alpha)}$$

is a trivial Kan fibration. Similarly, α is q-cocartesian if and only if the map

$$\mathscr{C}_{/\alpha} \to \mathscr{C}_{x/} \times_{\mathscr{D}_{q(x)/}} \mathscr{D}_{q(\alpha)/}$$

is a trivial Kan fibration.

For the proof we employ a specific deconstruction of the relevant horn inclusions.

lem:joyal3.3

Lemma 2.2 ([2, Lemma 3.3]). *For non-negative integers p and q, and $n = p+q+1$, the inclusions*

$$(\Lambda^p_0 \star \Delta^q) \coprod_{\Lambda^p_q \star \partial\Delta^q} (\Delta^p \star \partial\Delta^q) \to \Delta^p \star \Delta^q \cong \Delta^n$$

and

$$(\partial\Delta^p \star \Delta^q) \coprod_{\partial\Delta^p \star \Lambda^q_q} (\partial\Delta^p \star \Lambda^q_q) \to \Delta^p \star \Delta^q \cong \Delta^n$$

are identified with the inclusions of the extremal horns $\Lambda^n_0 \to \Delta^n$ and $\Lambda^n_n \to \Delta^n$ respectively.

One can see the text [2], or [5, 018N] for the details. We now proceed with the proof of Proposition 2.1.

Proof of Proposition 2.1. We address the cartesian situation, the cocartesian one being similar. Let $F : \mathscr{C}_{/\alpha} \to \mathscr{C}_{/y} \times_{\mathscr{D}_{/q(y)}} \mathscr{D}_{/q(\alpha)}$ denote the map under consideration. A solution to a lifting problems of the form

$$\begin{tikzcd} \partial\Delta^m \arrow[r] \arrow[d] & \mathscr{C}_{/\alpha} \arrow[d, "F"] \\ \Delta^m \arrow[r] \arrow[ur, dotted] & \mathscr{C}_{/y} \times_{\mathscr{D}_{/q(y)}} \mathscr{D}_{/q(\alpha)}, \end{tikzcd} \tag{3}$$

eq:262

with $m \geq 0$, admit a solution if and only if the equivalent lifting problem

$$\begin{tikzcd} (\partial\Delta^m \star \Delta^1) \cup (\Delta^m \star \Lambda^1_1) \arrow[r] \arrow[d] & \mathscr{C} \arrow[d, "q"] \\ \Delta^m \star \Delta^1 \arrow[r] \arrow[ur, dotted] & \mathscr{D} \end{tikzcd}$$

obtained by way of adjunction Lemma I-5.22 admits a solution. Via direct inspection the final edge $\Delta^1 \cong \emptyset \star \Delta^1 \to \mathscr{C}$ in the latter diagram is α, so that this diagram is identified, via Lemma 2.2, with a diagram of the form

$$\begin{tikzcd} \Lambda^n_n \arrow[r] \arrow[d] & \mathscr{C} \arrow[d, "q"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & \mathscr{D} \end{tikzcd} \tag{4}$$

eq:276

in which $n \geq 2$ the edge $\Delta^{\{n-1,n\}} \to \mathscr{C}$ is α. It follows that all lifting problems of the form (3) admit a solution if and only if all lifting problems of the form (4) admit a solution, i.e. that the map F is a trivial Kan fibration if and only if the map α is q-cartesian. $\square$

2.2. Cartesian morphisms via mapping spaces.

prop:cocart_maps

Proposition 2.3 ([5, 01TL])**.** *Consider an inner fibration $q : \mathscr{C} \to \mathscr{D}$, and a morphism $\alpha : x_1 \to x_2$ in $\mathscr{C}$ with image $\bar{\alpha} : \bar{x}_1 \to \bar{x}_2$ in $\mathscr{D}$. The morphism α is q-cartesian if and only if for each third object x_0 in $\mathscr{C}$, with corresponding triples $x : \{0,1,2\} \to \mathscr{C}$ and $\bar{x} : \{0,1,2\} \to \mathscr{D}$, the diagram*

$$\begin{tikzcd} \mathrm{Fun}(\Delta^2, \mathscr{C})_x \times_{\mathrm{Hom}_{\mathscr{C}}(x_1,x_2)} \{\alpha\} \arrow[r] \arrow[d, "q"'] & \mathrm{Hom}_{\mathscr{C}}(x_1, x_2) \arrow[d, "q"'] \\ \mathrm{Fun}(\Delta^2, \mathscr{D})_{\bar{x}} \times_{\mathrm{Hom}_{\mathscr{D}}(\bar{x}_1,\bar{x}_2)} \{\bar{\alpha}\} \arrow[r] & \mathrm{Hom}_{\mathscr{D}}(\bar{x}_1, \bar{x}_2) \end{tikzcd}$$

is a homotopy pullback diagram of Kan complexes.

We cover the proof of Proposition **??** in Section 2.3 below. Let us record now a number of examples.

sect:cocart_maps_proof

2.3. Proof of Proposition 2.3.

2.4. **Uniqueness for q-cocartesian lifts.**

prop:cocart_uniqueness

Proposition 2.4 ([5, 01VK]). *Let $q : X \to S$ be an inner fibration of simplicial sets, and let Y be the full simplicial subset in $\mathrm{Fun}(\Delta^1, X)$ whose vertices are q-cocartesian edges in X. Let Z be the full simplicial set in $\mathrm{Fun}(\{0\}, X) \times_{\mathrm{Fun}(\{0\},S)} \mathrm{Fun}(\Delta^1, S)$ whose edges lie in the image of the composition*

$$Y \to \mathrm{Fun}(\Delta^1, X) \to \mathrm{Fun}(\{0\}, X) \times_{\mathrm{Fun}(\{0\},S)} \mathrm{Fun}(\Delta^1, S).$$

Then the induced map $Y \to Z$ is a trivial Kan fibration. The analogous statement holds when we replace Y with the full simplicial subset of q-cartesian edges in $\mathrm{Fun}(\Delta^1, X)$ as well.

Said informally, Proposition 2.4 tells us that, if a cocartesian solution to the diagram

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d] & X \arrow[d, "q"] \\ \Delta^1 \arrow[r, "\alpha"'] & S \end{tikzcd}$$

exists, then that solution is unique. We note that in the case where q itself is cocartesian, the simplicial subset Z is all of $\mathrm{Fun}(\{0\}, X) \times_{\mathrm{Fun}(\{0\},S)} \mathrm{Fun}(\Delta^1, S)$.

The proof of Proposition 2.4 employs a decomposition of a certain inclusion of simplicial sets which we record here.

lem:simpl_328

Lemma 2.5 ([5, 00TH]). *At any positive integer n, the inclusion*

$$(\Delta^1 \times \partial\Delta^n) \coprod_{\{0\}\times\partial\Delta^n} (\{0\} \times \Delta^n) \to \Delta^1 \times \Delta^n$$

decomposes into a sequence of inclusions

$$(\Delta^1 \times \partial\Delta^n) \cup (\{0\} \times \Delta^n) = X(0) \to \cdots \to X(n) \to X(n+1) = \Delta^1 \times \Delta^n$$

in which each $X(i+1)$ fits into a pushout diagram

$$\begin{tikzcd} \Lambda^{n+1}_{n-i} \arrow[r] \arrow[d] & X(i) \arrow[d] \\ \Delta^{n+1} \arrow[r] & X(i+1) \end{tikzcd}$$

and the sequence

$$\Delta^{\{0,1\}} \to \Delta^{n+1} \to\to X(n+1) = \Delta^1 \times \Delta^{n+1}$$

is an isomorphism onto the edge $\Delta^1 \times \{0\}$ in $\Delta^1 \times \Delta^{n+1}$.

To be clear, our filtration is obtained by applying the opposite to the specific sequence from [5, 00TH].

Idea of proof. Consider the simplices $\sigma_i : \Delta^{n+1} \to \Delta^1 \times \Delta^n$ defined by taking $\sigma_i(j) = (0, j)$ if $j \leq n - i$ and $(1, j - 1)$ if $j > n - i$. We define sequentially $X(i + 1) = X(i) \cup \mathrm{im}(\sigma_i)$. We refer the reader to [5] for the specific details. $\square$

We now can prove our uniqueness result for cocartesian lefts.

Proof of Proposition 2.4. We deal with the case of cocartesian situation, the cartesian case following by taking opposites. We consider a lifting problem

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & Y \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & Z. \end{tikzcd} \tag{5}$$

eq:361

In the case $n = 0$, this problem admits a solution by the defintion of Z. In the case $n \geq 0$, solving this problem is equivalent to solving a lifting problem of the form

$$\begin{tikzcd} (\Delta^1 \times \partial\Delta^n) \cup (\{0\} \times \Delta^n) \arrow[r] \arrow[d, "\mathrm{incl}"'] & X \arrow[d] \\ \Delta^1 \times \Delta^n \arrow[r] \arrow[ur, dotted] & S \end{tikzcd} \tag{6}$$

eq:368

in which all of the constituent maps $\Delta^1 \times \{i\} \to X$ are q-cocartesian. In particular, the map $\Delta^1 \times \{0\} \to X$ is q-cocartesian. We decompose the map incl into a sequence of inclusion $X(i) \to X(i+1)$ as in Lemma 2.5, and produce sequential solutions to the problems

$$\begin{tikzcd} X(i) \arrow[r, "\sigma_i"] \arrow[d] & X \arrow[d, "q"] \\ X(i+1) \arrow[r] \arrow[ur, dotted, "\sigma_{i+1}"] & S \end{tikzcd}$$

for each $i < n$ since the inclusion $X(i) \to X(i+1)$ is inner anodyne in this case. For the final inclusion at $i = n$, we have the extended diagram

$$\begin{tikzcd} \Lambda^{n+1}_0 \arrow[r] \arrow[d] & X(n) \arrow[r, "\sigma_n"] \arrow[d] & X \arrow[d, "q"] \\ \Delta^{n+1} \arrow[r] \arrow[urr, dotted] & X(n+1) = \Delta^1 \times \Delta^n \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

and can solve the external problem since the initial edge $\Lambda^n_0 \to X$ has q-cocartesian image in X, and can therefore solve the internal lifting problem since the leftmost square is a pushout diagram. We therefore obtain a solution to our original problems (5) and (6). □

2.5. **Exponentiating cocartesian fibrations.**

prop:397

Proposition 2.6 ([5, 01VG])**.** *If $q : X \to S$ is a cocartesian fibration, then for any simplicial set K the map $q_* : \mathrm{Fun}(K, X) \to \mathrm{Fun}(K, S)$ is a cocartesian fibration. An edge $\xi : \Delta^1 \to \mathrm{Fun}(K, X)$ is q_*-cocartesian if and only if, at each v in K, the composite $v^*\xi : \Delta^1 \to X$ is q-cocartesian in X.*

The proof follows by a hands on analysis of certain lifting problems which we won't reproduce here. The reader can see the [5, 01VG & 01VM] for the details.

Let us recall that, for any inner fibration $q : X \to S$ and fixed map $\xi : A \to S$ the simplicial set $\mathrm{Fun}_S(A, X)$ is obtained as the fiber

$$\begin{tikzcd} \mathrm{Fun}_S(A,X) \arrow[r] \arrow[d] & \mathrm{Fun}(A,X) \arrow[d, "q_*"] \\ * \arrow[r, "\xi"'] & \mathrm{Fun}(A,S) \end{tikzcd}$$

Since the map q_* is an inner fibration (Corollary I-5.8) we understand that $\mathrm{Fun}_S(A, X)$ is an ∞-category. Of course, in the more restrictive case in which q is a cocartesian fibration, we have just seen that q_* is furthermore cocartesian.

thm:simp_lift

Theorem 2.7. *Let K be any simplicial set and $q : X \to S$ be a cocartesian fibration. Any lifting problem*

$$\begin{tikzcd} \{0\} \times K \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^1 \times K \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

admits a solution $\Delta^1 \times K \to X$ for which, at each vertex v in K, the composite

$$\Delta^1 \cong \Delta^1 \times \{v\} \to \Delta^1 \times K \to X$$

is a q-cocartesian edge in X. Furthermore, the full ∞-subcategory in $\mathrm{Fun}_S(\Delta^1 \times K, X)$ spanned by such solutions is a contractible Kan complex.

Proof. By Proposition 2.6, the map $q_* : \mathrm{Fun}(K, X) \to \mathrm{Fun}(K, S)$ is a cocartesian fibration and solutions to the above lifting problem are identified with q_*-cocartesian solutions $\widetilde{\xi} : \Delta^1 \to \mathrm{Fun}(K, X)$ to the associated lifting problem

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d] & \mathrm{Fun}(K,X) \arrow[d, "q_*"] \\ \Delta^1 \arrow[r] \arrow[ur, dotted] & \mathrm{Fun}(K,S). \end{tikzcd}$$

Existence and uniqueness of such solutions now follow by Proposition 2.4. □

In the event that $q : X \to S$ is a left fibration, all morphisms in X are cocartesian. So we see that there is a unique solution to the above lifiting problem.

cor:left_lift

Corollary 2.8. *Let K be any simplicial set and $q : X \to S$ be a left fibration. Any lifting problem*

$$\begin{tikzcd} \{0\} \times K \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^1 \times K \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

admits a solution $\Delta^1 \times K \to X$, and the collection of all such solutions $\mathrm{Fun}_S(\Delta^1 \times K, X)$ is a contractible Kan complex.

3. Morphisms of fibrations

3.1. **Categories of isofibrations.** For a fixed simiplicial set S, we consider the simplicial category $\underline{\mathrm{Isofib}}(S)$ whose objects are isofibrations $q : X \to S$ over S and morphisms complexes are the Kan complexes fiber

$$\mathrm{Fun}_S(X, Y) := \mathrm{Fun}(X, Y) \times_{\mathrm{Fun}(X,S)} \{q\}.$$

We recall that these mapping complexes are ∞-categories as the map

$$p_* : \mathrm{Fun}_S(X, Y) \to \mathrm{Fun}_S(X, S)$$

along the implicit isofibration $p_* : Y \to S$ is an isofibration, and in particular an inner fibration. The n-simplices in this complex are strictly commuting diagrams

$$\begin{tikzcd} \Delta^n \times X \arrow[rr] \arrow[dr, "q"'] & & Y \arrow[dl, "p"] \\ & S & \end{tikzcd}$$

where q denotes the composition of the projection $\Delta^n \times X \to X$ with q, by an abuse of notation.

We recall that a transformation $\zeta : \Delta^1 \times X \to Y$ is an isomorphism in the ∞-category $\mathrm{Fun}_S(X, Y)$ if and only it, at each vertex $s : * \to S$, the fiber

$$\zeta_s : \Delta^1 \times X_s \to Y_s$$

is an equivalence of ∞-categories (Proposition I-7.9).

Definition 3.1. We let $\underline{\mathrm{Isofib}}(S)^+$ denote the Kan enriched simplicial category with morphisms $\mathrm{Fun}_S(X, Y)^{\mathrm{Kan}}$, and take

$$\mathscr{I}sofib(S) = \mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Isofib}}(S)^+).$$

We let $\underline{\mathrm{Cocart}}(S)^+$ and $\underline{\mathrm{Cart}}(S)^+$ denote the non-full ∞-subcategories in $\underline{\mathrm{Isofib}}(S)$ whose objects are cocartesian and cartesian fibrations over S, respectively, and whose morphisms are the full subcomplexes in $\mathrm{Fun}_S(X, Y)^{\mathrm{Kan}}$ consisting of those functors which send cocartesian or cartesian edges in X to cocartesian or cartesian edges in Y, respectively. We take finally

$$\mathscr{C}ocart(S) = \mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Cocart}}(S)^+) \ \text{ and } \ \mathscr{C}art(S) = \mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Cart}}(S)^+).$$

An equivalence between isofibrations, or cocartesian fibrations, or cartesian fibrations, is an isomorphism in the corresponding ∞-category. In each case, an equivalence is explicitly given by a pair of maps of fibrations

$$F : X \to Y \ \text{ and } \ G : X \to Y$$

which admit natural isomorphisms of fibrations

$$\zeta : \Delta^1 \times X \to Y \ \text{ and } \ \eta : \Delta^1 \times Y \to X$$

with the apparent restrictions

$$\zeta|_{\{0\}} = GF, \quad \zeta_1|_{\{1\}} = id_X, \quad \eta|_{\{0\}} = FG, \quad \eta|_{\{1\}} = id_Y.$$

We note that for any map of simplicial sets $K \to S$ the pullback functor $(-)_K := - \times_S K$ provides a well-defined map of simplicial categories

$$(-)_K : \underline{\mathrm{Isofib}}(S) \to \underline{\mathrm{Isofib}}(K)$$

which necessarily restricts to a simplicial functor on the associated Kan enriched categories $(-)_K : \underline{\mathrm{Isofib}}(S)^+ \to \underline{\mathrm{Isofib}}(K)^+$. We apply the homotopy coherent nerve to obtain a map of ∞-categories

$$(-)_K : \mathscr{I}sofib(S) \to \mathscr{I}sofib(K).$$

Since any functor between ∞-categories preserves equivalences we see that pullback preserved equivalences between isofibrations.

prop:isoequiv_pull **Proposition 3.2.** *If $F : X \to Y$ is an equivalence between isofibrations over a given base S then, for any map of simplicial sets $K \to S$, the base change $F_K : X_K \to Y_K$ is also an equivalence of isofibrations.*

We recall that a map of simplicial sets $i : K \to L$ is called a categorical equivalence if, at each ∞-category $\mathscr{C}$, the induced map $i^* : \mathrm{Fun}(L, \mathscr{C}) \to \mathrm{Fun}(K, \mathscr{C})$ is an equivalence of ∞-categories (see Definition I-10.11 and I-10.12).

prop:equiv_catequiv **Proposition 3.3** ([5, 0285])**.** *A morphism between isofibrations*

$$\begin{tikzcd} X \arrow[rr, "F"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

is an equivalence if and only if F, considered simply as a map of simplicial sets, is a categorical equivalence.

We refer the reader to the text [5] for the details. (The proof that any equivalence of fibrations is a categorical equivalence is straightforward, and might be completed as an exercise. The converse claim is more subtle.) We only note that the proof uses a certain pushout construction and the following fact about isofibrations which we've not covered in this series [5, 01FR]: If $i : A \to B$ is an injective map of simplicial sets which is also a categorical equivalence, and $q : X \to S$ is a cocartesian fibration, then any lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "q"] \\ B \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

admits a solution.

One combines Propositions 3.2 and 3.3 to obtain the following.

Corollary 3.4. *Let $F : X \to Y$ be a map between isofibrations over a given base S, and suppose that F is a categorical equivalence. Then for each map of simplicial sets $K \to S$ the pullback $F_K : X_K \to Y_K$ is also a categorical equivalence.*

3.2. **Equivalences in the cocartesian setting.**

prop:318 **Proposition 3.5** ([5, 023R])**.** *Suppose that*

$$\begin{tikzcd} \mathscr{C} \arrow[rr, "F"] \arrow[dr, "q"'] & & \mathscr{D} \arrow[dl, "p"] \\ & S & \end{tikzcd} \tag{7}$$ eq:320

is a map of inner fibrations, and that F is an equivalence. Then a morphism α is $\mathscr{C}$ is q-cocartesian if and only if $F\alpha$ is p-cocartesian.

`cor:transf_fibrat` **Corollary 3.6.** *Suppose we have a diagram (7) in which F is an equivalence and one of p or q is a cocartesian fibration (resp. left fibration). Then p and q are both cocartesian fibrations (resp. left fibrations).*

Corollary 3.7. *Suppose we have a diagram (7), that p and q are cocartesian fibrations, and that F is an equivalence of inner fibrations. Then both p and q are cocartesian fibrations and F is an equivalence of cocartesian fibrations.*

`thm:equiv_fibers` **Theorem 3.8** ([5, 023M])**.** *Suppose*

$$\begin{tikzcd} X \arrow[rr, "F"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

is a map of cartesian or cocartesian fibrations over a simplicial set S. Then F is an equivalence if and only if, at each vertex $s : * \to S$, the fiber $F_s : X_s \to Y_s$ is an equivalence of ∞-categories.

`cor:equiv_char` **Corollary 3.9.** *Suppose*

$$\begin{tikzcd} X \arrow[rr, "F"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

is a map of cartesian or cocartesian fibrations over a simplicial set S. The following are equivalent:

(a) *F is an equivalence of cocartesian fibrations.*
(b) *F is an equivalence of isofibrations.*
(c) *F is a categorical equivalence.*
(c) *At each vertex $s : * \to S$ the fiber $F_s : X_s \to Y_s$ is an equivalence of ∞-categories.*
(d) *For each map of simplicial sets $K \to S$ the pullback $F_K : X_K \to Y_K$ is a categorical equivalence.*
(e) *For each map from an ∞-category $\mathscr{K} \to S$ the pullback $F_{\mathscr{K}} : X_{\mathscr{K}} \to Y_{\mathscr{K}}$ is an equivalence of ∞-categories.*
(f) *For each map from a Kan complex $\mathscr{S} \to S$ the pullback $F_{\mathscr{S}} : X_{\mathscr{S}} \to Y_{\mathscr{S}}$ is an equivalence of ∞-categories.*
(g) *For each n-simplex $\Delta^n \to S$ the pullback $F_{\Delta^n} : X_{\Delta^n} \to Y_{\Delta^n}$ is an equivalence of ∞-categories.*

Proof. The equivalences between (a) and (c)−−(g) are implied by Theorem 3.8. The equivalence between (b) and (c) is Proposition 3.3. The equivalence between (b) and (d) follows by Proposition 3.2. □

4. Directional fibrations and Kan complexes

4.1. **Exponentials for directional fibrations.**

Definition 4.1. A map of simplicial sets $A \to B$ is called left anodyne (resp. right anodyne) if any lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

in which f is a left (resp. right) fibration admits a solution.

One can show that the class of left anodyne maps is the saturated class generated by the horn inclusions $\Lambda^n_i \to \Delta^n$, where $0 \leq i < n$ [5, 0151]. One similarly characterizes right anodyne maps.

lem:328 **Lemma 4.2** ([5, kerodon]). *Let $i : A \to B$ and $j : K \to L$ be monomorphisms of simplicial sets. If one of i or j is left (resp. right) anodyne, then the induced map*

$$(B \times K) \coprod_{A \times K} (A \times L) \to B \times L$$

is left (resp. right) anodyne.

We refer the reader to Kerodon [5] for the proof.

prop:direct_tech **Proposition 4.3.** *Let $f : X \to S$ be a map of simplicial sets, and $j : K \to L$ be a monomorphism of simplicial sets. Consider the induced map on the functor complexes*

$$\rho : \mathrm{Fun}(L, X) \to \mathrm{Fun}(K, X) \times_{\mathrm{Fun}(K,S)} \mathrm{Fun}(L, S).$$

1. *If f is a left fibration, then ρ is a left fibration.*
2. *If f is a right fibration, then ρ is a right fibration.*
3. *If f is a left fibration and j is left anodyne, then ρ is a trivial Kan fibration.*
4. *If f is a right fibration and j is right anodyne, then ρ is a trivial Kan fibration.*

Proof. Solving a lifting problem of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathrm{Fun}(L, X) \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dotted] & \mathrm{Fun}(K, X) \times_{\mathrm{Fun}(K,S)} \mathrm{Fun}(L, S) \end{tikzcd}$$

is equivalent to solving the corresponding lifting problem

$$\begin{tikzcd} (B \times K) \amalg_{(A \times K)} (A \times L) \arrow[r] \arrow[d] & X \arrow[d] \\ B \times L \arrow[r] \arrow[ur, dotted] & S. \end{tikzcd}$$

So all of the claims follow from a consideration of Lemma 4.2. □

4.2. **Directional fibrations and Kan complexes.**

Proposition 4.4. *A cocartesian (resp. cartesian) fibration $f : X \to S$ is a left (resp. right) fibration if and only if all of the fibers X_s, at arbitrary $s : * \to S$, are Kan complexes.*

5. A deviation on $(\infty,2)$-categories

5.1. $(\infty,2)$-categories.

Definition 5.1. Let X be a simplicial set. A 2-simplex $\tau:\Delta^2\to X$ is called thin if any horn for any $n>2$, index $0<i<n$, and inner horn

$$\bar{\sigma}:\Lambda^n_i\to X \quad \text{with} \quad \bar{\sigma}|\Delta^{\{i-1,i,i+1\}}=\tau,$$

the lifting problem

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "\bar{\sigma}"] \arrow[d] & X \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & * \end{tikzcd}$$

admits a solution.

One sees immediately that every 2-simplex in an ∞-category is thin, for example.

Recall our notation $s_i:[n]\to[n-1]$ for the weakly increasing surjection with $s_i(i)=s_i(i+1)=i$, for $0\le i\le n-1$, and the corresponding degeneracies $s_i^*:\Delta^n\to\Delta^{n-1}$. We call an n-simplex $\sigma:\Delta^n\to X$ in a simplicial set *left degenerate* if σ factors through the degeneracy $s_0^*:\Delta^n\to\Delta^{n-1}$, and *right degenerate* if σ factors through the degeneracy $s_{n-1}^*:\Delta^n\to\Delta^{n-1}$.

def:infty2 **Definition 5.2** ([5, 01W9, 01Y5])**.** A simplicial set X is called an $(\infty,2)$-category if the following hold:

(a) Any horn $\Lambda^2_1\to X$ admits an extension to a thin 2-simplex.
(b) Every degenerate 2-simplex in X is thin.
(c.l) For $n>2$, any horn $\bar{\sigma}:\Lambda^n_0\to X$ in which the 2-simplex $\bar{\sigma}|\Delta^{\{0,1,n\}}$ is left degenerate admits an extension to an n-simplex in X.
(c.r) For $n>2$, any horn $\bar{\sigma}':\Lambda^n_n\to X$ in which the 2-simplex $\bar{\sigma}'|\Delta^{\{0,n-1,n\}}$ is right degenerate admits an extension to an n-simplex in X.

A functor, or map, between $(\infty,2)$-categories is a map of simplicial sets $F:X\to Y$ which preserves thin 2-simplices.

Remark 5.3. Having introduced this notion, let us recall that the term ∞-category is used interchangeably with the term $(\infty,1)$-category.

If we consider an ∞-category $\mathscr{C}$, then in any horn $\Lambda^n_0\to\mathscr{C}$ as in (c.l) the initial edge $\Delta^{\{0,1\}}\to\mathscr{C}$ is degenerate, and hence an isomorphism in $\mathscr{C}$. Hence we have the proposed completion to an n-simplex $\Delta^n\to\mathscr{C}$, by Proposition I-5.33. Similarly any horn $\Lambda^n_n\to\mathscr{C}$ as in (c.r) completes to an n-simplex as well. So we observe the following.

Lemma 5.4. *Any ∞-category is an $(\infty,2)$-category. Furthermore, an $(\infty,2)$-category X is an ∞-category if and only if every 2-simplex in X is thin.*

Recall that each simplex Δ^n is an ∞-category, and hence an $(\infty,2)$-category.

Example 5.5. Since any degenerate 2-simplex in an $(\infty,2)$-category is thin, any map of simplicial sets $*=\Delta^0\to X$ is a map of $(\infty,2)$-categories. Similarly, any map of simplicial sets $\Delta^1\to X$ is a map of $(\infty,2)$-categories.

One has the following practical check for maps between $(\infty,2)$-categories.

prop:infty2_check

Proposition 5.6 ([5, 01YC])**.** *Let X and Y be $(\infty,2)$-categories, and $F : X \to Y$ be a map of simplicial sets. Then F is a functor, i.e. preserves thin 2-simplexes, if and only if any horn $\Lambda^2_1 \to X$ can be competed to a thin simplex with thin image in Y.*

Idea of proof. The result is a consequence of stability of thin simplices under various conditions. Namely one establishes an inner-exchange property for thin simplices, which we recall below, and a 4-of-5 property which one can find at [5, 01XX]. □

5.2. **The pith of an $(\infty,2)$-category.**

Definition 5.7. Given an $(\infty,2)$-category X, the pith in X is the simplicial subset $X^{\mathrm{Pith}} \subseteq X$ whose simplices $\Delta^n \to X^{\mathrm{Pith}}$ consist of all simplices $\sigma : \Delta^n \to X$ in which each restriction along a 2-simplex

$$\Delta^2 \to \Delta^n \overset{\sigma}{\to} X$$

is thin.

Since functors between $(\infty,2)$-categories preserve thin simplices, by defintion, we see that any map $F : \mathscr{C} \to X$ from an ∞-category to an $(\infty,2)$-category factors through the pith.

lem:360

Lemma 5.8 ([5, 01XL], Inner-exchange property)**.** *Consider a 3-simplex $\sigma : \Delta^3 \to X$ in an $(\infty,2)$-category, and suppose that the associated 2-simplices $\sigma|\Delta^{\{1,2,3\}}$ and $\sigma|\Delta^{\{0,1,2\}}$ are thin. Then the 2-simplex $\sigma|\Delta^{\{0,2,3\}}$ is thin if and only if the 2-simplex $\sigma|\Delta^{\{0,1,3\}}$ is thin.*

The proof employs certain facts about interior fibrations (see below), and is omitted. From Lemma 5.8 the proof of the following is immediate.

Proposition 5.9. *For any $(\infty,2)$-category X, the subcomplex X^{Pith} is an ∞-category.*

Proof. For any completion $\Delta^3 \to X$ of an inner horn $\Lambda^3_i \to X$ in which all of the associated face $\Delta^2 \to \Lambda^3_i \to X$ are thin, the final face $\Delta^{[3]\setminus\{i\}} \to X$ is also thin, by Lemma 5.8. This shows that the pith is stable under the completion of inner horns $\Lambda^3_i \to X^{\mathrm{Pith}}$. Stability under completion of all inner horns $\Lambda^n_i \to X^{\mathrm{Pith}}$ with $n > 3$ is immediate, since the horn Λ^n_i already contains all 2-faces in Δ^n in this case. Taken together with condition (a) of Definition 5.2, we see that any lifting problem

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d] & X^{\mathrm{Pith}} \arrow[d] \\
\Delta^n \arrow[r] & *
\end{tikzcd}$$

with $0 < i < n$ admits a solution, as required. □

5.3. **$(\infty,2)$-category via simplicial categories.** Recall that one can associate to any simplicial category $\underline{S}$ its associated homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}(\underline{S})$ Section I-2.7. The n-simplices in $\mathrm{N}^{\mathrm{hc}}(\underline{S})$ are simplicial functors from the path category

Path$[n]$. For $S = \mathrm{N}^{\mathrm{hc}}(\underline{S})$ we have in low dimension

$$S[0] = \{ \text{ objects in } \underline{S} \ \}$$

$$S[1] = \{ \text{ pairs of object } (x_0, x_1) \text{ along with a map } f \in \underline{\mathrm{Hom}}_{\underline{S}}(x_0, x_1)[0] \ \}$$

$$S[2] = \left\{ \begin{array}{l} \text{triples of objects } (x_0, x_1, x_2), \text{ maps } f_{ij} : x_i \to x_j \text{ for each } i < j, \text{ and} \\ \text{a 1-simplex } h : \Delta^1 \to \underline{\mathrm{Hom}}_{\underline{S}}(x_0, x_2) \text{ with } h|_0 = f_{02},\ h|_1 = f_{12}f_{01} \end{array} \right\}$$

Lemma I-2.19. We take the following theorem for granted.

thm:hc_infty2

Theorem 5.10 ([5, 01YM]). *Let $\underline{S}$ be a simplicial category in which, at each pair of objects x and y in $\underline{S}$, the mapping complex $\underline{\mathrm{Hom}}_{\underline{S}}(x, y)$ is an ∞-category. Then the homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}(\underline{A})$ is an $(\infty, 2)$-category.*

What we are most interested in here is the $(\infty, 2)$-category of ∞-categories. Recall that for any ∞-categories $\mathscr{C}$ and $\mathscr{D}$ the simplicial set of functors $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$, whose simplicial are as expected

$$\mathrm{Fun}(\mathscr{C}, \mathscr{D})[n] = \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n \times \mathscr{C}, \mathscr{D}),$$

form another ∞-category Corollar I-5.8. With these morphisms we obtain the simplicial category $\underline{\mathrm{Cat}}_\infty$ of ∞-categories and their functor categories. We note that $\underline{\mathrm{Cat}}_\infty$ is a full simplicial subcategory in the ambient category $\underline{\mathrm{sSet}}$ of simplicial sets.

5.4. **The $(\infty, 2)$-category of ∞-categories.**

Theorem 5.11. *The homotopy coherent nerve*

$$\mathfrak{Cat}_\infty := \mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Cat}}_\infty)$$

is an $(\infty, 2)$-category.

According to the above analysis the 0-simplices in $\mathfrak{Cat}_\infty$ are ∞-categories, the 1-simplices are functors between ∞-categories, and 2-simplices are triples of functors and a natural transformation

$$\begin{tikzcd}
 & \mathscr{C}_2 & & \mathscr{C}_2 \\
\mathscr{C}_1 \arrow[ur, "F_{12}"] & & {} \arrow[r, "\zeta"] & {} \\
 & \mathscr{C}_0 \arrow[ul, "F_{01}"] & & \mathscr{C}_0. \arrow[uu, "F_{02}"]
\end{tikzcd}$$

Definition 5.12. The $(\infty, 2)$-category $\mathfrak{Cat}_\infty$ is called the $(\infty, 2)$-category of ∞-categories.

Remark 5.13. The $(\infty, 2)$-category $\mathfrak{Cat}_\infty$ is in our universe of "large" sets, which is strictly larger than our universe of "normal sized" set in which all other ∞-categories are assumed to live.

We recall our ∞-category $\mathscr{C}at_\infty$ of ∞-categories, which we obtain by restricting the morphisms $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$ to the associated Kan can complex $\mathrm{Fun}(\mathscr{C}, \mathscr{D})^{\mathrm{Kan}}$ then applying the simplicial nerve. The inclusions of ∞-categories

$$\mathrm{Fun}(\mathscr{C}, \mathscr{D})^{\mathrm{Kan}} \to \mathrm{Fun}(\mathscr{C}, \mathscr{D})$$

imply an inclusion of $(\infty, 2)$-categories $\mathscr{C}at_\infty \to \mathfrak{Cat}_\infty$, and hence an inclusion into the pith

$$\mathscr{C}at_\infty \to (\mathfrak{Cat}_\infty)^{\mathrm{Pith}}. \tag{8}$$ eq:444

By a general result one finds that this inclusion is an equality.

Proposition 5.14 ([5, 01YT])**.** *The inclusion* (8) *is an equality,* $\mathscr{C}at_\infty = (\mathfrak{Cat}_\infty)^{\mathrm{Pith}}$.

5.5. **Interior fibrations.**

def:interior **Definition 5.15.** A map of simplicial sets $q : X \to S$ is called an interior fibration if the following hold:

(a) At each 0-simplex x in X, the identity $id_x : x \to x$ is both q-cartesian and cocartesian.
(b) For any lifting problem

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^n \arrow[r, "\sigma"'] \arrow[ur, dotted] & S \end{tikzcd} \tag{9}$$ eq:400

in which $0 < i < n$ and $\sigma|\Delta^{\{i-1,i,i\}}$ is thin in S, (9) admits a solution.

It is clear that if $f : S' \to S$ is a map of simplicial sets which preserves thin 2-simplices, and the diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "q'"'] & X \arrow[d, "q"] \\ S' \arrow[r, "f"] & S \end{tikzcd}$$

is a pullback diagram of simplicial sets in which q is an interior fibration, then the map $q' : X' \to S'$ is an interior fibration as well.

One also observes the following.

lem:495 **Lemma 5.16.** *If S is an $(\infty, 2)$-category, and $q : X \to S$ is an interior fibration, then X is also an $(\infty, 2)$-category and q is a functor between $(\infty, 2)$-categories.*

Proof. One sees via the lifting property for q that any 2-simplex $\Delta^2 \to X$ which has thin image in S is thin in X. From this we see that any horn $\Lambda^2_1 \to X$ can be completed to a thin 2-simplex in X. One obtains this thin completion by lifting a thin completion $\Delta^2 \to S$. We are left to prove that any appropriate degenerate horn $\Lambda^n_0 \to X$ or $\Lambda^n_n \to X$, at $n > 2$, completes to an n-simplex. However this follows from the fact the fact that identity maps in X are both q-cartesian and cocartesian, and the fact that the corresponding horns in S admit completions. We now see that X is an $(\infty, 2)$-category. One sees that q is a functor, i.e. preserves thin 2-simplices, by applying Proposition 5.6. □

As we see in the above proof, given an interior fibration $q : X \to S$ over an $(\infty, 2)$-category, one can detect thin simplices in X by considering their images in S along q.

lem:505 **Lemma 5.17.** *If $q : X \to S$ is an interior fibration then a 2-simplex in X is thin if and only if its image in S is thin.*

cor:interior_pullback

Corollary 5.18. *Consider a pullback diagram*

$$\begin{tikzcd} Z \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "q"] \\ Y \arrow[r, "F"'] & S \end{tikzcd}$$

in which q is an interior fibration and F is a map between $(\infty, 2)$-categories. Then Z is an $(\infty, 2)$-category, p_1 is an interior fibration, and p_2 is a map of $(\infty, 2)$-categories.

Proof. The fact that Z is an $(\infty, 2)$-category and p_1 is a map of $(\infty, 2)$-categories follows by Lemma 5.16. As for p_1, we consider a thin 2-simplex $\Delta^2 \to Z$, and note that its image in Y is thin. Hence its image in S is thin, and so its image in X is thin by Lemma 5.17. It follows that p_2 is a map of $(\infty, 2)$-categories, by definition. □

We are especially interested in the fiberings of interior fibrations over ∞-categories.

lem:502

Lemma 5.19. *Let $\mathscr{C}$ be an ∞-category. A map of simplicial sets $q : X \to \mathscr{C}$ is an interior fibration if and only if it is an inner fibration.*

Proof. If q is an interior fibration then it is an inner fibration since all 2-simplices in $\mathscr{C}$ are thin. Conversely, if q is an inner fibration then X is an ∞-category and q is therefore an inner fibration between ∞-categories. Condition (a) of Definition 5.15 now follows from the fact that the identity in an ∞-category is an isomorphism, and an application of Proposition I-5.33. □

One combines Lemma 5.19 with the above discussion of fiber products to obtain the following corollary.

Corollary 5.20. *Consider an interior fibration $q : X \to S$ over an $(\infty, 2)$-category S.*

(a) *For any ∞-category $\mathscr{C}$, and any functor of $(\infty, 2)$-categories $\mathscr{C} \to S$, the fiber product $X \times_S \mathscr{C}$ is an ∞-category. Furthermore, the projection $X \times_S \mathscr{C} \to \mathscr{C}$ is an inner fibration.*
(b) *At each point $s : * \to S$ the fiber X_s is an ∞-category.*

cor:interior_pith

Corollary 5.21. *If $q : X \to S$ is an interior fibration over an $(\infty, 2)$-category S then the diagram*

$$\begin{tikzcd} X^{\mathrm{Pith}} \arrow[r] \arrow[d] & X \arrow[d] \\ S^{\mathrm{Pith}} \arrow[r] & S \end{tikzcd}$$

is a pullback diagram, and the map $X^{\mathrm{Pith}} \to S^{\mathrm{Pith}}$ is an interior fibration.

Proof. In this case the pullback $X \times_S S^{\mathrm{Pith}}$ is an ∞-category and the projection to X is a map of $(\infty, 2)$-categories. So the identification

$$X^{\mathrm{Pith}} = X \times_S S^{\mathrm{Pith}}$$

follows via an application of the universal property for the pullback and the universal property for the pith. □

5.6. **Undercategories and overcategories and pointed ∞-categories.** In the $(\infty,2)$-setting we can define overcategories and undercategories exactly as in the ∞-setting. Namely, for a map of simplicial sets $p: K \to X$ the overcategory $X_{p/}$ is the simplicial set with n-simplices provided by the join

$$X_{p/}[n] := \mathrm{Hom}_{\mathrm{sSet}}(K \star \Delta^n, X)_p,$$

and similarly for the undercategory

$$X_{/p}[n] := \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n \star K, X)_p$$

Section I-5.7. In the case in which X is an ∞-category, we saw that the forgetful functors

$$\begin{tikzcd} X_{p/} \arrow[dr] & & X_{/p} \arrow[dl] \\ & X & \end{tikzcd}$$

obtained by restricting along the inclusions $\Delta^n \to \Delta^n \star K$ and $\Delta^n \to K \star \Delta^n$ are directional fibrations, and in particular isofibrations. We have a similar result in the 2-categorical context.

prop:504 **Proposition 5.22** ([5, 01WU])**.** *Let X be an $(\infty,2)$-category and $p: K \to X$ be a map of simplicial sets. The the forgetful maps*

$$X_{p/} \to X \quad \text{and} \quad X_{/p} \to X$$

are both interior fibrations.

At this point we'll begin to leave many of the details unaccounted for. In particular, we direct the reader to the original text [5] for the details on Proposition 5.22. In any case, we record some corollaries.

Corollary 5.23. *For an $(\infty,2)$-category X and a diagram $p: K \to X$, the simplical sets $X_{p/}$ and $X_{/p}$ are $(\infty,2)$-category and the forgetful maps are both functors between $(\infty,2)$-categories.*

cor:601 **Corollary 5.24.** *Let $p: K \to X$ be a map from a simplicial set into an $(\infty,2)$-category. At any point $x: * \to X$ the fibers $(X_{p/})_x$ and $(X_{/p})_x$ are both ∞-categories.*

We apply this corollary in the case where the diagram p is a point $x: * \to X$ to obtain mapping categories for any $(\infty,2)$-category X.

Definition 5.25. For any $(\infty,2)$-category X, and objects $x, y: * \to X$, the left pinched mapping ∞-category is the fiber

$$\mathrm{Hom}^{\mathrm{L}}_X(x,y) := (X_{x/}) \times_X \{y\}.$$

Similarly,the right pinched mapping ∞-category is the fiber

$$\mathrm{Hom}^{\mathrm{R}}_X(x,y) = \{x\} \times_X (X_{/y}).$$

As with any interior fibration, we can restrict the forgetful functor to the piths to obtain inner fibrations of ∞-categories

$$(X_{p/})^{\mathrm{Pith}} \to X^{\mathrm{Pith}} \quad \text{and} \quad (X_{/p})^{\mathrm{Pith}} \to X^{\mathrm{Pith}}.$$

In this particular instance one can observe a stronger characterization of these functors.

prop:over_cartesian

Proposition 5.26 ([5, 01YE])**.** *For X and $p : K \to X$ as above, the restrictions of the forgetful functors*

$$(X_{p/})^{\mathrm{Pith}} \to X^{\mathrm{Pith}} \quad \textit{and} \quad (X_{/p})^{\mathrm{Pith}} \to X^{\mathrm{Pith}}.$$

are, respectively, a cocartesian fibration and a cartesian fibration.

One might view this result in analogy with the ∞-setting, where the forgetful functors were observed to be right and left fibrations Corollary I-5.27.

5.7. **Mapping categories in the homotopy coherent nerve.** Let $\underline{S}$ be a simplicial category whose morphism complexes are weak Kan complexes, and let S be the homotopy coherent nerve, $S = \mathrm{N}^{\mathrm{hc}}(\underline{S})$. We recall that S is an $(\infty, 2)$-category in this case. By an abuse of notation take

$$\underline{\mathrm{Hom}}_S(x, y) = \underline{\mathrm{Hom}}_{\underline{S}}(x, y)$$

for any given pair of objects in S. We construct a map of simplicial sets

$$\theta : \underline{\mathrm{Hom}}_S(x, y) \to \mathrm{Hom}^{\mathrm{L}}_S(x, y)$$

[5, 01LD] which is subsequently found to be an equivalence of ∞-categories.

To begin, for any simplicial set K we consider the simplicial category $E(K)$ with objects x_- and x_+ and morphisms

$$\mathrm{Hom}_{E(K)}(x_-, x_-) = \mathrm{Hom}_{E(K)}(x_+, x_+) = * \quad \text{and} \quad \mathrm{Hom}_{E(K)}(x_-, x_+) = K.$$

We consider the $(n+1)$-simplex $\{-1\} \star \Delta^n \cong \Delta^{n+1}$ and the simplicial path category $\mathrm{Path}(\{-1\} \star \Delta^n)$ whose morphisms are given by the nerves

$$\mathrm{Hom}_{\mathrm{Path}(\{-1\}\star\Delta^n)}(l, m) = \mathrm{N}(\mathrm{Subsets}^{\mathrm{op}}_{l,m})$$

where $\mathrm{Subsets}_{l,m}$ is the partially ordered set of subsets $S \subseteq [n]$ with $\min S = l$ and $\max S = m$, ordered by inclusion.

At each integer n we have a simiplicial functor

$$\theta^*_n : \mathrm{Path}(\{-1\} \star \Delta^n) \to E\Delta^n$$

which is define on objects by taking $\theta^*_n(-1) = x_-$, and $\theta^*_n(i) = x_+$ for all $i \geq 0$, and defined on morphisms by the simplicial map

$$\theta^*_n : \mathrm{Hom}_{\mathrm{Path}(\{-1\}\star\Delta^n)}(l, m) = \mathrm{N}(\mathrm{Subsets}^{\mathrm{op}}_{l,m}) \to \mathrm{Hom}_E(x_-, x_+) = \Delta^n = \mathrm{N}([n])$$

associated to the functor $\mathrm{Subsets}^{\mathrm{op}}_{l,m} \to [n]$ which sends each subset $S = \{l < s_1 < \dots s_r < m\}$ to s_1 and each inclusion $S' \supseteq S$ to the inequality $s'_1 \leq s_1$.

For objects x and y in S, n-simplices in $\underline{\mathrm{Hom}}_S(x, y)$ are identified with simplicial functors $\mathrm{Fun}_{\mathrm{sCat}}(E\Delta^n, \underline{S})$ in the fiber over (x, y) in $\mathrm{Fun}(E\emptyset, \underline{S})$. Each such functor now defined an $(n+1)$-simplex in S via a consideration of the identification

$$S[n+1] = \mathrm{Fun}_{\mathrm{sCat}}(\mathrm{Path}(\{-1\} \star \Delta^n), \underline{S})$$

and restricting along θ^*_n. One sees, by the definiton of θ^*_n that this associated $(n+1)$-simplex has initial vertex x and all other vertices y, and restricts trivially to $\Delta^n \subseteq \{-1\} \star \Delta^n$. So we obtain a map of sets

$$\theta_n : \underline{\mathrm{Hom}}_S(x, y)[n] \to (S_{x/}) \times_S \{y\} = \mathrm{Hom}^{\mathrm{L}}_S(x, y)[n],$$

$$(f : E\Delta^n \to \underline{S}) \mapsto (f\theta^*_n : \mathrm{Path}\{-1\} \star \Delta^n \to \underline{S}).$$

One observes directly that any increasing function $t : [n] \to [n']$ produces a commutative diagram

$$\begin{tikzcd} \mathrm{Path}(\{-1\} \star \Delta^n) \arrow[r, "\theta_n"] \arrow[d, "t_*"'] & E\Delta^n \arrow[d, "t_*"] \\ \mathrm{Path}(\{-1\} \star \Delta^{n'}) \arrow[r, "\theta_{n'}"'] & E\Delta^{n'}, \end{tikzcd}$$

from which we see that the θ_n assemble into a map of simplicial sets, or a map of ∞-categories,

$$\theta : \underline{\mathrm{Hom}}_S(x, y) \to \mathrm{Hom}^{\mathrm{L}}_S(x, y).$$

thm:pinched_simplicial

Theorem 5.27 ([5, 01LG])**.** *Let $\underline{S}$ be a simplicial category whose morphism complexes are ∞-categories. Take $S = \mathrm{N}^{\mathrm{hc}}(\underline{S})$. For any objects $x, y : * \to S$ there is a natural equivalence of ∞-categories*

$$\theta : \underline{\mathrm{Hom}}_S(x, y) \to \mathrm{Hom}^{\mathrm{L}}_S(x, y).$$

We do not cover the details, and refer the reader to the text [5].

cor:simplicial_pullback

Corollary 5.28. *Take $\underline{S}$ and S as above. For any pair of points $x, y : * \to S$ there is a categorical pullback diagram*

$$\begin{tikzcd} \underline{\mathrm{Hom}}_S(x, y) \arrow[r, "\theta"] \arrow[d] & (S_{x/})^{\mathrm{Pith}} \arrow[d] \\ * \arrow[r, "y"'] & S^{\mathrm{Pith}}. \end{tikzcd} \tag{10}$$

eq:704

Proof. By Corollary 5.18 and Proposition 5.22 the projection map $\mathrm{Hom}^{\mathrm{L}}_S(x, y) \to S_{x/}$ has image in the Pith $(S_{x/})^{\mathrm{Pith}}$. Applying this fact in conjunction with Corollary 5.21, we observe a pullback diagram of ∞-categories

$$\begin{tikzcd} \mathrm{Hom}^{\mathrm{L}}_S(x, y) \arrow[r, "\theta"] \arrow[d] & (S_{x/})^{\mathrm{Pith}} \arrow[d] \\ * \arrow[r, "y"] & S^{\mathrm{Pith}}. \end{tikzcd}$$

in which the right-hand map is an inner fibration. This diagram is additionally a categorical pullback square by Corollary I-6.22 and Proposition 5.26. Since $\theta : \underline{\mathrm{Hom}}_S(x, y) \to \mathrm{Hom}^{\mathrm{L}}_S(x, y)$ is an equivalence of ∞-categories it follows that the corresponding diagram (10) is a categorical pullback square as well (see Proposition I-6.23). □

sect:transport

6. Transport I: Classifying functors

6.1. **Preliminary discussion.** In analogy with the plain categorical setting, we claim that cocartesian fibrations $q : \mathscr{E} \to \mathscr{C}$ over a given ∞-category are "the same thing" as functors into the ∞-category of ∞-categories $F : \mathscr{C} \to \mathscr{C}at_\infty$. In our imaginations, the functor F should evaluate as the fibers $F(x) \cong \mathscr{E}_x$ and the image of a given map $\alpha : x \to y$ should be some kind of pushforward functor $\alpha_* : \mathscr{E}_x \to \mathscr{E}_y$ which "moves along" cartesian lifts $\widetilde{\alpha} : \widetilde{x} \to \widetilde{y}$, so that $\alpha_*(\widetilde{x}) \cong \widetilde{y}$.

Of course one can not simply construct the desired functor $F : \mathscr{C} \to \mathscr{C}at_\infty$ by hand. We (or rather, Lurie) instead proceed(s) by establishing a universal cocartesian fibration over ∞-categories

$$U : \mathscr{Z} \to \mathscr{C}at_\infty.$$

It is then shown that each cocartesian fibration is realized as a (categorical) pullback along U,

$$\begin{tikzcd} \mathscr{E} \arrow[r] \arrow[d, "q"'] & \mathscr{Z} \arrow[d] \\ \mathscr{C} \arrow[r, "F"'] & \mathscr{C}at_\infty, \end{tikzcd}$$

and furthermore that the space of such pullback diagrams assembles into a contractible space. We refer to this uniquely determined functor F as the covariant transport functor along q, or as the functor which classifies q. One obtains a completely similar analysis of cartesian fibrations and classification via an applications of the opposite involution.

In this section we outline the above construction. Unlike at other points in this text we are not especially concerned with (all of) the technical details, and seek only to provide a coherent narrative which explains clearly what's going on and how this stuff works.

We begin with a detour into $(\infty, 2)$-categories. We then construct the universal cocartesian fibration $\mathscr{Z}$ via a certain "category of objects", and explain how each fiber $\mathscr{Z}_\mathscr{E}$ over a given ∞-category $\mathscr{E} : * \to \mathscr{C}at_\infty$ reproduces $\mathscr{E}$ itself, up to equivalence. We define the space $\mathscr{T}\mathscr{W}it(q)$ of classifying diagrams and recall the contractibility of this space from [5]. The section concludes with a description of the pushforward functors α_* appearing the transport F.

sect:univ_fib

6.2. **Categories with objects and the universal cocartesian fibration.** From the $(\infty, 2)$-category $\mathfrak{Cat}_\infty$ we can produce the $(\infty, 2)$-category of pointed ∞-categories

$$(\mathfrak{Cat}_\infty)_{*/} = (\mathfrak{Cat}_\infty)_{\Delta^0/}.$$

By Proposition 5.22 the simplicial set $(\mathfrak{Cat}_\infty)_{*/}$ is in fact an $(\infty, 2)$-category and the forgetful functor $(\mathfrak{Cat}_\infty)_{*/} \to \mathfrak{Cat}_\infty$ is a interior fibration.

Definition 6.1. The ∞-category of ∞-categories with a distinguished object is the pith of the $(\infty, 2)$-category of pointed ∞-categories,

$$\mathscr{P}.\mathscr{C}at_\infty := \left((\mathfrak{Cat}_\infty)_{*/}\right)^{\mathrm{Pith}}.$$

Remark 6.2. The $\mathscr{P}$ suffix stands for "pointed", though we heed the warning from [5, 020W] and do not label this ∞-category as such.

Remark 6.3. There is a comparison functor $\mathscr{P}.\mathscr{C}at_\infty \to (\mathscr{C}at_\infty)_{*/}$ which is, apparently, bijective on objects. However this map is not bijective on 1-morphisms so that it is not an isomorphism [5, 020Z].

Via an application of Corollary 5.21 we see that the forgetful functor restricts to provide a pullback diagram

$$\begin{tikzcd} \mathscr{P}.\mathscr{C}at_\infty \arrow[r] \arrow[d] & (\mathfrak{Cat}_\infty)_{*/} \arrow[d] \\ \mathscr{C}at_\infty \arrow[r] & \mathfrak{Cat}_\infty. \end{tikzcd}$$

The forgetful functor $\mathscr{P}.\mathscr{C}at_\infty \to \mathscr{C}at_\infty$ is furthermore seen to be a cocartesian fibration in this case, via an application of Proposition 5.26. We record this result.

Proposition 6.4 ([5, 0213]). *The forgetful functor $\mathscr{P}.\mathscr{C}at_\infty \to \mathscr{C}at_\infty$ is a cocartesian fibration.*

We call the above forgetful functor the *universal cocartesian fibration*, for reasons which will be apparent shortly.

Definition 6.5. We let univ : $\mathscr{P}.\mathscr{C}at_\infty \to \mathscr{C}at_\infty$ denote the cocartesian fibration induced by the forgetful functor $(\mathfrak{Cat}_\infty)_{*/} \to \mathfrak{Cat}_\infty$, as considered above.

At a baseline, objects in the ∞-category $\mathscr{P}.\mathscr{C}at_\infty$ are simply pointed ∞-categories $x : * \to \mathscr{C}$. A map in $\mathscr{P}.\mathscr{C}at_\infty$ between two objects $(\mathscr{C}, x)$ and $(\mathscr{D}, y)$ consists of a functor $F : \mathscr{C} \to \mathscr{D}$ along with a map

$$\alpha : \Delta^1 \to \mathscr{D}$$

with $\alpha(0) = F(x)$ and $\alpha(1) = y$, i.e. a choice of a morphism $\alpha : F(x) \to y$ in $\mathscr{D}$. A 2-simplex in $\mathscr{P}.\mathscr{C}at_\infty$ consists of a choice of the following data:

- Three functors with moprhisms $F_{ij} : \mathscr{C}_i \to \mathscr{C}_j$ for $1 \leq i < j \leq 3$ and a natural isomorphism $\beta : F_{23}F_{12} \to F_{13}$,
- Morphisms $\alpha_{ij} : F_{ij}(x_i) \to x_j$ in $\mathscr{C}_j$.
- Morphisms $\alpha^1_{123} : F_{32}F_{12}(x_1) \to F_{13}(x_1)$, $\alpha^2_{123} : F_{32}F_{12}(x_2) \to F_{23}(x_2)$, and $\alpha^3_{123} : F_{23}F_{12}(x_1) \to x_3$ in $\mathscr{C}_3$ with $\alpha^1 = \beta(x_1)$.
- Two 2-simplices $\sigma_k : \Delta^2 \to \mathscr{C}_3$ with $\sigma_k|_{\Delta^{\{0,1\}}} = \alpha^k_{123}$, $\sigma_k|_{\Delta^{\{1,2\}}} = \alpha_{k3}$, and $\sigma_k|_{\Delta^{\{0,3\}}} = \alpha^3_{123}$.

We have the following characterization of univ-cocartesian edges.

`prop:univ_cocart` **Proposition 6.6** ([5, 026X,01YE]). *A morphism $(F, \alpha) : (\mathscr{C}, x) \to (\mathscr{D}, y)$ in $\mathscr{P}.\mathscr{C}at_\infty$ is cocartesian for the universal fibration* univ : $\mathscr{P}.\mathscr{C}at_\infty$ *if and only if the underlying map $\alpha : F(x) \to y$ is an isomoprhism in $\mathscr{D}$.*

6.3. **A remark on notation.** Our $(\infty, 2)$-category $\mathfrak{Cat}_\infty$ is the $(\infty, 2)$-category denoted by a bold $\mathcal{QC}$ in [5, 020K]. Our $(\mathfrak{Cat}_\infty)_{*/}$ is the $(\infty, 2)$-category denoted by a bold $\mathcal{QC}_{\mathrm{Obj}}$ in [5, 0210]. The associated piths, which we've denoted $\mathscr{C}at_\infty$ and $\mathscr{P}.\mathscr{C}at_\infty$ respectively, are the non-bolded ∞-categories $\mathcal{QC}$ and $\mathcal{QC}_{\mathrm{Obj}}$ in [5].

6.4. **Fibers of the map** univ : $\mathscr{P}.\mathscr{C}at_\infty \to \mathscr{C}at_\infty$**.** In considering the universal cocartesian fibration $\mathscr{P}.\mathscr{C}at_\infty \to \mathscr{C}at$, any point $e : \Delta^0 \to \mathscr{C}at$ corresponds to an ∞-category $\mathscr{E} = e(0)$ and we have the pullback

$$\mathscr{P}.\mathscr{C}at_\infty \times_{\mathscr{C}at_\infty} \{e\}$$

which is some other ∞-category. Now, objects in this fiber are simply maps of ∞-categories $* \to \mathscr{E}$, and hence are identified with objects in $\mathscr{E}$. Similarly, 1-simplices in the fiber are identified 2-simplices in the ∞-category of ∞-categories

$$\begin{tikzcd} & * \arrow[dl, "x"'] \arrow[dr, "y"] & \\ \mathscr{E} \arrow[rr, "="'] & & \mathscr{E}. \end{tikzcd}$$

These are, by definition, natural transformations $\alpha \in \mathrm{Hom}_{\mathrm{sSet}}(\Delta^1, \mathscr{E})$ with $\alpha|_0 = x$ and $\alpha|_1 = y$, i.e. 1-simplices $\alpha : x \to y$. So we observe an identification of 1-skeleta

$$\mathscr{E}[\leq 1] \;=\; \mathscr{P}.\mathscr{C}at_\infty \times_{\mathscr{C}at_\infty} \{e\}[\leq 1].$$

As an application of Theorem 5.27 and Corollary 5.28, we see that this direct identification of simplices in low-dimension expands to an equivalence of ∞-categories which calculates the fiber.

prop:univ_fibs **Proposition 6.7.** *For any ∞-category $\mathscr{E}$, which we can understand as a point $\mathscr{E} : * \to \mathscr{C}at_\infty$, we have a categorical pullback square*

$$\begin{tikzcd} \mathscr{E} \arrow[r] \arrow[d] & \mathscr{P}.\mathscr{C}at_\infty \arrow[d] \\ * \arrow[r, "\mathscr{E}"'] & \mathscr{C}at_\infty, \end{tikzcd}$$

and a corresponding equivalence between ∞-categories $\theta : \mathscr{E} \xrightarrow{\sim} \mathscr{P}.\mathscr{C}at_\infty \times_{\mathscr{C}at_\infty} \{\mathscr{E}\}$.

6.5. **Covariant transport: classifying cocartesian fibrations.**

def:transp_witness **Definition 6.8.** Let $q : X \to S$ be a cocartesian fibration. We say a diagram of simplicial sets

$$\begin{tikzcd} X \arrow[r, "\widetilde{F}"] \arrow[d, "q"'] & \mathscr{P}.\mathscr{C}at_\infty \arrow[d, "\mathrm{univ}"] \\ S \arrow[r, "F"'] & \mathscr{C}at_\infty \end{tikzcd} \tag{11}$$ eq:672

witnesses the functor $F : S \to \mathscr{C}at_\infty$ as a classifying functor, or covariant transport functor, for q if the corresponding map to the fiber product

$$X \to \mathscr{P}.\mathscr{C}at_\infty \times_{\mathscr{C}at_\infty} S$$

is an equivalence of cocartesian fibrations over X. We say a functor $F : S \to \mathscr{C}at_\infty$ classifies the cocartesian fibration $q : X \to S$, or is a transport functor for q, if there exists a diagram (11) which witnesses F as a classifying functor for q.

The fiber product considered above is often denoted

$$\int_S F := S \times_{\mathscr{C}at_\infty} \mathscr{P}.\mathscr{C}at_\infty.$$

Implicit in Definition 6.8 is the requirement that the functor $\widetilde{F}$ preserves cocartesian edges. Indeed, this is equivalent to the fact that the associated morphism to the pullback preserves cocartesian edges.

Remark 6.9. The term “classifies” is used with some frequency in the works [3, 4]. However, [5] seems to prefer the term “transport representation” for a functor F as above. We will usually just refer to F as a, or the, transport functor for q.

We have an alternate characterization of transport functors in the case where the base S is an ∞-category.

Proposition 6.10. *Let $q : \mathscr{E} \to \mathscr{C}$ be a cocartesian fibration over an ∞-category $\mathscr{C}$. A diagram (11) witnesses the functor $F : \mathscr{C} \to \mathscr{C}at_\infty$ as a transport functor for q if and only if the diagram (11) is a categorical pullback diagram*

Proof. This is an immediate consequence of Corollary I-6.22 and Proposition I-6.23. □

cor:c_univ_cocart **Corollary 6.11.** *Consider a functor of ∞-categories $F : S \to \mathscr{C}at_\infty$ and $q : \int_S F \to S$ be the corresponding cocartesian fibration. An edge*

$$(\zeta, \alpha) : (x, a : * \to F(x)) \to (y, b : * \to F(y))$$

in $\int_S F$ is q-cocartesian if and only if the underlying morphism $\alpha : F(\zeta)(a) \to b$ is an isomorphism in the ∞-category $F(b)$.

One observes that transport functors are stable under restriction.

lem:687 **Lemma 6.12.** *Suppose we have a pullback square*

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "q'"'] & X \arrow[d, "q"] \\ S' \arrow[r, "f"'] & S \end{tikzcd} \tag{12}$$ eq:689

in which q and q' are are cocartesian fibrations between ∞-categories and g preserves cocartesian edges. Consider a diagram of the form (11) which witnesses a functor $F : S \to \mathscr{C}at_\infty$ as transport along q. Then for $\widetilde{F}' = \widetilde{F}g$ and $F' = Ff$, the diagram

$$\begin{tikzcd} X' \arrow[r, "\widetilde{F}'"] \arrow[d, "q'"'] & \mathscr{P}.\mathscr{C}at_\infty \arrow[d] \\ S' \arrow[r, "F'"'] & \mathscr{C}at_\infty. \end{tikzcd}$$

witnesses F' as a transport functor along q'.

Proof. Our hypothesis is that the induced map $X \to \int_S F$ is an equivalence of cocartesian fibrations over S. By the formula $\int_{S'} F' = S' \times_S (\int_S F)$, Proposition 3.2, and Corollary 3.9, it follows that the induced map $S' \to \int_{S'} F'$ is an equivalence as well. □

Given a cocartesian fibration we can now consider the simplicial subset in the functor category

$$\mathrm{Fun}(S, \mathscr{C}at_\infty) \times_{\mathrm{Fun}(X, \mathscr{C}at_\infty)} \mathrm{Fun}(X, \mathscr{P}.\mathscr{C}at_\infty) \tag{13}$$ eq:710

which consists of diagrams witnessing transport for a given cocartesian fibration $q : X \to S$.

def:transp_space

Definition 6.13. For a given cocartesian fibration $q : X \to S$, we let $\mathscr{T}\mathscr{W}it(q)$ denote the simplicial subset in the fiber product (13) whose simplices correspond to diagrams

$$\begin{tikzcd} \Delta^n \times X \arrow[r, "\widetilde{F}"] \arrow[d, "\Delta^n \times q"'] & \mathscr{P}.\mathscr{C}at_\infty \arrow[d] \\ \Delta^n \times S \arrow[r, "F"'] & \mathscr{C}at_\infty. \end{tikzcd}$$

which witness F as a covariant transport functor along $\Delta^n \times q$.

Stability of such diagrams under restriction (Lemma 6.12) assures us that $\mathscr{T}\mathscr{W}it(q)$ is in fact a simplicial subset in the given fiber product.

thm:transport

Theorem 6.14 (Universality theorem [5, 02SC])**.** *For any cocartesian fibration $q : \mathscr{E} \to \mathscr{C}$, the simplicial set $\mathscr{T}\mathscr{W}it(q) \to *$ is a contractible Kan complex.*

Note that a contractible Kan complex is, by definition, a Kan complex for which the terminal morphism $\mathscr{X} \to *$ is a trivial Kan fibration. In particular, a contractible Kan complex is nonempty. So, Theorem 6.14 says that any cocartesian fibration q admits a covariant transport functor $F : \mathscr{C} \to \mathscr{C}at_\infty$, and that this functor is uniquely determined up to a contractible space of choices.

6.6. **Transport for cartesian fibrations.** Given any cartesian fibration $p : Y \to S$, we have the associated cocartesian fibration $p^{\mathrm{op}} : Y^{\mathrm{op}} \to S^{\mathrm{op}}$. So our analysis of classifying functors for cocartesian fibrations dualizes in the obvious ways to provide an analysis of classifying functors for cartesian fibrations.

Definition 6.15. Let $p : Y \to S$ be a cartesian fibration of ∞-categories. We say a functor $F : \mathscr{C}^{\mathrm{op}} \to \mathscr{C}at_\infty$ classifies the cartesian fibration p, or is a contravariant transport functor for p, if it a classifying functor for the corresponding cocartesian fibration p^{op}.

Gives a cartesian fibration $p : Y \to S$, we define the space of transport functors with witness in the obvious way $\mathscr{T}\mathscr{W}it(p) := \mathscr{T}\mathscr{W}it(p^{\mathrm{op}})$. Theorem 6.14 implies contractibility of this space.

thm:uniq_transp

Theorem 6.16 (Contravariant universality)**.** *For any cartesian fibration $p : Y \to S$, the simplicial set $\mathscr{T}\mathscr{W}it(p)$ is a contractible Kan complex.*

Again this establishes both the existence and uniqueness of contravariant transport.

6.7. **Classification of left and right fibrations.** We have the simplicial subcategory $\underline{\mathrm{Kan}} \to \underline{\mathrm{Cat}}_\infty$ and subsequent simplicial subset $\mathscr{K}an \subseteq \mathfrak{Cat}_\infty$. This simplicial subset is the full $(\infty, 2)$-subcategory whose objects are precisely those ∞-categories which are Kan complexes, and so the inclusion preserves thin 2-simplices. We now have the full $(\infty, 2)$-subcategory $\mathscr{K}an_{*/} \to (\mathfrak{Cat}_\infty)_{*/}$ of pointed Kan complexes and the pullback diagram

$$\begin{tikzcd} \mathscr{K}an_{*/} \arrow[r] \arrow[d] & (\mathfrak{Cat}_\infty)_{*/} \arrow[d] \\ \mathscr{K}an \arrow[r] & \mathfrak{Cat}_\infty \end{tikzcd}$$

which restricts to a pullback diagram into the piths

$$\begin{tikzcd} \mathscr{K}an_{*/} \arrow[r] \arrow[d] & (\mathfrak{Cat}_\infty)_{*/} \arrow[d] \\ \mathscr{K}an \arrow[r] & \mathfrak{Cat}_\infty. \end{tikzcd}$$

We recall that the map $\mathscr{K}an_{*/} \to \mathscr{K}an$ is a left fibration, by Corollary I-**??**.

prop:kan_transp

Proposition 6.17. *A cocartesian fibration $q : X \to S$ is a left fibration if and only if the corresponding transport functor $F : \mathscr{C} \to \mathscr{C}at_\infty$ has image in $\mathscr{K}an$. Similarly, a cartesian fibration $p : X \to S$ is a right fibration if and only if the corresponding transport functor $G : \mathscr{C}^{\mathrm{op}} \to \mathscr{C}at_\infty$ has image in $\mathscr{K}an$.*

Proof. By Proposition **??**, a cocartesian (resp. cartesian) fibration $X \to S$ is a left (resp. right) fibration if and only if its fibers over object sin $\mathscr{C}$ are Kan complexes. So the result follows by the calculation of the fibers of the pullback fibration $\int_S F \to \mathscr{C}$ provided in Proposition 6.7. □

This proposition tells us that any left fibration $q : X \to S$ fits into a square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "q"'] & \mathscr{K}an_{*/} \arrow[d] \\ S \arrow[r, "F"'] & \mathscr{K}an \end{tikzcd}$$

for which the induced map to the pullback $X \to \mathscr{K}an_{*/} \times_{\mathscr{K}an} S$ is an equivalence of left fibrations over S. In this way left fibrations are classified by maps into the ∞-category of Kan complexes. One obtains similar statements for right fibrations by applying the opposite functor.

sect:weight_nerv

6.8. **Weighted nerves.** In the following two subsections we provide a complete and explicit description of the fibration $\int_{\mathbb{A}} F \to \mathbb{A}$ in the case where $\mathbb{A}$ is a discrete category and F comes from a simplicial functor. We subsequently obtain an explicit classification of cocartesian fibrations over discrete categories, up to equivalence.

Definition 6.18. Let K be a simplicial set. A strictly commuting diagram in $\mathscr{C}at_\infty$ is a functor $p : K \to \mathscr{C}at_\infty$ which admits a factorization $K \to \mathrm{Cat}_\infty \to \mathscr{C}at_\infty$.

We note that Cat_∞ is a simplicial subset in $\mathscr{C}at_\infty$, so that, if such a factorization exists then it is unique. We are also interested in functors $F : \mathbb{A} \to \mathscr{C}at_\infty$ which admit such a (unique) factorization. These functors are also, formally speaking in terms of our definition, strictly commuting diagrams.

def:weight_nerv

Definition 6.19. Let $\mathbb{A}$ be a discrete category and $F : \mathbb{A} \to \mathscr{C}at_\infty$ be a functor which factors through the discrete category Cat_∞. We define the weighted nerve $\mathrm{N}^F(\mathbb{A})$ to be the simplicial set with n-simplices

$$\mathrm{N}^F(\mathbb{A})[n] = (\sigma : \Delta^n \to \mathbb{A}, \tau_i : \Delta^i \to F(a_i) : 0 \le i \le n)$$

where in the above expression the a_i are the object $\sigma(i)$ in $\mathbb{A}$, the τ_i are required to fit into strictly commuting diagram

$$\begin{tikzcd} \Delta^i \arrow[r] \arrow[d, "\tau_i"'] & \Delta^{i+1} \arrow[d, "\tau_{i+1}"] \\ F(a_i) \arrow[r, "F(i<i+1)"'] & F(a_{i+1}) \end{tikzcd}$$

at all $i < n$, and where each inclusion $\Delta^i \to \Delta^{i+1}$ is induced by the inclusion $[i] \to [i+1]$.

Below we may write τ for the tuple $\tau = \{\tau_i : \Delta^i \to F(a_i) : 0 \leq i \leq n\}$, so that an n-simplex in the weighted nerve appears as a pair (σ, τ).

As for the restriction maps, given a non-decreasing function $t : [m] \to [n]$ the restriction function t^* sends a pair (σ, τ) to the pair of the m-simplex σt^* in $\mathbb{A}$ with the tuple of simplices

$$\tau'_j : \Delta^j \to \Delta^{t(j)} \to F(a_{t(j)})$$

where the first map is simply given by restricting t to a function $t : [j] \to [t(j)]$. We have the obvious forgetful map

$$\mathrm{N}^F(\mathbb{A}) \to \mathbb{A}, \quad (\sigma, \tau) \mapsto \sigma.$$

At the lowest levels, one sees that vertices $* \to \mathrm{N}^F(\mathbb{A})$ consist of a choice of object $\bar{x}$ in $\mathbb{A}$ and an object $x : * \to F(\bar{x})$ in the ∞-category over $\bar{x}$. An edge $\Delta^1 \to \mathrm{N}^F(\mathbb{A})$ over a morphism $\alpha : \bar{x} \to \bar{y}$ in $\mathbb{A}$ consists of a choice of objects $x : * \to F(\bar{x})$ and $y : * \to F(y)$, along with with a morphism $\xi : F(\alpha)(x) \to y$ in the ∞-category $F(y)$.

ex:point_nerv

Example 6.20 ([5, 025Y])**.** Let $\mathscr{C} : * \to \underline{\mathrm{Cat}}_\infty$ be the function which just picks an ∞-category $\mathscr{C}$. Then the first term in an n-simplex (σ, τ) in $\mathrm{N}^{\mathscr{C}}(*)$ is trivial, and the second term determines an increasing sequence of simplices in $\mathscr{C}$,

$$\begin{tikzcd} \Delta^0 \arrow[r] \arrow[drr, "\tau_0"'] & \Delta^1 \arrow[r] \arrow[dr, "\tau_1"] & \cdots \arrow[r] & \Delta^{n-1} \arrow[r] \arrow[dl, "\tau_{n-1}"'] & \Delta^n \arrow[dll, "\tau_n"] \\ & & \mathscr{C} & & \end{tikzcd}.$$

So we see that the functions $\phi[n] : \mathrm{N}^{\mathscr{C}}(*)[n] \to \mathscr{C}[n]$, $(\sigma, \tau) \mapsto \tau_n$ determine a isomorphism of simplicial sets $\mathrm{N}^{\mathscr{C}}(*) \overset{\cong}{\to} \mathscr{C}$.

Example 6.21. For the constant functor $* : \mathbb{A} \to \underline{\mathrm{Cat}}^+_\infty$, which sends all object to the terminal space $*$, the second factor in any n-simplex (σ, τ) in the weighted nerve is completely determined, so that we obtain an isomorphism $\mathrm{N}^*(\mathbb{A}) \cong \mathbb{A}$.

We note that the weighted nerve construction is functorial in the obvious ways. Namely, if $F : \mathbb{A} \to \mathscr{C}at_\infty$ is a functor which factors through Cat_∞ and $\phi : \mathbb{B} \to \mathbb{A}$ is a functor between ∞-categories, then we have a map of simplicial sets

$$\mathrm{N}(\phi) : \mathrm{N}^{F\phi}(\mathbb{B}) \to \mathrm{N}^F(\mathbb{A})$$

which just composes simplices in the first factor $\sigma \mapsto \sigma\phi$, and is the identity in the second factor $\tau \mapsto \tau$. The following calculation is immediate.

lem:1277 **Lemma 6.22.** *Given a functor between discrete categories $\phi : \mathbb{B} \to \mathbb{A}$ and a functor $F : \mathbb{A} \to \mathscr{C}at_\infty$ which factors through Cat_∞, the weighted nerves fit into a pullback diagram*

$$\begin{tikzcd} \mathrm{N}^{F\phi}(\mathbb{B}) \arrow[r, "\mathrm{N}(\phi)"] \arrow[d] & \mathrm{N}^F(\mathbb{A}) \arrow[d] \\ \mathbb{B} \arrow[r, "\phi"'] & \mathbb{A}. \end{tikzcd}$$

Our main objective for the remainder of the subsection is to prove the following result.

prop:cocart_relnev **Proposition 6.23.** *Let $F, G : \mathbb{A} \to \mathscr{C}at_\infty$ be functors which admit factorizations through Cat_∞, and $\xi : F \to G$ be a natural transformation which also factors through Cat_∞. Suppose that at each object a in $\mathbb{A}$ the morphism $\xi(a) : F(a) \to G(a)$ is a cocartesian fibration, and that for any morphism $t : a \to b$ the map $F(t) : F(a) \to F(b)$ preserves ξ-cocartesian maps. Then the following hold:*

1. *The induced map $\mathrm{N}^\xi : \mathrm{N}^F(\mathbb{A}) \to \mathrm{N}^G(\mathbb{A})$ is a cocartesian fibration.*
2. *An edge $\lambda : F(\alpha)(x) \to y$ in $\mathrm{N}^F(\mathbb{A})$, over an edge $t : a \to b$ in $\mathbb{A}$, is N^ξ-cocartesian if and only if the underlying map $\lambda : \Delta^1 \to F(b)$ is $\xi(b)$-cocartesian.*

Here N^ξ is the obvious map, i.e. the map which sends an n-simplex $(\sigma, \tau_i : 0 \leq i \leq n)$ in $\mathrm{N}^F(\mathbb{A})$ to $\sigma : \Delta^n \to \mathbb{A}$ paired with the composites $\xi(a_i)\tau_i : \Delta^i \to F(a_i) \to G(a_i)$. In the case of the constant functor $G : \mathbb{A} \to \mathscr{C}at_\infty$, with $G(a) = *$ at all a in $\mathbb{A}$, Proposition 6.23 appears as follows.

Corollary 6.24. *The forgetful functor $q : \mathrm{N}^F(\mathbb{A}) \to \mathbb{A}$ is a cocartesian fibration, and for any map $t : a \to b$ in $\mathbb{A}$, a morphism $\lambda : F(\alpha)(x) \to y$ over t in $\mathrm{N}^F(\mathbb{A})$ is q-cocartesian if and only if λ is an isomorphism in $F(b)$.*

For the proof of Proposition 6.23 we employ a relative join construction $\mathscr{C} \star_{\mathscr{T}} \mathscr{D}$. Here we consider the ∞-categories over $\mathscr{T}$, we have the unique map

$$\Delta^1 \times \mathscr{T} \to \mathscr{T} \star \mathscr{T}$$

with $\{0\} \times \mathscr{T}$ mapping identically to $\mathscr{T} \star \emptyset = \mathscr{T}$ and $\{1\} \times \mathscr{T}$ identically to $\emptyset \star \mathscr{T} = \mathscr{T}$, and we consider the fiber product

$$\mathscr{C} \star_{\mathscr{T}} \mathscr{D} := (\mathscr{C} \star \mathscr{D}) \times_{\mathscr{T} \star \mathscr{T}} (\Delta^1 \times \mathscr{T}).$$

lem:1307 **Lemma 6.25.** *Let $\mathscr{C} \to \mathscr{T}$ and $\mathscr{D} \to \mathscr{T}$ be maps of ∞-categories. The map $\mathscr{C} \star_{\mathscr{T}} \mathscr{D} \to \mathscr{C} \star \mathscr{D}$ is an inner fibration.*

Proof. It suffices to show that the map $\Delta^1 \times \mathscr{T} \to \mathscr{T} \star \mathscr{T}$ is an inner fibration. This follows from the fact that the composite

$$\Delta^1 \times \mathscr{T} \to \mathscr{T} \star \mathscr{T} \to \Delta^0 \star \Delta^0 = \Delta^1,$$

which one sees is just the projection onto the first factor, is an inner fibration. Indeed, for a given lifting problem for an inner horn

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & \Delta^1 \times \mathscr{T} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & \mathscr{T} \star \mathscr{T} \end{tikzcd} \tag{14}$$ eq:1317

either Δ^n has image in one of the two $\mathscr{T}$ factors in $\mathscr{T} \star \mathscr{T}$, in which case the problem has a unique solution, or else a solution to the associated lifting problem

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & \Delta^1 \times \mathscr{T} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & \mathscr{T} \star \mathscr{T} \end{tikzcd}$$

solves (14). □

It is relatively easy to see that the map from the usual join $\mathscr{C} \star \mathscr{D} \to \Delta^0 \star \Delta^0 = \Delta^1$ is an inner fibration, so that $\mathscr{C} \star \mathscr{D}$ is in particular an ∞-category. It follows from Lemma 6.25 that the relative join is an ∞-category as well.

Corollary 6.26. *For ∞-categories $\mathscr{C}$ and $\mathscr{D}$ over another ∞-category $\mathscr{T}$, the relative join $\mathscr{C} \star_{\mathscr{T}} \mathscr{D}$ is also an ∞-category.*

To describe the relative join, we have the projection

$$\mathscr{C} \star_{\mathscr{T}} \mathscr{D} \to \mathscr{C} \star \mathscr{D} \to \Delta^1$$

and one the fibers over $\{0\}$ and $\{1\}$ are copies of $\mathscr{C}$ and $\mathscr{D}$ respectively. Furthermore, for any map $\alpha : \Delta^1 \to \mathscr{C} \star_{\mathscr{T}} \mathscr{D}$ with $\alpha|_{\{0\}} = x$ in $\mathscr{C}$ and $\alpha|_{\{1\}} = y$ in $\mathscr{D}$ one calculates

$$\mathrm{Hom}_{\mathscr{C} \star_{\mathscr{T}} \mathscr{D}}(x, y) = \mathrm{Hom}_{\Delta^1 \times \mathscr{T}}((0, \bar{x}), (1, \bar{y})),$$

where $\bar{x}$ and $\bar{y}$ are the images of x and y in $\mathscr{T}$, respectively. This latter mapping space is the product

$$\mathrm{Hom}_{\Delta^1}(0, 1) \times \mathrm{Hom}_{\mathscr{T}}(\bar{x}, \bar{y}) = \{*\} \times \mathrm{Hom}_{\mathscr{T}}(\bar{x}, \bar{y}).$$

So in total we calculate

$$\mathrm{Hom}_{\mathscr{C} \star_{\mathscr{T}} \mathscr{D}}(x, y) = \begin{cases} \mathrm{Hom}_{\mathscr{C}}(x, y) & \text{if } x \text{ and } y \text{ are in } \mathscr{C} \\ \mathrm{Hom}_{\mathscr{D}}(x, y) & \text{if } x \text{ and } y \text{ are in } \mathscr{D} \\ \mathrm{Hom}_{\mathscr{T}}(\bar{x}, \bar{y}) & \text{if } x \text{ is in } \mathscr{C} \text{ and } y \text{ in } \mathscr{D}. \end{cases}$$

The mapping spaces vanish when x is in $\mathscr{D}$ and y is in $\mathscr{C}$, as there are simply no 1-simplices which begin at x and end in y by the definition of the join.

The proof of Proposition 6.23 relies on the following generic observation.

`lem:reljoin_cocart` **Lemma 6.27** ([5, 02RH]). *Consider a diagram of ∞-categories*

$$\begin{tikzcd} \mathscr{C} \arrow[r, "F"] \arrow[d, "q"'] & \mathscr{T} \arrow[d, "p"] \\ \mathscr{D} \arrow[r] & \mathscr{V} \end{tikzcd}$$

in which q and p are cocartesian fibrations and F preserved cocartesian morphisms. The the induced map $z : \mathscr{C} \star_{\mathscr{T}} \mathscr{T} \to \mathscr{D} \star_{\mathscr{V}} \mathscr{V}$ is also a cocartesian fibration. Furthermore, a map $\alpha : x \to y$ in $\mathscr{C} \star_{\mathscr{T}} \mathscr{T}$ is z-cocartesian if and only if one of the following two conditions holds:

(a) *Both x and y are in $\mathscr{C}$, and α is q-cocartesian.*
(b) *One of x or y is not in $\mathscr{C}$, and the image of α along the map $\mathscr{C} \star_{\mathscr{T}} \mathscr{T} \to \Delta^1 \times \mathscr{T} \to \mathscr{T}$ is p-cocartesian.*

While the proof is not extraordinarily complications, but requires one to touch on various points regarding the relative join. We refer the reader to [5], and in particular [5, 0241], for the details.

We now provide our argument for Proposition 6.23.

Proof of Proposition 6.23. Via Lemma 6.22 we reduce to the case of an n-simplex $\mathbb{A} = \Delta^n$, and we proceed by induction to observe the conclusions of Proposition 6.23 over such a base. In the case $n = 0$, the functors $F : G : * \to \mathscr{C}at_\infty$ just choose ∞-categories and the transformation ξ is just a map of ∞-categories $\xi : F(*) \to G(*)$ which is specifically a cocartesian fibration. Under the identifications

$$\mathrm{N}^F(*) = F(*) \quad \text{and} \quad \mathrm{N}^G(*) = G(*)$$

of Example 6.20 we have $\mathrm{N}^\xi = \xi$. Since ξ is a cocartesian fibration by hypothesis we obtain condition (1). Condition (2) demands that a map $\alpha : x \to y$ in $F(*)$ is N^ξ-cocartesian if and only if it is ξ-cocartesian, which is a tautology and in particular is true.

Suppose now that the result holds over $\mathbb{A}_0 = \Delta^{n-1}$, and consider $\mathbb{A} = \Delta^n = \mathbb{A}_0 \star \{n\}$. We take two functors

$$F, G : \mathbb{A} \to \mathscr{C}at_\infty$$

and a transformation $\xi : F \to G$ as prescribed. We have natural decompositions

$$\mathrm{N}^{F_0}(\mathbb{A}) \cong \mathrm{N}^F(\mathbb{A}_0) \star_{F(n)} F(n) \quad \text{and} \quad \mathrm{N}^{G_0}(\mathbb{A}) \cong \mathrm{N}^G(\mathbb{A}_0) \star_{G(n)} G(n)$$

where the map $\mathrm{N}^F(\mathbb{A}) \to F(n)$ and $\mathrm{N}^G(\mathbb{A}) \to G(n)$ are provided by the structural morphisms $F(m \leq n) : F(m) \to F(n)$ and $G(m \leq n) : G(m) \to G(n)$. Here also F_0 and G_0 are the obvious restrictions.

The map $N^{\xi_0} : \mathrm{N}^{F_0}(\mathbb{A}_0) \to \mathrm{N}^{G_0}(\mathbb{A}_0)$ is a cocartesian fibration by our induction hypothesis, and $\xi(n) : F(n) \to G(n)$ is a cocartesian fibration by assumption. Also by assumption the map $N^{F_0}(\mathbb{A}_0) \to F(n)$ preserves cocartesian edges. It follows that the map in question

$$\mathrm{N}^\xi = \mathrm{N}^\xi \star_{\xi(n)} \xi(n) : \mathrm{N}^F(\mathbb{A}_0) \star_{F(n)} F(n) \to \mathrm{N}^G(\mathbb{A}_0) \star_{G(n)} G(n)$$

is a cocartesian fibration by Lemma 6.27. Lemma 6.27 also verifies the proposed description of cocartesian morphisms in $\mathrm{N}^F(\mathbb{A})$. □

sect:weighted_fib

6.9. Fibrations over discrete categories via weighted nerves.

thm:weight_nerv_univ

Theorem 6.28 ([5, 027J])**.** *Let $\mathbb{A}$ be a discrete category and $F : \mathbb{A} \to \mathscr{C}at_\infty$ be a functor which factors through the discrete category $\mathscr{C}at_\infty$. Then there is an equivalence of cocartesian fibrations*

$$\begin{tikzcd} \mathrm{N}^F(\mathbb{A}) \arrow[rr, "\sim", "\mu"'] \arrow[dr] & & \int_{\mathbb{A}} F \arrow[dl] \\ & \mathbb{A} & \end{tikzcd}$$

We describe the functor $\mu : \mathrm{N}^F(\mathbb{A}) \to \int_{\mathbb{A}} F$, but leave the verification that it is an equivalence to the text [5]. Our job here is simple–given an n-simplex in the weighted nerve we need to produce an n-simplex in the ∞-category $\int_{\mathbb{A}} F$.

We consider an n-simplex $\omega : \Delta^n \to \mathrm{N}^F(\mathbb{A})$, which is specified by a pair $(\sigma : \Delta^n \to \mathbb{A}, \tau_i : \Delta^i \to F(a_i))$, where $a_i = \sigma(i)$. From ω we produce a diagram

$$\omega' : \{-1\} \star \Delta^n \to \mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{Cat}}_\infty) = \mathbf{Cat}_\infty$$

with $\omega'|_{\Delta^n} = f(\sigma)$ and $\omega'(-1) = *$. Such a diagram corresponds to an n-simplex in the undercategory $(\mathfrak{Cat}_\infty)_{*/}$ so that we have a diagram

$$\begin{tikzcd} \Delta^n \arrow[r, "\omega'"] \arrow[d] & (\mathfrak{Cat}_\infty)_{*/} \arrow[d] \\ \mathbb{A} \arrow[r, "F"'] & \mathfrak{Cat}_\infty. \end{tikzcd}$$

Since $\mathbb{A}$ has image in the pith $\mathscr{C}at_\infty = (\mathfrak{Cat}_\infty)^{\mathrm{Pith}}$, any such lift ω' also has image in the pith $\mathscr{P}.\mathscr{C}at_\infty = ((\mathfrak{Cat}_\infty)_{*/})^{\mathrm{Pith}}$ by Corollary 5.21 and Proposition 5.22. Hence ω' defines a map into the fiber product $\Delta^n \to \int_{\mathbb{A}} F$, and we denote this n-simplex simply ω' by an abuse of notation.

Let us proceed with the construction of the associated simplex $\omega' : \{-1\} \star \Delta^n \to \mathfrak{Cat}_\infty$. Below we take $\underline{\mathbb{A}}$ the underlying simplicial category for $\mathbb{A}$, so that $\mathbb{A} = \mathrm{N}^{\mathrm{hc}}(\underline{\mathbb{A}})$. We define, following [5, 027A], ω' as the unique simplicial functor

$$\omega' : \mathrm{Path}(\{-1\} \star \Delta^n) \to \underline{\mathrm{Cat}}_\infty$$

with

$$\omega'|_{\mathrm{Path}(\Delta^n)} = F \circ \sigma : \mathrm{Path}(\Delta^n) \to \underline{\mathbb{A}} \to \underline{\mathrm{Cat}}_\infty$$

and $\omega'(-1) = *$ and each map

$$\underline{\mathrm{Hom}}(-1, i) = \mathrm{N}(\mathrm{Subsets}^{\mathrm{op}}_{-1,i}) \to \mathrm{Fun}(*, F(a_i)) = F(a_i)$$

given as the composite

$$\mathrm{N}(\mathrm{Subsets}^{\mathrm{op}}_{-1,i}) \xrightarrow{\rho} \Delta^i \xrightarrow{\tau_i} F(a_i),$$

where ρ is induced by the map of partially ordered sets

$$\mathrm{Subsets}^{\mathrm{op}}_{-1,i} \to \Delta^i, \quad S \mapsto \min(S - \{-1\}).$$

One can check that ω' is in fact a well-defined simplicial functor, and given a diagram

$$\begin{tikzcd} \Delta^n \arrow[r, "\omega"] & \mathrm{N}^F(\mathbb{A}) \\ \Delta^m \arrow[u] \arrow[r, "\nu"'] & \mathrm{N}^F(\mathbb{A}) \arrow[u, "="'] \end{tikzcd}$$

one checks directly from the definition the corresponding diagram

$$\begin{tikzcd} \mathrm{Path}\{-1\} \star \Delta^n \arrow[r, "\omega'"] & \underline{\mathrm{Cat}}_\infty \\ \mathrm{Path}\{-1\} \star \Delta^m \arrow[u] \arrow[r, "\nu'"'] & \underline{\mathrm{Cat}}_\infty. \arrow[u, "="'] \end{tikzcd}$$

In this way we obtain maps $\mathrm{N}^F(\mathbb{A})[n] \to (\int_{\mathbb{A}} F)[n]$ which assemble into a map of simplicial sets $\mu : \mathrm{N}^F(\mathbb{A}) \to \int_{\mathbb{A}} F$.

Finally, it is argued in [5, Proof of] that the map μ is an equivalence by first noting that the induced maps on each fiber

$$\mu_a : F(a) \cong \mathrm{N}^F(\mathbb{A})_a \to (\int_{\mathbb{A}} F)_a$$

is an equivalence, which is certainly expected from the fiber calculations of 6.7, and als that μ preserved cocartesian edges. This latter point is determined via the explicit descriptions of cocartesian edges in each ∞-category provided in Proposition 6.23 and Corollary 6.11.

lem:simplicial_fun

Lemma 6.29. *Let $\mathbb{A}$ be a discrete category. Any functor $F : \mathbb{A} \to \mathscr{C}at_\infty$ is isomorphic to one a functor $F' : \mathbb{A} \to \mathscr{C}at_\infty$ which factors through the discrete subcategory* Cat_∞.

Proof. Let $\mathrm{Cat}_{\mathrm{Kan}}$ be the category of simplicial categories which are enriched in Kan complexes. By [3, Theorem 2.2.5.1] the homotopy coherent nerve provides an equivalence of homotopy categories

$$\mathrm{h}\,\mathrm{N}^{\mathrm{hc}} : \mathrm{h}\,\mathrm{Cat}_{\mathrm{Kan}} \overset{\sim}{\to} \mathrm{h}\,\mathscr{C}at_\infty.$$

In particular, every functor between homotopy coherent nerves $F : \mathrm{N}^{\mathrm{hc}}(\underline{A}) \to \mathrm{N}^{\mathrm{hc}}(\underline{B})$ is, up to natural isomorphism, identified with $\mathrm{N}^{\mathrm{hc}}(F')$ for some simplicial functor F'. We note finally that any simplicial functor $\underline{\mathbb{A}} \to \underline{\mathrm{Cat}}^+_\infty$ necessarily factors through the discrete category $\mathrm{Cat}_\infty = \underline{\mathrm{Cat}}^+_\infty[0]$. $\square$

From Theorem 6.28 in conjunction with a result from Section **??** below, Theorem **??** and Proposition **??**, we obtain a classification of all cocartesian fibrations over a discrete category.

thm:discrete_cocart

Theorem 6.30. *Let $\mathbb{A}$ be a discrete category and $q : \mathscr{E} \to \mathbb{A}$ be an arbitrary cocartesian fibration. The there is a functor $F : \mathbb{A} \to \mathrm{Cat}_\infty \subseteq \mathscr{C}at_\infty$ for which we have an equivalence $\mathscr{E} \overset{\sim}{\to} \mathrm{N}^F(\mathbb{A})$ of cocartesian fibrations over $\mathbb{A}$.*

sect:htf

7. Transport II: Homotopy representations

7.1. **Homotopy transport representations.** Consider a cocartesian fibration $q : X \to S$ and an edge $\alpha : s \to t$ in S. We then have the fibers X_s and X_t over these respective points, both of which are ∞-categories.

We consider the diagram

$$\begin{tikzcd} \{0\} \times X_s \arrow[r] \arrow[d] & X \arrow[d, "q"] \\ \Delta^1 \times X_s \arrow[r] & S \end{tikzcd}$$

where the top arrow is the inclusion and the bottom arrow is the composite of the projection with α,

$$\Delta^1 \times X_s \to \Delta^1 \times \{s\} \overset{ev_\alpha}{\to} S.$$

By Theorem 2.7 the above diagram is split by a transformation

$$\xi_\alpha : \Delta^1 \times X_s \to X$$

which has $\alpha_!|_{\{0\}\times X_s}$ equal to the inclusion and has

$$\xi_\alpha|_{\Delta^1 \times \{s'\}} : \Delta^1 \times \{s'\} \to X$$

a q-cocartesian morphism over α. In particular, the restriction at 1 produces a functor

$$\alpha_! := \xi_\alpha|_{\{1\}\times X_s} : X_s \to X_t.$$

Furthermore, this transformation ξ_α is uniquely determined up to a contractible space of choices, so that $\alpha_!$ is similarly uniquely determined up to a contractible space as well.

def:homotop_transp **Definition 7.1.** Given a cocartesian fibration $q : X \to S$ and any edge $\alpha : s \to t$ in the base, we let $\alpha_! : X_s \to X_t$ denote the uniquely determined functor which comes equipped with a cocartesian transformation ξ_α over α, as above. We call $\alpha_! : X_s \to X_t$ the homotopy transport functor over α.

prop:1297 **Proposition 7.2.** *Let $X \to S$ be a cocartesian fibration. Suppose that we have a 2-simplex $A : \Delta^2 \to S$ and take $\alpha_{ij} = A|_{\Delta^{\{i,j\}}} : s_i \to s_j$. Then there is an isomorphism*

$$(\alpha_{02})_! \cong (\alpha_{12})_!(\alpha_{01})_!$$

in $\mathrm{Fun}(X_{s_0}, X_{s_2})$.

Proof. Take $X_i = X_{s_i}$. Consider a diagram $\widetilde{A} : \Delta^1 \times \Delta^1 \to S$ which appears as

$$\begin{tikzcd}
s_2 \arrow[r, "id"] & s_2 \\
s_0 \arrow[u, "\alpha_{02}"] \arrow[ur, "\alpha_{02}"] \arrow[r, "\alpha_{01}"'] & s_1, \arrow[u, "\alpha_{12}"']
\end{tikzcd}$$

which we might obtain by expanding A for example. Then we have a lifting problem

$$\begin{tikzcd}
\{0\} \times \Delta^1 \times X_0 \arrow[r, "\xi_{\alpha_{02}}"] \arrow[d] & X \arrow[d] \\
\Delta^1 \times \Delta^1 \times X_0 \arrow[r, "\widetilde{A} \times \{s_0\}"'] \arrow[ur, dotted] & S
\end{tikzcd}$$

and can take $\xi_A : \Delta^1 \times \Delta^1 \times X_0 \to X$ to be the unique cocartesian solution. Restricting ξ_A to $\Delta^1 \times \{0\} \times X_0$ provides a cocartesian lift of α_{01} and so is identified with $\xi_{\alpha_{01}}$, and similarly restricting ξ_A to the edge $\Delta^1 \times \{1\} \times X_0$ is identified with $id_{\alpha_{02}}$. By the 2-of-3 property for q-cocartesian maps, we also see that the restriction of ξ_A to the diagonal $\Delta^1 \times X_0$ is a cocartesian lift of α_{02} and so is identified with $\xi_{\alpha_{02}}$ as well.

We have only the edge $\{1\} \times \Delta^1 \times X_0 \to X$ to be identified. By the 2-of-3 property again we see that F_A provides a cocartesian solution to the diagram

$$\begin{tikzcd}
\{1\} \times \{0\} \times X_0 \arrow[r, "(\alpha_{01})_!"'] \arrow[d] \arrow[rr, bend left, "\xi_A|_{\dots}"] & \{0\} \times X_1 \arrow[r] \arrow[d, dashed] & X \arrow[d] \\
\{1\} \times \Delta^1 \times X_0 \arrow[r, "id \times (\alpha_{01})_!"'] \arrow[urr, "\xi_A|_{\dots}"] & \Delta^1 \times X_1 \arrow[r] & S.
\end{tikzcd}$$

However we have the alternate cocartesian lift

$$\begin{tikzcd}
\{1\}\times\{0\}\times X_0 \arrow[r, "(\alpha_{01})_!"'] \arrow[d] & \{0\}\times X_1 \arrow[r] \arrow[d, dashed] & X \arrow[d] \\
\{1\}\times\Delta^1\times X_0 \arrow[r, "id\times(\alpha_{01})_!"'] & \Delta^1\times X_1 \arrow[r] \arrow[ur, "\xi_{\alpha_{12}}"] & S
\end{tikzcd}$$

so that there is a unique isomorphism

$$\xi_A|_{\{1\}\times\Delta^1\times X_0}\cong \xi_{\alpha_{12}}(id\times(\alpha_{01})_!).$$

So we restrict further to $\{1\}\times\{1\}\times X_0$ to get

$$(\alpha_{02})_! = \xi_{\alpha_{02}}|_{\{1\}}\cong \xi_A|_{\{1\}\times\{1\}}\cong \xi_{\alpha_{12}}(id\times(\alpha_{01})_!)|_{\{1\}} = (\alpha_{12})_!(\alpha_{01})_!.$$

$\square$

Corollary 7.3. *Let $q:\mathscr{E}\to\mathscr{C}$ be a cocartesian fibration over an ∞-category $\mathscr{C}$. The functors $\alpha_!:X_s\to X_t$ assemble into a functor into the homotopy category of ∞-categories $\bar{q}_!:\mathrm{h}\mathscr{C}\to\mathrm{h}\mathscr{C}at_\infty$.*

Definition 7.4. Let $q:\mathscr{E}\to\mathscr{C}$ be a cocartesian fibration over an ∞-category $\mathscr{C}$. The homotopy transport representation for q is the functor on homotopy categories

$$\bar{q}_!:\mathrm{h}\mathscr{C}\to\mathrm{h}\mathscr{C}at_\infty$$

is the functor whose value at each object $x:*\to\mathscr{C}$ is the fiber $\bar{q}_!(x)=\mathscr{E}_x$, and whose value at any morphism $\alpha:x\to y$ in $\mathscr{C}$ is the associated homotopy transport functor $\alpha_!:\mathscr{E}_x\to\mathscr{E}_y$.

More generally, we call any functor $F:\mathrm{h}\mathscr{C}\to\mathrm{h}\mathscr{C}at_\infty$ which comes equipped with a natural isomorphism $\zeta:F\overset{\sim}{\to}\bar{q}_!$ a homotopy transport representation for q.

One observes that the homotopy transport representation is natural in diagrams of cocartesian fibrations.

lem:1366 **Lemma 7.5.** *Let*

$$\begin{tikzcd}
Y \arrow[r, "F"] \arrow[d] & X \arrow[d] \\
T \arrow[r, "f"'] & S
\end{tikzcd}$$

be a diagram of cocartesian fibrations, i.e. a diagram in which F preserves cocartesian maps. Then for any edge $\alpha:t\to t'$ in T the homotopy transport functors fit into a diagram

$$\begin{tikzcd}
Y_t \arrow[r, "\alpha_!"] \arrow[d, "F"'] & Y_{t'} \arrow[d, "F"] \\
X_{f(t)} \arrow[r, "f(\alpha)_!"'] & X_{f(t')}
\end{tikzcd}$$

in $\mathscr{C}at_\infty$.

Proof. Follows from the fact that both $F\xi_\alpha$ and $\xi_{f(\alpha)}f$ provide cocartesian lifts for the diagram

$$\begin{tikzcd} \{0\}\times Y_t \arrow[r] \arrow[d] & \{0\}\times X_{f(t)} \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^1\times Y_t \arrow[r] & \Delta^1\times X_{f(t)} \arrow[r] & S. \end{tikzcd}$$

$\square$

7.2. **The $\mathrm{h}\mathscr{K}an$-enriched category of ∞-categories.** Let $\underline{A}$ be a simplicial category whose morphism complexes are all Kan complexes. Via the functor to the homotopy category $\pi : \mathrm{Kan} \to \mathrm{h}\mathscr{K}an$ we obtain a new category $\pi\underline{A}$ which is enriched in $\mathrm{h}\mathscr{K}an$. (Here we note that the usual product of Kan complexes endows $\mathrm{h}\mathscr{K}an$ with a unique symmetric monoidal structure under which the projection $\pi : \mathrm{Kan} \to \mathrm{h}\mathscr{K}an$ is symmetric monoidal.) We compare this Kan enriched category to the Kan enriched category $\pi\,\mathrm{N}^{\mathrm{hc}}(\underline{A})$ obtained via the mapping spaces in the homotopy coherent nerve and their associated composition functions of Section I-9.

prop:pi_hcnerve

Proposition 7.6 ([5, 02LN])**.** *Let $\underline{A}$ be a simplicial category whose morphism compelxes are Kan complexes, and let $\mathscr{A} = \mathrm{N}^{\mathrm{hc}}(\underline{A})$ denote its associated ∞-category. Then the natural equivalences*

$$\underline{\mathrm{Hom}}_{\underline{A}}(x,y) \overset{\sim}{\to} \mathrm{Hom}^{\mathrm{L}}_{\mathscr{A}}(x,y) \overset{\sim}{\to} \mathrm{Hom}_{\mathscr{A}}(x,y)$$

supplied by Theorems 5.27 and I-10.3 define an equivalence of $\mathrm{h}\mathscr{K}an$-enriched categories $\pi\underline{A} \to \pi\mathscr{A}$.

sect:ehtf

7.3. **Enriched homotopy transport.** Given a simplicial set S and vertices $s,t : * \to S$ we take $\mathrm{Hom}_S(s,t) = \mathrm{Fun}(\Delta^1,S)\times_{S\times S}\{(s,t)\}$. For any cocartesian fibration $q : X \to S$ we consider the evaluation map

$$\Delta^1 \times \mathrm{Hom}_S(s,t) \times X_s \to \Delta^1 \times \mathrm{Fun}(\Delta^1,S) \times \{s\} \overset{ev}{\to} S$$

and the diagram

$$\begin{tikzcd} \{0\}\times \mathrm{Hom}_S(s,t)\times X_s \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^1\times \mathrm{Hom}_S(s,t)\times X_s \arrow[r] & S. \end{tikzcd}$$

The top map is the composite of the projection to X_s with the inclusion $X_s \to X$. By Theorem 2.7 there is a unique functor

$$\xi : \Delta^1 \times \mathrm{Hom}_S(s,t) \times X_s \to X$$

which splits the above diagram and sends each edge $\Delta^1 \times \{(\alpha,x)\}$ to a q-cocartesian morphism in X. The uniqueness claim of Theorem 2.7 tells us that, and each α in $\mathrm{Hom}_S(s,t)$, ξ restricts to the transformation ξ_α appearing in the defintion of the homotopy transport functor $\alpha_!$. So the map

$$\xi|_{\{1\}} : \mathrm{Hom}_S(s,t) \times X_s \to X_t$$

provides a paramterized family of morphisms whose fibers are the homotopy transport functors $\alpha_!$.

def:pht **Definition 7.7.** Given a cocartesian fibration $q : X \to S$, we call the functor $\xi|_{\{1\}} : \mathrm{Hom}_S(s,t) \times X_s \to X_t$ constructed above the parametrized homotopy transport functor for q.

Note that we can view $\xi|_{\{1\}}$ as a functor

$$q_{s,t} : \mathrm{Hom}_S(s,t) \to \mathrm{Fun}(X_s, X_t)$$

via adjunction. In the case that S is an ∞-category, we note that $\widetilde{q}_{s,t}$ is a functor between ∞-categories.

Consider now a cocartesian fibration $\mathscr{E} \to \mathscr{C}$ over an ∞-category $\mathscr{C}$. Recall our notation $\pi\mathscr{C}$ for the $\mathrm{h}\mathscr{K}an$-enriched category with By similar arguments to those employed in our analysis of the homotopy transport functors $\alpha_! : X_s \to X_t$, one sees that these maps assemble into a functor of $\mathrm{h}\mathscr{K}an$-enriched categories

$$\pi\mathscr{C} \to \pi\underline{\mathrm{Cat}}^+_\infty$$

which lifts the homotopy transport functor $\bar{q}_!$ of Section 7.1.

def:param_transp **Definition 7.8.** Given a cocartesian fibration $q : \mathscr{E} \to \mathscr{C}$ over an ∞-category $\mathscr{C}$, we let

$$q_! : \pi\mathscr{C} \to \pi\mathscr{C}at_\infty$$

denote the $\mathrm{h}\mathscr{K}an$-enriched functor whose value at any object $x : * \to \mathscr{C}$ is the fiber $\mathscr{E}_x$, and whose values on morphisms

$$q_! : \mathrm{Hom}_{\mathscr{C}}(x,y) \to \mathrm{Fun}(\mathscr{E}_x, \mathscr{E}_y)^{\mathrm{Kan}} \cong \mathrm{Hom}_{\mathscr{C}at_\infty}(\mathscr{E}_x, \mathscr{E}_y)$$

are the functors induced by parametrized homotopy transport. We call $q_!$ the enriched homotopy transport representation associated to q.

We generally, we call any enriched functor $F : \pi\mathscr{C} \to \pi\mathscr{C}at_\infty$ which comes equipped with a natural isomorphism $\zeta : F \overset{\sim}{\to} q_!$ a homotopy transport representation for q.

Note that here we've employed the natural identification $\pi\underline{\mathrm{Cat}}^+_\infty \cong \mathscr{C}at_\infty$ provided by Proposition 7.6 here when we replace the functor categories with the Hom-spaces for $\mathscr{C}at_\infty$.

One again sees that the enriched homotopy transport representation is natural

lem:enriched_pullback **Lemma 7.9.** *Consider a diagram of ∞-categories*

$$\begin{tikzcd} \mathscr{E} \arrow[r, "G"] \arrow[d, "q"'] & \mathscr{K} \arrow[d, "p"] \\ \mathscr{C} \arrow[r, "F"'] & \mathscr{D} \end{tikzcd} \tag{15}$$ eq:1464

in which p and q are cocartesian fibrations and G preserves cocartesian maps. Suppose also that $\mathscr{C}$ and $\mathscr{D}$ are ∞-categories. Then the maps $G|_{\mathscr{E}_x} : \mathscr{E}_x \to \mathscr{K}_{F(x)}$ define a natural transformation between the enriched homotopy transport representations

$$G_! : q_! \to p_! F.$$

The proof is the same as that of Lemma 7.5. We note that when the diagram (15) is a categorical pullback diagram then the maps $G|_{\mathscr{E}_x} : \mathscr{E}_x \to \mathscr{K}_{F(x)}$ are isomorphisms in $\mathrm{h}\mathscr{K}an$, so that $\widetilde{G}_!$ is a natural isomorphism of $\mathrm{h}\mathscr{K}an$-enriched functors.

Lemma 7.10. *If a diagram* (15) *is a pullback diagram of cocartesian fibrations, then the composite functor*

$$\pi\mathscr{C} \xrightarrow{F} \pi\mathscr{D} \xrightarrow{p_!} \pi\mathscr{C}at_\infty$$

is an enriched homotopy transport functor for q.

7.4. **Transport functors induce homotopy transport.** We have the following fundamental result concerning homotopy transport.

hm:transport_v_transport **Theorem 7.11** ([5, 02S5])**.** *Consider the universal cocartesian fibration $U : \mathscr{P}.\mathscr{C}at_\infty \to \mathscr{C}at_\infty$. The equivalences*

$$\theta : \mathscr{C} \to (\mathscr{P}.\mathscr{C}at_\infty)_{\mathscr{C}}$$

from Corollary 5.28 define a natural isomorphism $id_{\pi\mathscr{C}at_\infty} \xrightarrow{\sim} U_!$. This isomorphism realizes the identity functor $id : \pi\mathscr{C}at_\infty \to \pi\mathscr{C}at_\infty$ as enriched homotopy transport for the universal fibration.

The proof proceeds by a more general analysis of homotopy transport for cocartesian fibrations of the form

$$\left(\mathrm{N}^{\mathrm{hc}}(\underline{A})_{x/}\right)^{\mathrm{Pith}} \to \mathrm{N}^{\mathrm{hc}}(\underline{A})^{\mathrm{Pith}},$$

where $\underline{A}$ a simplicial category which is enriched in ∞-categories [5, 02RZ]. We omit the details and refer the reader instead to the text [5].

Now, given an arbitrary cocartesian fibration $q : \mathscr{E} \to \mathscr{C}$ we have a categorical pullback diagram

$$\begin{tikzcd} \mathscr{E} \arrow[r] \arrow[d, "q"'] & \mathscr{P}.\mathscr{C}at_\infty \arrow[d] \\ \mathscr{C} \arrow[r, "F"'] & \mathscr{C}at_\infty \end{tikzcd} \tag{16}$$ eq:1525

which identifies a given functor F as the transport functor for q. Since homotopy transport is preserved under categorical pullback, we conclude that transport functors always induce enriched homotopy transport at the level of the enriched homotopy category.

or:transport_v_transport **Corollary 7.12.** *For any cocartesian fibration $q : \mathscr{E} \to \mathscr{C}$, and classifying functor $F : \mathscr{C} \to \mathscr{C}at_\infty$, the induced map on enriched homotopy categories*

$$\pi F : \pi\mathscr{C} \to \pi\mathscr{C}at_\infty$$

is an enriched homotopy transport functor for q. More specifically, the isomorphisms $\mathscr{E}_x \to F(x)$ in $\mathrm{h}\mathscr{K}an$ provided by the diagram (16) *and Theorem 5.27 define an isomorphism of enriched functors $\pi F \xrightarrow{\sim} q_!$.*

7.5. **Homotopy transport and enriched Hom functors.** For an ∞-category $\mathscr{C}$ and an object $x : * \to \mathscr{C}$, we recall the oriented fiber product $\{x\} \times^{\mathrm{or}}_{\mathscr{C}} \mathscr{C}$, which is the explicitly the isofibration

$$\{x\} \times_{\mathrm{Fun}(\{0\},\mathscr{C})} \mathrm{Fun}(\Delta^1, \mathscr{C}) \to \mathrm{Fun}(\{1\}, \mathscr{C}) = \mathscr{C}.$$

We have the equivalence of isofibrations

$$\begin{tikzcd} \mathscr{C}_{x/} \arrow[rr, "\sim"] \arrow[dr] & & \{x\} \times^{\mathrm{or}}_{\mathscr{C}} \mathscr{C} \arrow[dl] \\ & \mathscr{C} & \end{tikzcd}$$

of Theorem I-10.15, from which we conclude that the projection $\{x\} \times_{\mathscr{C}} \mathscr{C} \to \mathscr{C}$ is in fact a left fibration.

We assess homotopy transport $\mathscr{C}_{x/} \to \mathscr{C}$ by considering this equivalent fibration. We note that the fibers of $\{x\} \times^{\mathrm{or}}_{\mathscr{C}} \mathscr{C}$ over $\mathscr{C}$ are simply the mapping spaces $\mathrm{Hom}_{\mathscr{C}}(x, y)$.

prop:comp_transp

Proposition 7.13. *Let $\mathscr{C}$ be an ∞-category and $x_0 : * \to \mathscr{C}$ be any object. The composition functions*

$$\circ : \mathrm{Hom}_{\mathscr{C}}(x_1, x_2) \times \mathrm{Hom}_{\mathscr{C}}(x, x_1) \to \mathrm{Hom}_{\mathscr{C}}(x, x_2)$$

from Section I-9.1 are parametrized homotopy transport for the left fibration $\{x\} \times^{\mathrm{or}}_{\mathscr{C}} \mathscr{C} \to \mathscr{C}$.

Proof. Take $x_0 = x$ and $\vec{x} = (x_0, x_1, x_2)$. A morphism $\Delta^1 \times \mathrm{Hom}(x_1, x_2) \times \mathrm{Hom}(x_0, x_1) \to \{x_0\} \times^{\mathrm{or}}_{\mathscr{C}} \mathscr{C}$ is equivalent to a choice of a morphism

$$\Delta^1 \times \Delta^1 \times \mathrm{Hom}(x_1, x_2) \times \mathrm{Hom}(x_0, x_1) \to \mathscr{C}$$

whose restriction to $\{0\}$ in the first argument is of constant value x_0. Let $h : \Delta^1 \times \Delta^1 \to \Delta^2$ be the map which sends $(0, j)$ to 0 and $(1, j)$ to $j + 1$ and let

$$\omega : \mathrm{Hom}(x_1, x_2) \times \mathrm{Hom}(x_0, x_1) \to \mathrm{Fun}(\Delta^2, \mathscr{C})_{\vec{x}}$$

be any section of the trivial Kan fibration

$$\mathrm{Fun}(\Delta^2, \mathscr{C})_{\vec{x}} \to \mathrm{Fun}(\Lambda^2_1, \mathscr{C})_{\vec{x}} = \mathrm{Hom}(x_1, x_2) \times \mathrm{Hom}(x_0, x_1).$$

We consider the composite

$$\begin{aligned} \Delta^1 \times \Delta^1 \times \mathrm{Hom}(x_1, x_2) \times \mathrm{Hom}(x_0, x_1) &\overset{h\times id}{\to} \Delta^2 \times \mathrm{Hom}(x_1, x_2) \times \mathrm{Hom}(x_0, x_1) \quad (17) \\ &\overset{id\times\omega}{\to} \Delta^2 \times \mathrm{Fun}(\Delta^2, \mathscr{C})_{\vec{x}} \overset{ev}{\to} \mathscr{C}. \end{aligned}$$ eq:1582

One sees directly that this composite is of constant value x_0 when restricted to $\{0\}$ in the first argument, and the restriction to $\{1\}$ in the first argument yields the map

$$\Delta^1 \times \mathrm{Hom}(x_1, x_2) \times \mathrm{Hom}(x_0, x_1) \to \Delta^1 \times \mathrm{Hom}(x_1, x_2) \overset{ev}{\to} \mathscr{C}$$

since ω is a section of the aforementioned fibration. This implies commutativity of the diagram

$$\begin{tikzcd} \{0\} \times \mathrm{Hom}(x_1, x_2) \times \mathrm{Hom}(x_0, x_1) \arrow[r] \arrow[d] & \{x\} \times^{\mathrm{or}}_{\mathscr{C}} \mathscr{C} \arrow[d] \\ \Delta^1 \times \mathrm{Hom}(x_1, x_2) \times \mathrm{Hom}(x_0, x_1) \arrow[r] \arrow[ur, "\xi"] & \mathscr{C}, \end{tikzcd}$$

where ξ is adjoint to the composite (17). So therefore realize the restriction

$$\xi|_{\{1\}} : \Delta^1 \times \mathrm{Hom}(x_1, x_2) \times \mathrm{Hom}(x_0, x_1) \to \mathrm{Hom}(x_0, x_2)$$

as enriched transport for the given fibration, which one checks directly is simply the composition function for $\mathrm{Hom}_{\mathscr{C}}$, i.e. the uniquely determined composite

$$\mathrm{Hom}(x_1, x_2) \times \mathrm{Hom}(x_0, x_1) \to \mathrm{Fun}(\Delta^2, \mathscr{C})_{\vec{x}} \to \mathrm{Fun}(\Delta^{\{0,1\}}, \mathscr{C})_{(x_0, x_2)} = \mathrm{Hom}(x_0, x_2)$$

in $\mathrm{h}\mathscr{K}an$. $\square$

We now find that the Hom-functor

$$\mathrm{Hom}_{\mathscr{C}}(x,-):\pi\mathscr{C}\to\pi\mathscr{K}an$$

is the enriched homotopy transport representation for the oriented fiber product $\{x\}\times^{\mathrm{or}}\times^{\mathrm{or}}_{\mathscr{C}}\mathscr{C}\to\mathscr{C}$, and hence also for the fibration $\mathscr{C}_{x/}\to\mathscr{C}$.

Corollary 7.14. *Let $\mathscr{C}$ be any ∞-category and $x:*\to\mathscr{C}$ be any object. The* Hom*-functor*

$$\mathrm{Hom}_{\mathscr{C}}(x,-):\pi\mathscr{C}\to\pi\mathscr{K}an$$

is an enriched homotopy transport functor for the left fibration $\mathscr{C}_{x/}\to\mathscr{C}$.

8. Transport III: Naturality of transport

8.1. **Foundations: Weighted nerves in the simplicial setting.** In Section 6.8 we introduced the weighted nerve, which produces a cocartesian fibration

$$\mathrm{N}^F(\mathbb{A})\to\mathrm{N}^G(\mathbb{A})$$

for each pair of functors $F,G:\mathbb{A}\to\mathrm{Cat}_\infty\subseteq\mathscr{C}at_\infty$ from a discrete category $\mathbb{A}$ and transformation $F\to G$ which is appropriately cocartesian. Since the contant functor $*:\mathbb{A}\to\mathrm{Cat}_\infty$ is terminal, and ∞-categories are cocartesian fibrations over a point, we have the canonical fibration

$$\mathrm{N}^F(\mathbb{A})\to\mathrm{N}^*(\mathbb{A})=\mathbb{A}$$

associated to any discrete functor into Cat_∞. We recall here a relative version of this construction.

For any functor $F:\mathbb{A}\to\mathrm{sSet}$ from a discrete category we can define the simplicial set $\mathrm{N}^F(\mathbb{A})$ exactly as in Definition 6.19. So, n-simplices in $\mathrm{N}^F(\mathbb{A})$ consist of the choice of an n-simplex $\sigma:\Delta^n\to\mathbb{A}$ along with an expanding sequence of n-simplices $\tau_i:\Delta^i\to F(a_i)$, where $a_i=\sigma(i)$ at each i and these τ_i are assumed to fit into diagrams

$$\begin{tikzcd}
\Delta^i \arrow[r, "\mathrm{incl}"] \arrow[d, "\tau_i"'] & \Delta^{i+1} \arrow[d, "i+1"] \\
F(a_i) \arrow[r, "F\sigma(i<i+1)"'] & F(a_{i+1}).
\end{tikzcd}$$

We have the following apparent generalization of Proposition 6.23.

prop:relnev_simp

Proposition 8.1 ([5, 046X])**.** *Let $F,G:\mathbb{A}\to\mathrm{sSet}$ be functors from a discrete category $\mathbb{A}$, and $\xi:F\to G$ be a natural transformation. Suppose that at each object a in $\mathbb{A}$ the morphism $\xi(a):F(a)\to G(a)$ is a cocartesian fibration, and that for any morphism $t:a\to b$ the map $F(t):F(a)\to F(b)$ preserves cocartesian edges. Then the following hold:*

1. *The induced map $\mathrm{N}^\xi:\mathrm{N}^F(\mathbb{A})\to\mathrm{N}^G(\mathbb{A})$ is a cocartesian fibration.*
2. *An edge $\lambda:F(\alpha)(x)\to y$ in $\mathrm{N}^F(\mathbb{A})$, over an edge $t:a\to b$ in $\mathbb{A}$, is N^ξ-cocartesian if and only if the underlying map $\lambda:\Delta^1\to F(b)$ is $\xi(b)$-cocartesian.*

For the proof one employs the simplicial generalization of the relative join

$$K\star_T L=(K\star L)\times_{T\star T}(\Delta^1\times T),$$

observes a simplicial generalization [5, 02RH] of Lemma 6.27, and argues exactly as in the proof of Proposition 6.23.

8.2. **Foundation: Lifting witnessing data.** Given a cocartesian fibration $q : X \to S$ over a simplicial set S, we have the space of transport functors with witnessing data $\mathscr{TW}it(q)$, as defined in Definition 6.13. This is the space of diagrams

$$\begin{tikzcd} X \arrow[r, "\widetilde{F}"] \arrow[d, "q"'] & \mathscr{P}.\mathscr{C}at_\infty \arrow[d, "U"] \\ S \arrow[r, "F"'] & \mathscr{C}at_\infty \end{tikzcd}$$

which witness F as a transport functor for q, or equivalently the space of functors $F : S \to \mathscr{C}at_\infty$ which are paired with a diagram

$$\begin{tikzcd} X \arrow[rr, "G"] \arrow[dr, "q"'] & & \int_S F \arrow[dl] \\ & S & \end{tikzcd}$$

in which G is an equivalence of cocartesian fibrations.

prop:tdata_res **Proposition 8.2** ([5, 02SK])**.** *Let $q : X \to S$ be a cocartesian fibration, $i : K \to S$ be an inclusion of simplicial sets, and $q_K : X_K \to K$ be the corresponding pullback fibration. Suppose that every vertex in S is in the image of i. Then the restriction functor $i^* : \mathscr{TW}it(q) \to \mathscr{TW}it(q_K)$ is a trivial Kan fibration.*

We first note that the restriction functor

$$\begin{aligned} i^* : \mathrm{Fun}(X, \mathscr{C}at_\infty) \times_{\mathrm{Fun}(S,\mathscr{C}at_\infty)} \mathrm{Fun}(X, \mathscr{P}.\mathscr{C}at_\infty \\ \to \mathrm{Fun}(X_K, \mathscr{C}at_\infty) \times_{\mathrm{Fun}(K,\mathscr{C}at_\infty)} \mathrm{Fun}(X_K, \mathscr{P}.\mathscr{C}at_\infty) \end{aligned}$$

does in fact preserve the subcomplexes of transport data, by Lemma 6.12. So the functor $i^* : \mathscr{TW}it(q) \to \mathscr{TW}it(q_K)$ does in fact exist, and is well-defined. We provide the (somewhat involved) proof of Proposition 8.2 in Section **??** below.

8.3. **Naturality of transport.**

prop:tnat **Proposition 8.3.** *Consider a map of cocartesian fibrations*

$$\begin{tikzcd} X' \arrow[rr, "F"] \arrow[dr, "q'"'] & & X \arrow[dl, "q"] \\ & S & \end{tikzcd},$$

and let $T' : S \to \mathscr{C}at_\infty$ and $T : S \to \mathscr{C}at_\infty$ be transport functors for q' and q respectively. Consider also the relative join $Y = X' \star_X X$ and let $p_F : Y \to S \star_S S = \Delta^1 \times S$ be the map induced by q' and q. The following hold:

(1) *p_F is a cocartesian fibration.*

(2) *There are canonical maps $X' \to Y_0$ and $X \to Y_1$ to the fibers $Y_i = Y \times_{\Delta^1 \times S} (\{i\} \times S)$ which are both isomorphisms of cocartesian fiberations over $\mathscr{C}$.*

(3) *There exists a transport functor $\zeta_F : \Delta^1 \times \mathscr{C} \to \mathscr{C}at_\infty$ for p_F which satisfies $(\zeta_F)_0 = T'$ and $(\zeta_F)_1 = T$.*

Here when we say the maps $X' \to Y_0$ and $X \to Y_1$ are isomorphisms, we mean they are isomorphisms in the discrete category of cocartesian fibrations over S. In particular, they are isomorphisms of simplicial sets.

Proof. (1) Follows from Lemma 6.27, or rather from its simplicial generalization [5, 02RH]. We argue points (2) and (3).

The transport functors F and F' are equivalently a transport functor $[T' \ T] : S \amalg S \to \mathscr{C}at_\infty$ for the cocartesian fibration $q' \amalg q : X' \amalg X \to S \amalg S$. We consider the inclusion $i : S \amalg S = \partial\Delta^1 \times S \to \Delta^1 \times S$. After specifying witnessing data for F' and F, ζ_F is obtained by lifting along the pullback map $i^* : \mathscr{T}\mathscr{W}it(p_F) \to \mathscr{T}\mathscr{W}it(q' \amalg q)$,

$$\begin{tikzcd} & \mathscr{T}\mathscr{W}it(p) \arrow[d, "i^*"] \\ * \arrow[ur, dashed, "\zeta_F"] \arrow[r, "{[F' \ F]}"'] & \mathscr{T}\mathscr{W}it(q' \amalg q). \end{tikzcd}$$

We note that such a lift exists since i^* is a trivial Kan fibration, by Proposition 8.2. ☐

We observe that the transformation ζ_F is unique.

thm:uniq_tnat **Theorem 8.4.** *Consider a map of cocartesian fibrations*

$$\begin{tikzcd} X' \arrow[rr, "F"] \arrow[dr, "q'"'] & & X \arrow[dl, "q"] \\ & S & \end{tikzcd},$$

and let $T' : S \to \mathscr{C}at_\infty$ and $T : S \to \mathscr{C}at_\infty$ be transport functors for q' and q respectively. Suppose we have specified witnessing data

$$\mu' : X' \xrightarrow{\sim} \int_S T' \quad \text{and} \quad \mu : X \xrightarrow{\sim} \int_S T$$

as well (Definition 6.8), and consider the corresponding fibration $q' \amalg q : X' \amalg X \to S \amalg S$ with transport $[T' \ T] : S \amalg S \to \mathscr{C}at_\infty$. Let $p_F : Y \to \Delta^1 \times S$ be the fibration from Proposition 8.3 and $i : S \amalg S \cong \partial\Delta^1 \times S \to \Delta^1 \times S$ be the inclusion. Then the space

$$\mathscr{T}\mathscr{W}it(p_F) \times_{\mathscr{T}\mathscr{W}it(q' \amalg q)} \{[\mu' \ \mu]\}$$

of transformations $\zeta_F : \Delta^1 \to \mathscr{C}at_\infty$ with witnessing data $\mu_F : Y \xrightarrow{\sim} \int_{\Delta^1} \zeta_F$ and specified restrictions is a contractible Kan complex.

Proof. Follows from the fact that the map $i^* : \mathscr{T}\mathscr{W}it(p_F) \to \mathscr{T}\mathscr{W}it(q' \amalg q)$ is a trivial Kan fibration, by Proposition 8.2, and the fact that the class of trivial Kan fibrations is stable under pullback. ☐

def:zeta_F **Definition 8.5.** Consider a map of cocartesian fibrations

$$\begin{tikzcd} X' \arrow[rr, "F"] \arrow[dr, "q'"'] & & X \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

and suppose $T' : S \to \mathscr{C}at_\infty$ and $T : S \to \mathscr{C}at_\infty$ are transport functors with respective witnessing data

$$\mu' : X' \overset{\sim}{\to} \int_S T' \ \text{ and } \ \mu : X \overset{\sim}{\to} \int_S T.$$

A transformation $\zeta_F : T' \to T$ induced by F is any transformation in the fiber

$$\mathscr{T}\mathscr{W}it(p_F) \times_{\mathscr{T}\mathscr{W}it(q' \amalg q)} \{[\mu' \ \mu]\}$$

from Theorem 8.4.

As demonstrated in Theorem 8.4, the given fiber is contractible, so that the transformation ζ_F is uniquely specified.

8.4. **Does the witnessing data actually matter?** One might ask the question: In determining the transformation $\zeta_F : T' \to T$ associated to a given functor F, does the choice of witnessing data for T' and T actually matter? The short answer to this question is, in a generic sense no, but in a specific sense yes. The no part of this answer comes from the observation that, as one varies witnessing data for T' and T then one can lift the corresponding paths in $\mathscr{T}\mathscr{W}it(q' \amalg q)$ to paths in $\mathscr{T}\mathscr{W}it(p_K)$ along the trivial Kan fibration

$$i^* : \mathscr{T}\mathscr{W}it(p_K) \to \mathscr{T}\mathscr{W}it(q' \amalg q).$$

However, in following this lifted path one actually *changes* the underlying transformation ζ_F, though one obviously does stay within a single isomorphism class for such transformations. To illustrate this one can consider the case of a point $S = *$.

In order to keep our presentation at least slightly efficient, we leave the proof of the following to the interested reader.

Proposition 8.6. *Let $F : \mathscr{C} \to \mathscr{D}$ be functor between ∞-categories with corresponding fibration $p_F : \mathscr{C} \star_{\mathscr{D}} \mathscr{D} = \mathrm{N}^F(\Delta^1) \to \Delta^1$. Note that we have an identification $(\mathscr{C} \star_{\mathscr{D}} \mathscr{D}) \times_{\Delta^1} \partial\Delta^1 = \mathscr{C} \amalg \mathscr{D}$, and consider these underlying categories as functors $\mathscr{C}, \mathscr{D} : * \to \mathscr{C}at_\infty$. Consider also the sequence of maps*

$$\mathscr{T}\mathscr{W}it(p_F) \to \mathscr{T}\mathscr{W}it(\mathscr{C} \amalg \mathscr{D}) \to \mathrm{Fun}(\partial\Delta^1, \mathscr{C}at_\infty).$$

Given another functor $F' : \Delta^1 \to \mathscr{C}at_\infty$ with $F'(0) = \mathscr{C}$ and $F'(1) = \mathscr{D}$, F' and F both lie in the image of the fiber

$$\mathscr{T}\mathscr{W}it(p_F) \times_{\mathrm{Fun}(\partial\Delta^1, \mathscr{C}at_\infty)} \{(\mathscr{C}, \mathscr{D})\} \to \mathrm{Fun}(\Delta^1, \mathscr{C}at_\infty)$$

if and only if there exist equivalences $\alpha : \mathscr{D} \to \mathscr{D}$ and $\beta : \mathscr{C} \to \mathscr{C}$ at which one has a natural isomorphism $F' \overset{\sim}{\to} \alpha F \beta^{-1}$.

For fixed witnessing data $\mu_{\mathscr{C}} : \mathscr{C} \to \int_* \mathscr{C}$ and $\mu_{\mathscr{D}} : \mathscr{D} \to \int_* \mathscr{D}$, F' and F lie in the image of the fiber

$$\mathscr{T}\mathscr{W}it(p_F) \times_{\mathscr{T}\mathscr{W}it(\mathscr{C} \amalg \mathscr{D})} \{(\mu_{\mathscr{C}}, \mu_{\mathscr{D}})\} \to \mathrm{Fun}(\Delta^1, \mathscr{C}at_\infty)$$

if and only if there is a natural isomorphism $F' \overset{\sim}{\to} F$.

We note also that the fiber

$$\mathscr{T}\mathscr{W}it(p_F) \times_{\mathrm{Fun}(\partial\Delta^1, \mathscr{C}at_\infty)} \{(\mathscr{C}, \mathscr{D})\},$$

though it is a Kan complex, needn't be contractible. Indeed, when $\mathscr{C} = \mathscr{D}$ and F is an automorphism, the forgetful functor

$$\begin{aligned}&\mathscr{TWit}(p_F) \times_{\mathrm{Fun}(\partial\Delta^1, \mathscr{C}at_\infty)} \{(\mathscr{C}, \mathscr{C})\} \\ &\to \mathrm{Fun}(\Delta^1, \mathscr{C}at_\infty^{\mathrm{Kan}})^{\mathrm{Kan}} \times_{\mathrm{Fun}(\partial\Delta^1, \mathscr{C}at_\infty)} \{(\mathscr{C}, \mathscr{C})\} = \mathrm{Aut}_{\mathscr{C}at_\infty}(\mathscr{C})\end{aligned}$$

identifies the components of the fiber with the components of the automorphism group

$$\pi_0 \,\mathrm{Aut}_{\mathrm{sCat}_\infty}(\mathscr{C}) = \mathrm{Aut}_{\mathrm{h}\,\mathscr{C}at_\infty}(\mathscr{C}).$$

In particular the "witnessless fiber" is not even connected when $\mathscr{C}$ non-homotopic automorphisms. One can consider the case $\mathscr{C} = \mathrm{Sing}(T^n)$ for example, where one has necessarily non-homotopic automorphisms which exchange the generators in the fundamental group.

8.5. **Equivalences and natural isomorphisms.** The following can be observed by checking on the fibers over S and the characterization

Proposition 8.7. *Consider a map of cocartesian fibrations*

$$\begin{tikzcd} X' \arrow[rr, "F"] \arrow[dr, "q'"'] & & X \arrow[dl, "q"] \\ & S & \end{tikzcd},$$

and let $T' : S \to \mathscr{C}at_\infty$ and $T : S \to \mathscr{C}at_\infty$ be transport functors for q' and q respectively. The transformation $\zeta_F : \Delta^1 \times S \to \mathscr{C}at_\infty$ induced by F is a natural isomorphism between T' and T if and only if F is an equivalence of cocartesian fibrations.

Given naturally isomorphic maps of cocartesian fibrations $F_0, F_1 : X' \to X$, with corresponding natural isomorphism

$$\begin{tikzcd} \Delta^1 \times X' \arrow[rr, "\bar{\xi}"] \arrow[dr] & & X \arrow[dl] \\ & S & \end{tikzcd},$$

we can replace $\bar{\xi}$ with the corresponding map $\xi : \Delta^1 \times X' \to \Delta^1 \times X$ over $\Delta^1 \times S$ where ξ is specifically obtained as the composite

$$\Delta^1 \times X' \xrightarrow{\delta \times 1} \Delta^1 \times \Delta^1 \times X' \xrightarrow{1 \times \bar{\xi}} \Delta^1 \times X.$$

We note that, on objects, we have $\xi(i, x) = (i, \bar{\xi}(i, x))$ and on morphisms we have

$$\xi(0 < 1, id_x) = (0 < 1, \bar{\xi}_x) \quad \xi(i, \alpha) = (i, F_i(\alpha)).$$

Since the maps $\bar{\xi}_x$ are isomorphisms in the fiber X_s, over a specific point in S, we observe from the above description that ξ preserves cocartesian edges.

Taking the weighted nerve then provides a cocartesian fibration

$$q_\xi : \mathrm{N}^\xi(\Delta^1) \to \Delta^1 \times (\Delta^1 \times S) = \Delta^1 \times \Delta^1 \times S \xrightarrow{\mathrm{symm} \times 1} \Delta^1 \times \Delta^1,$$

where symm is the symmetry (factor swapping) on sSet. The fibers of q_ξ along the inclusions $\{i\} \times \Delta^1 \times S \to \Delta^1 \times \Delta^1 \times S$ recover the nerves $\mathrm{N}^{F_i}(\Delta^1)$, and the fibers along inclusions $\Delta^1 \times \{(i, s)\}$ recover the nerves

$$\Delta^1 \times X'_s = \mathrm{N}^{id_{X'_s}}(\Delta^1) \ \text{ and } \ \Delta^1 \times X_s = \mathrm{N}^{id_{X_s}}(\Delta^1).$$

From this information, and Proposition 8.2, one sees that the associated transport functor

$$\zeta_{\bar{\xi}} : \Delta^1 \times \Delta^1 \times S \to \mathscr{C}at_\infty$$

provides a natural isomorphism $\zeta_{F_0} \overset{\sim}{\to} \zeta_{F_1}$. So we observe the following.

prop:resp_isoms **Proposition 8.8.** *Consider maps of cocartesian fibrations*

$$\begin{tikzcd} X' \arrow[rr, shift left, "F_0"] \arrow[rr, shift right, "F_1"'] \arrow[dr, "q'"'] & & X \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

and transport functors $T', T : S \to \mathscr{C}at_\infty$ for q' and q. Suppose that F_0 and F_1 are isomorphic as maps between isofibrations over S. Then the corresponding transformations $\zeta_{F_0}, \zeta_{F_1} : T' \to T$ are naturally isomorphic as well.

8.6. **Composition and homotopy straightening.**

prop:resp_comps **Proposition 8.9.** *Consider maps of cocartesian fibrations*

$$\begin{tikzcd} X'' \arrow[r, "F'"] \arrow[dr, "q''"'] & X' \arrow[r, "F"] \arrow[d, dashed, "q'"] & X \arrow[dl, "q"] \\ & S & \end{tikzcd} \tag{18}$$

eq:2301

and trasport functors $T'', T', T : S \to \mathscr{C}at_\infty$ for q'', q', and q respectively. The induced transformations $\zeta_{F'}$, ζ_F and $\zeta_{FF'}$ fit into a 2-simplex

$$\begin{tikzcd} & T' \arrow[dr, "\zeta_F"] & \\ T'' \arrow[ur, "\zeta'_F"] \arrow[rr, "\zeta_{FF'}"'] & & T \end{tikzcd}$$

in $\mathrm{Fun}(S, \mathscr{C}at_\infty)$. *This is to say, $\zeta_{FF'}$ is a composite $\zeta_{FF'} = \zeta_F \circ \zeta_{F'}$.*

Proof. The sequence of maps $FF' : X'' \to X' \to X$ define a 2-simplex $\sigma : \Delta^2 \to$ sSet which factors through $\mathrm{Cocart}(S)$, and the diagram (18) defines a natural transformation to the constant diagram $\sigma \to \underline{S}$ which evaluates at each vertex $i : * \to \Delta^2$ to a cocartesian fibration. Hence the induced map

$$\mathrm{N}^\sigma(\Delta^2) \to \mathrm{N}^{id_S}(\Delta^2) = \Delta^2 \times S$$

is a cocartesian fibration by Proposition 8.1. The transport functors for the restrictions

$$\mathrm{N}^\sigma(\Delta^2)_{\Delta^{\{0,1\}}} = \mathrm{N}^{F'}(\Delta^2), \quad \mathrm{N}^\sigma(\Delta^2)_{\Delta^{\{1,2\}}} = \mathrm{N}^{F}(\Delta^2), \quad \mathrm{N}^\sigma(\Delta^2)_{\Delta^{\{0,2\}}} = \mathrm{N}^{FF'}(\Delta^2)$$

are $\zeta_{F'}$, ζ_F, and $\zeta_{FF'}$ respectively. These transport functors collectively define a functor

$$\partial\zeta_\sigma : \partial\Delta^2 \times S \to \mathscr{C}at_\infty$$

which we fill, using Proposition 8.2, to the proposed 2-simplex $\zeta_\sigma : \Delta^2 \times S \to \mathscr{C}at_\infty$. □

We combine Propositions 8.8 and 8.9 to obtain a functor at the homotopical level.

cor:htop_st

Corollary 8.10. *For any simplicial set S, the assignments*

$$\{\ q : X \to S\ \} \mapsto \{\ \text{a chosen trasport functor } T : S \to \mathscr{C}at_\infty\ \}$$

$$\left\{ \begin{tikzcd} X' \arrow[rr, "F"] \arrow[dr, "q'"'] & & X \arrow[dl, "q"] \\ & S & \end{tikzcd} \right\} \mapsto \{\ \text{the induced transformation } \zeta_F : T' \to T\ \}$$

provides a well-defined functor $\mathrm{h}\,\mathrm{St} : \mathrm{h}\,\mathscr{C}ocart(S) \to \mathrm{h}\,\mathrm{Fun}(S, \mathscr{C}at_\infty)$.

Of course, here we have made some choices in order to define the functor $\mathrm{h}\,\mathrm{St}$. Specifically, we have specified a transport functor for each cocartesian fibration. However the resulting functor uniquely determined up to a unique natural isomorphism via contractibility of the space of such choices. Alternatively, one can replace the category $\mathscr{C}ocart(S)$ with an equivalent category of pairs consisting of a cocartesian fibration over S with a specific choice of transport functor with witnessing data.

It's clear that the functor $\mathrm{h}\,\mathrm{St}$ is essentially surjective, and it is relatively easy to argue that it is fully faithful as well. Hence it is an equivalence. The inverse to $\mathrm{h}\,\mathrm{St}$ is provided by a functor $\mathrm{h}\,\mathrm{Un}$ which simply pulls back along the universal fibration $\mathrm{univ} : \mathscr{P}.\mathscr{C}at_\infty \to \mathscr{C}at_\infty$, i.e. by the assignment $T \mapsto \int_S T$, and applies cocartesian lifts

$$\begin{tikzcd} \{0\} \times \int_S T_0 \arrow[r] \arrow[d] & \int_{\Delta^1 \times S} \xi \arrow[d] \\ \Delta^1 \times \int_S T_0 \arrow[r] \arrow[ur, "\exists!"] & \Delta^1 \times S \end{tikzcd} \quad \Rightarrow \quad F_\xi : \int_S T_0 \to \int_S T_1$$

to transformations $\xi : \Delta^1 \times S \to \mathscr{C}at_\infty$.

One can show that these functors lift to equivalences of ∞-categories.

thm:str_unstr

Theorem 8.11. *The functors* $\mathrm{h}\,\mathrm{St}$ *and* $\mathrm{h}\,\mathrm{Un}$ *lift to mutually inverse equivalences of ∞-categories* $\mathrm{St} : \mathscr{C}ocart(S) \to \mathrm{Fun}(S, \mathscr{C}at_\infty)$ *and* $\mathrm{Un} : \mathrm{Fun}(S, \mathscr{C}at_\infty) \to \mathscr{C}ocart(S)$.

As we won't use this refinement of Corollary **??**, we confine the proof to the appendix.

Remark 8.12. One of the most fundamental results in the text [3] is the production of the straightening and unstraightening equivalences [3, Theorem 3.2.0.1]. On objects it is clear that our functors St and Un agree with those from [3]. We've not been able to explicitly compare our functors with those of [3] at the level of morphisms or mapping spaces. We expect, however, that this topic will be reformulated essentially completely in the text [5], in accordance with the general philosophy exemplified throughout [5, 027M, 028K, 028M].

8.7. **Transformations over a varying base.** Consider a diagram

$$\begin{tikzcd} X' \arrow[r, "F"] \arrow[d, "q'"'] & X \arrow[d, "q"] \\ S \arrow[r, "f"'] & U \end{tikzcd}$$

in which q and q' are cocartesian fibrations and F preserves cocartesian edges. Then we have the corresponding map of cocartesian fibrations to the pullback

$$\begin{tikzcd} X' \arrow[rr, "F_S"] \arrow[dr, "q"'] & & X_S \arrow[dl, "q_S"] \\ & S & \end{tikzcd}$$

Consider transport functors $T' : S \to \mathscr{C}at_\infty$ and $T : U \to \mathscr{C}at_\infty$ for q' and q with respective witnessing data

$$\mu' : X' \overset{\sim}{\to} \int_S T' \quad \text{and} \quad \mu : X \overset{\sim}{\to} \int_U T.$$

We recall from Lemma 6.12 that the composite $Tf : S \to \mathscr{C}at_\infty$ is a transport functor for the pullback q_S, along with the pulled back data $\mu_S : X_S \to \int_S Tf$. Take

$$p_F : X' \star_{X_S} X_S \to \Delta^1$$

the cocartesian fibration associated to F_S (Lemma 6.27).

def:zeta_Frel **Definition 8.13.** For a diagram of cocartesian fibrations as above, with specified transport functors etc. The transformation $\zeta_F : T' \to Tf$ induced by F is any element in the contractible space

$$\mathscr{T}\mathscr{W}it(p_F) \times_{\mathscr{T}\mathscr{W}it(q' \amalg q_S)} \{(\mu', \mu_S)\}.$$

Rather the transformation induced by F is, in the sense of Definition 8.5, the transformation induced by the pullback $F_S : X' \to X_S$.

8.8. **At the homotopy level.** Consider a diagram of cocartesian fibrations

$$\begin{tikzcd} \mathscr{E}' \arrow[rr, "F"] \arrow[dr, "q'"'] & & \mathscr{E} \arrow[dl, "q"] \\ & \mathscr{C} & \end{tikzcd}$$

over an ∞-category $\mathscr{C}$, and the corresponding fibration $\mathrm{N}^F(\Delta^1) \to \Delta^1 \times S$. The enriched homotopy transport representation

$$\Delta^1 \times \pi\mathscr{C} = \pi(\Delta^1 \times \mathscr{C}) \to \pi\mathscr{C}at_\infty$$

restricts to recover the enriched homotopy transport functors $q'_!$ and $q_!$ at the points 0 and $1 : * \to \Delta^1$ respectively (Definition 7.8), and at each object $x : * \to \mathscr{C}$ a morphism $\mathrm{transf}_x : \mathscr{E}'_x \to \mathscr{E}_x$ which fits into a diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{C}}(x,y) \arrow[r, "q_!"] \arrow[d, "q'_!"'] & \mathrm{Fun}(\mathscr{E}_x, \mathscr{E}_y) \arrow[d, "(\mathrm{transf}_x)^*"] \\ \mathrm{Fun}(\mathscr{E}'_x, \mathscr{E}'_y) \arrow[r, "(\mathrm{transf}_x)_*"'] & \mathrm{Fun}(\mathscr{E}'_x, \mathscr{E}_y) \end{tikzcd} \tag{19}$$

eq:2458

over arbitrary points $x, y : * \to \mathscr{C}$.

One calculates the transformation in question by evaluating the unique cocartesian solution

$$\begin{tikzcd} \{0\} \times \mathscr{E}'_x \arrow[r] \arrow[d] & \mathrm{N}^{F_x}(\Delta^1) \arrow[d] \\ \Delta^1 \times \mathscr{E}_x \arrow[r] \arrow[ur, dashed, "\exists!\ \widetilde{F}_x"] & \Delta^1 \end{tikzcd} \tag{20}$$

eq:2468

at $\{1\} \times \mathscr{E}_x$.

Lemma 8.14. *In the above situation, there is a cocartesian solution $\widetilde{F}_x : \Delta^1 \times \mathscr{E}'_x \to \mathrm{N}^{F_x}(\Delta^1)$ to the lifting problem (20) which satisfies $\widetilde{F}_x|_{\{1\}} = F_x : \mathscr{E}'_x \to \mathscr{E}_x$.*

Proof. We consider the identification $\mathrm{N}^F(\Delta^1) = \mathscr{E}'_x \star_{\mathscr{E}_x} \mathscr{E}_x$. The unique map $\Delta^1 \times \mathscr{E}'_x \to \mathscr{E}'_x \star \mathscr{E}_x$ which restricts to the identity on $\{0\} \times \mathscr{E}'_x$ and F_x on $\{1\} \times \mathscr{E}'_x$, and the map $id_{\Delta^1} \times F_x : \Delta^1 \times \mathscr{E}'_x \to \Delta^1 \times \mathscr{E}_x$, define a map of simplicial sets to the weighted nerve

$$\widetilde{F}_x : \Delta^1 \times \mathscr{E}'_x \to \mathrm{N}^F(\Delta^1).$$

By construction $\widetilde{F}_x$ is a map over Δ^1. Furthermore the restriction $\widetilde{F}_x : \Delta^1 \times \{e\} \to \mathrm{N}^F(\Delta^1)$ recovers the cocartesian edge $(e, id_{F(e)} : F(e) \to F(e))$ in the nerve. Hence $\widetilde{F}_x$ provides the unique cocartesian solution to the lifting problem (20). Finally, we have by construction $\widetilde{F}_x|_{\{0\}} = id_{\mathscr{E}'_x}$ and $\widetilde{F}_x|_{\{1\}} = F_x$. $\square$

This lemma tells us that the mystery transformations transf_x and transf_y appearing in (19) are explicitly given by the fibers F_x and F_y respectively. We now have a complete description of the enriched homotopy transport representation for the fibration $\mathrm{N}^F(\Delta^1) \to \Delta^1 \times \mathscr{C}$, and hence for the enriched transformation induced by F.

cor:enrich_zeta_F

Corollary 8.15. *Consider a diagram of ∞-categories*

$$\begin{tikzcd} \mathscr{E}' \arrow[r, "F"] \arrow[d, "q'"'] & \mathscr{E} \arrow[d, "q"] \\ \mathscr{C}' \arrow[r, "f"'] & \mathscr{C} \end{tikzcd}$$

in which q' and q are cocartesian fibrations and F preserves cocartesian edges. Let $T' : \mathscr{C}' \to \mathscr{C}at_\infty$ and $T : \mathscr{C} \to \mathscr{C}at_\infty$ be transport functors, with specified witnessing data, and $\zeta_F : T' \to Tf$ be the transformation induced by F. Then under the identifications $\pi T' \cong q'_! : \pi\mathscr{C}' \to \pi\mathscr{C}at_\infty$ and $\pi T \cong q_! : \pi\mathscr{C} \to \pi\mathscr{C}at_\infty$ from Corollary 7.12, the transformation $\pi\zeta_F : q'_! \to q_! f$ is calculated by the fibers of F along $\mathscr{C}$,

$$(\pi\zeta_F)_x = F_x : \mathscr{E}'_x \to \mathscr{E}_{F(x)}.$$

Example 8.16. Consider a functor between ∞-categories $f : \mathscr{C} \to \mathscr{D}$ and the corresponding map of left fibrations

$$\begin{tikzcd} \{x\} \times^{\mathrm{or}}_{\mathscr{C}} \mathscr{C} \arrow[r, "F"] \arrow[d] & \{F(x)\} \times^{\mathrm{or}}_{\mathscr{D}} \mathscr{D} \arrow[d] \\ \mathscr{C} \arrow[r, "f"'] & \mathscr{D}, \end{tikzcd}$$

at a given object $x : * \to \mathscr{C}$. By Proposition 7.13 the associated enriched homotopy transport functors are the corepresentable functors

$$\mathrm{Hom}_{\mathscr{C}}(x, -) : \pi\mathscr{C} \to \pi\mathscr{K}an \quad \text{and} \quad \mathrm{Hom}_{\mathscr{D}}(fx, -) : \pi\mathscr{D} \to \pi\mathscr{K}an$$

respectively, and by Corollary 8.15 the transformation induced by F evaluates at each object y to return the expected map

$$\pi\zeta_F = f : \mathrm{Hom}_{\mathscr{C}}(x, y) \to \mathrm{Hom}_{\mathscr{D}}(fx, fy).$$

9. Initial and terminal objects

Before beginning with our study in earnest, with the introduction of Hom functors and the Yoneda embedding for ∞-categories, we discuss the notions of initial and terminal objects in an ∞-category.

9.1. **Initial and terminal basics.**

Definition 9.1. Let $\mathscr{C}$ be an ∞-category. An object x in $\mathscr{C}$ is called initial if, for each object z in $\mathscr{C}$, the mapping space $\mathrm{Hom}_{\mathscr{C}}(x, z)$ is contractible. An object z in $\mathscr{C}$ is called terminal if, for each object z in $\mathscr{C}$, the space $\mathrm{Hom}_{\mathscr{C}}(z, y)$ is is contractible.

One sees that an object x is initial (resp. terminal) in $\mathscr{C}$ if and only if x is terminal (resp. initial) in the opposite category $\mathscr{C}^{\mathrm{op}}$. So we can freely translate between results for initial versus terminal objects. Note also that we can replace the mapping space $\mathrm{Hom}_{\mathscr{C}}(x, y)$ with either the left or right pinched spaces when evaluating initial-ness or terminal-ness of objects.

lem:init_unique **Lemma 9.2.** *Let $\mathscr{C}$ be an ∞-category, and let $\mathscr{C}_{\mathrm{Init}}$ and $\mathscr{C}_{\mathrm{Term}}$ denote the full ∞-subcategories whose objects are the initial and terminal objects in $\mathscr{C}$, respectively. Then each of the categories $\mathscr{C}_{\mathrm{Init}}$ and $\mathscr{C}_{\mathrm{Term}}$ is either empty or a contractible Kan complex.*

This is to say, the initial (or terminal) object in an ∞-category $\mathscr{C}$ is unique, provided any such object exists.

Proof. We only consider the case of $\mathscr{C}_{\mathrm{Init}}$. Let us suppose that this subcategory is nonempty. Via contractibility of the mapping spaces we conclude that the functor $\mathscr{C}_{\mathrm{Init}} \to *$ is fully faithful and essentially surjective, and hence an equivalence of ∞-categories. So $\mathscr{C}_{\mathrm{Init}}$ is a contractible Kan complex. □

lem:2181 **Lemma 9.3.** *If x is initial (resp. terminal) in $\mathscr{C}$, then another object x' is initial (resp. terminal) in $\mathscr{C}$ if and only if x' is isomorphic to x.*

Proof. For any isomorphism $\alpha : x \to x'$ the induced maps

$$\alpha^* : \mathrm{Hom}_{\mathscr{C}}(x, y) \to \mathrm{Hom}_{\mathscr{C}}(x', y) \quad \text{and} \quad \alpha_* : \mathrm{Hom}_{\mathscr{C}}(y, x) \to \mathrm{Hom}_{\mathscr{C}}(y, x')$$

are isomorphisms in $\mathrm{h}\mathscr{K}an$, at all y in $\mathscr{C}$. So contractibility of the left-hand spaces implies contratibility of the right-hand spaces. $\square$

One also sees that equivalences of ∞-categories preserve initial and terminal objects.

lem:equiv_initial

Lemma 9.4. *If $F : \mathscr{C} \to \mathscr{D}$ is an equivalence between ∞-categories, and x is initial (resp. terminal) in $\mathscr{C}$, then $F(x)$ is initial (resp. terminal) in $\mathscr{D}$*

Proof. Suppose that x is initial in $\mathscr{C}$. First note that any isomorphism $\beta : y \to y'$ in $\mathscr{D}$ induces isomorphisms

$$\beta_* : \mathrm{Hom}_{\mathscr{D}}(z, y) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{D}}(z, y')$$

in the homotopy category of Kan complexes. So an object z in $\mathscr{D}$ is initial if and only if the relevant mapping spaces are contractible at a dense collection of objects in $\mathscr{D}$. (By a dense collection we mean a collection which contains a representative for every isoclass in $\mathscr{D}$.) Since any equivalence is both fully faithful and essentially surjective, we have that the mapping spaces $\mathrm{Hom}_{\mathscr{D}}(Fx, y)$ are contractible at all y in the image of $\mathscr{C}$, and hence at all y in $\mathscr{D}$. So $F(x)$ is initial in $\mathscr{D}$. The case where x is terminal is proved similarly. $\square$

Warning 9.5. Initial and terminal objects are not well-behaved under fibering. Consider for example the cone $C = \{x^2 + y^2 = z : x, y, z \in \mathbb{R}\}$ and its projection onto the z-axis line $R_z \cong \mathbb{R}$. The projection $\mathrm{Sing}(C) \to \mathrm{Sing}(R_z)$ is a Kan fibration and the objects $\vec{1} = (1, 1, 1)$ and 1 are both initial and terminal in $\mathrm{Sing}(C)$ and $\mathrm{Sing}(R_z)$ respectively, since these spaces are contractible. However, $\vec{1}$ is not initial or terminal in the fiber $\mathrm{Sing}(C)_1 = \mathrm{Sing}(S^1)$. In fact, this fiber admits no such objects.

9.2. **Aside: trivial fibrations via the fibers.** For the analysis that follows, it is convenient to have a characterization of trivial Kan fibrations which can be checked on the fibers.

prop:triv_fibs

Proposition 9.6. *A map of simplicial sets $f : \mathscr{C} \to S$ is a trivial Kan fibration if and only if f is a left (or right) fibration and, at each point $s : * \to S$, the fiber $\mathscr{C}_s$ is a contractible Kan complex.*

Sketch proof. If f is a trivial Kan fibration then it is both a left and right fibration, and all of its fibers are contractible. As for the other direction, assume now that f is a left fibration and that all of its fibers are contractible. (The case of a right fibrations is then obtained by taking opposites.)

We must show that each lifting problem of the form

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \mathscr{C} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

admits a solution. In the case that $n = 0$, such a solution exists since the fibers $\mathscr{C}_s$ are all non-empty. So we assume $n > 0$. By replacing S with Δ^n, and $\mathscr{C}$ with $\Delta^n \times_S \mathscr{C}$, we may assume also that both S and $\mathscr{C}$ are ∞-categories. For fun we can finally replace this lifting problem with the related lifting problem

$$\begin{tikzcd} \{0\}\times\partial\Delta^n \arrow[r, "\bar{\sigma}"] \arrow[d] & \mathscr{C} \arrow[d] \\ \Delta^1\times\Delta^n \arrow[r, "\sigma"'] \arrow[ur, dotted] & S, \end{tikzcd} \tag{21}$$

eq:1722

which is obtained by restricting along the projections

$$p:\Delta^1\times\Delta^n\to\Delta^n,\quad p(0,i)=i,\ p(1,i)=n,$$

and which recovers our original problem after restricting to $\{0\}\times\Delta^n$. It suffices to solve this second problem.

By [5, 0153] the class of left anodyne maps is stable under the cartesian action of sSet on itself, so that the inclusion $\{0\}\times\partial\Delta^n\to\Delta^1\times\partial\Delta^n$ is left anodyne. Since the map f is left anodyne, it follows that the lifting problem (21) extends to a problem

$$\begin{tikzcd} \Delta^1\times\partial\Delta^n \arrow[r, "\bar{\sigma}"] \arrow[d] & \mathscr{C} \arrow[d] \\ \Delta^1\times\Delta^n \arrow[r, "\sigma"'] \arrow[ur, dotted] & S, \end{tikzcd}$$

Since the fibers of f are trivial Kan fibrations, the above problem extends further to a problem of the form

$$\begin{tikzcd} Y(0) \arrow[r, "\bar{\sigma}"] \arrow[d] & \mathscr{C} \arrow[d] \\ \Delta^1\times\Delta^n \arrow[r, "\sigma"'] \arrow[ur, dotted] & S, \end{tikzcd}$$

where $Y(0)$ is the pushout

$$Y(0)=(\Delta^1\times\partial\Delta^n)\coprod_{\{1\}\times\partial\Delta^n}(\{1\}\times\Delta^n).$$

By [5, Proof of 00TH] the inclusion $Y(0)\to\Delta^1\times\Delta^n$ can be factored into a sequence $Y(0)\to Y(1)\to\cdots\to Y(n+1)=\Delta^1\times\Delta^n$ with each $Y(i+1)$ fitting into a pushout diagram

$$\begin{tikzcd} \Lambda^{n+1}_{i+1} \arrow[r] \arrow[d] & Y(i) \arrow[d] \\ \Delta^{n+1} \arrow[r] & Y(i+1). \end{tikzcd}$$

Furthermore, this sequence can be constructed so that at $Y(n)=\Delta^1\times\Delta^n$ the sequence

$$\Delta^1\cong\Delta^{\{n,n+1\}}\to\Delta^{n+1}\to\Delta^1\times\Delta^n$$

recovers the edge $\Delta^1\times\{n\}$.

Now, since f is inner anodyne we can solve, in order, the lifting problems

$$\begin{tikzcd} Y(i-1) \arrow[r] \arrow[d] & \mathscr{C} \arrow[d] \\ Y(i) \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

at all $0 < i \leq n$. For the final lifting problem, along the inclusion $Y(n) \to Y(n+1)$, we need to solve a lifting problem of the form

$$\begin{tikzcd} \Lambda^{n+1}_{n+1} \arrow[r] \arrow[d] & \mathscr{C} \arrow[d] \\ \Delta^{n+1} \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

in which $\sigma|\Delta^{\{n,n+1\}}$ is of a constant value s in S. Since the fiber $\mathscr{C}_s$ is a Kan complex, the morphism $\Delta^{\{n,n+1\}} \to \Lambda^{n+1}_{n+1} \to \mathscr{C}$ is an isomorphism in $\mathscr{C}$. We can therefore solve this final lifting problem, by Proposition I-5.33, and hence obtain the desired solution to the problem (21). □

When applied to the case of a Kan fibration we have the following, which can also be deduced from Propositions I-4.9 and I-4.21.

Corollary 9.7. *A map between Kan complexes $f : \mathscr{X} \to \mathscr{Y}$ is a trivial Kan fibration if and only if it is a Kan fibration and, at each point $y : * \to \mathscr{Y}$, the fiber $\mathscr{X}_y$ is contractible.*

We recall that a map between Kan complexes is a trivial Kan fibration if and only if it is a Kan fibration and an equivalence. This is Proposition I-4.9. We have the following variant in the ∞-setting as another corollary.

`cor:triv_equiv_infty` **Corollary 9.8.** *A left (or right) fibration of ∞-categories $F : \mathscr{C} \to \mathscr{D}$ is a trivial Kan fibration if and only if it is an equivalence.*

Proof. Consider the diagram of fibrations

$$\begin{tikzcd} \mathscr{C} \arrow[rr, "F"] \arrow[dr, "F"'] & & \mathscr{D} \arrow[dl] \\ & \mathscr{D} & \end{tikzcd}.$$

From Theorem 3.8 we conclude that F is an equivalence if and only if, at each point $y : * \to \mathscr{D}$, the fiber $\mathscr{C}_y \to *$ is an equivalence of Kan complexes, i.e. if and only if each fiber $\mathscr{C}_y$ is contractible. By Proposition 9.6 this occurs if and only if F is a trivial Kan fibration. □

9.3. **Initial objects and undercategories.**

`prop:terminal_over` **Proposition 9.9.** *An object in an ∞-category $x : * \to \mathscr{C}$ is initial if and only if the forgetful functor $\mathscr{C}_{x/} \to \mathscr{C}$ is a trivial Kan fibration. Dually, an object $y : * \to \mathscr{C}$ is terminal if and only if the functor $\mathscr{C}_{/y} \to \mathscr{C}$ is a trivial Kan fibration.*

Proof. If x is initial then all of the left pinched mapping spaces are contractible, so that all of the fibers of the left fibration $\mathscr{C}_{x/} \to \mathscr{C}$ are contractible. It follows that this map is a trivial Kan fibration. For the converse, we simply note that trivial

Kan fibrations are stable under pullback. The arguments in the terminal case are similar. □

Let us now give a technical lemma.

lem:1699 **Lemma 9.10.** *For each positive integer n, the map*

$$(\Delta^1 \star \partial\Delta^n) \coprod_{(\{0\}\star\partial\Delta^n)} \{0\} \star \Delta^n \to \Delta^1 \star \Delta^n \cong \Delta^{n+2}$$

induced by the respective inclusions is an isomorphism onto the horn Λ_0^{n+2}.

See [5] for the proof. We have the following characterization of isomorphisms via initial and terminal objects.

prop:isom_initial **Proposition 9.11.** *For a map $\alpha : x \to y$ in an ∞-category $\mathscr{C}$, the following are equivalent:*

(a) *α is an isomorphism.*
(b) *α is initial when considered as an object in the undercategory $\mathscr{C}_{x/}$.*
(c) *α is terminal when considered as an object in the overcategory.*

Proof. We prove the equivalence between (a) and (b). The equivalence between (a) and (c) is obtained by taking opposites. The implication (b) $\Rightarrow$ (a) just follows by considering maps between α and id_x in the undercategory. So suppose that α is an isomorphism. By Proposition 9.9, α is initial in $\mathscr{C}_{x/}$ if and only if the forgetful functor

$$\mathscr{C}_{\alpha/} \cong (\mathscr{C}_{x/})_{\alpha/} \to \mathscr{C}_{x/}$$

is a trivial Kan fibration. Now, via a consideration of the identification from Lemma 9.10, solving a lifting problem of the form

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \mathscr{C}_{\alpha/} \arrow[d] \\ \Delta^n \arrow[r] & \mathscr{C}_{x/} \end{tikzcd}$$

is equivalent to solving the corresponding lifting problem

$$\begin{tikzcd} \Lambda_0^{n+2} \arrow[r] \arrow[d] & \mathscr{C} \arrow[d] \\ \Delta^{n+2} \arrow[r] & * \end{tikzcd}$$

in which the initial edge $\Delta^{\{0,1\}} \to \mathscr{C}$ is α. Such a problem admits a solution by Proposition I-5.33, so that the forgetful functor is seen to be a trivial Kan fibration. □

We take a moment to discuss some examples before returning to the theoretical foundations of this topic.

9.4. **Initial and terminal objects in simplicial nerves.**

Definition 9.12. An object x in a simplicial category $\underline{A}$ is called initial (resp. terminal) if, for each y in $\underline{A}$, the mapping complex $\underline{\mathrm{Hom}}_{\underline{A}}(x,y)$ (resp. $\underline{\mathrm{Hom}}_{\underline{A}}(y,x)$) is a contractible Kan complexes.

The easiest way for this to occur is if the relevant mapping complexes are just points. For example, one sees immediately that $\emptyset$ and $*$ are initial and terminal in $\underline{\mathrm{Kan}}$, respectively.

For $\underline{A}$ enriched in Kan complexes, and $\mathscr{A} = \mathrm{N}^{\mathrm{hc}}(\underline{A})$, the equivalence

$$\underline{\mathrm{Hom}}_{\underline{A}}(x,y) \xrightarrow{\sim} \mathrm{Hom}^{\mathrm{L}}_{\mathscr{A}}(x,y)$$

of Theorem 5.27 tells us that an object x is initial (resp. terminal) in $\underline{A}$ if and only if x is initial (resp. terminal) when considered as an object in the ∞-category $\mathscr{A}$. The analogous claim is seen to hold for terminal objects via a consideration of the opposite categories.

Lemma 9.13. *Let $\underline{A}$ be a simplicial category which is enriched in Kan complexes. Then an object x is initial (resp. terminal) in $\underline{A}$ if and only if the corresponding object x is initial (resp. terminal) in $\mathrm{N}^{\mathrm{hc}}(\underline{A})$.*

The following corollary is not an immediate consequence of triviality of the mapping categories $\mathrm{Fun}(\emptyset,\mathscr{C})$ and $\mathrm{Fun}(\mathscr{C},*)$, when $\mathscr{C}$ is an ∞-category.

cor:1761 **Corollary 9.14.** *The empty set $\emptyset$ is initial in both $\mathscr{K}an$ and $\mathscr{C}at_\infty$. The point $*$ is terminal in both $\mathscr{K}an$ and $\mathscr{C}at_\infty$.*

9.5. **Zero objects in pointed spaces.**

Though we will not use the term explicitly, a zero object in an ∞-category is an object which is simultaneously initial and terminal. Such objects are familiar from our studies of abelian categories. In the ∞-setting, the theory of abelian categories is, to some extent and in an indirect manner, reflected in the theory of stable categories. In the stable setting one again demands the existence of a zero object.

prop:tzero_under **Proposition 9.15.** *If x is terminal in an ∞-category $\mathscr{C}$, then x is both initial and terminal in the category $\mathscr{C}_{x/}$.*

By x in $\mathscr{C}_{x/}$ we mean any morphism $x \to x$. Since x is terminal, this lift of x to an object in $\mathscr{C}_{x/}$ is uniquely determined up to a contractible space. Practically speaking, we can just take this lift to be $id_x : x \to x$.

Proof. The fact that x is initial in $\mathscr{C}_{x/}$ follows by Proposition 9.11. For terminality, we consider the forgetful functor

$$\mathscr{C}_{x//x} \to \mathscr{C},$$

where $\mathscr{C}_{x//x} = (\mathscr{C}_{x/})_{/x} = (\mathscr{C}_{/x})_{x/}$. For any inclusion of simplicial sets $A \to B$, the existence of a solution to a lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathscr{C}_{x//x} \arrow[d] \\ B \arrow[r] & \mathscr{C}_{x/} \end{tikzcd}$$

is equivalent to the existence of a solution to the corresponding lifting problem

$$\begin{tikzcd} \{x\} \star A \arrow[r] \arrow[d] & \mathscr{C}_{/x} \arrow[d] \\ \{x\} \star B \arrow[r] & \mathscr{C}. \end{tikzcd}$$

By Proposition 9.9 a solution to the latter problem exists, since x is terminal in $\mathscr{C}$. It follows that the map $\mathscr{C}_{x//x} \to \mathscr{C}_{x/}$ is a trivial Kan fibration, and hence that x is terminal in $\mathscr{C}_{x/}$, by Proposition 9.9. □

Recall form Corollary 9.14 that the 1-point space $*$ is terminal in $\mathscr{K}an$.

Corollary 9.16. *The* 1*-point space* $*$ *is both initial and terminal in the* ∞*-category* $\mathscr{K}an_{*/}$ *of pointed Kan complexes.*

9.6. **Zero objects in derived categories.**

Definition 9.17. An object x in a dg category $\mathbf{A}$ is said to be initial (resp. terminal) if, at each y in $\mathbf{A}$, the Hom complex $\mathrm{Hom}^*_{\mathbf{A}}(x,y)$ (resp. $\mathrm{Hom}^*_{\mathbf{A}}(y,x)$) is acyclic.

Recall our calculation of the mapping spaces in the dg nerve $\mathscr{A} = \mathrm{N}^{\mathrm{dg}}(\mathbf{A})$ via the Hom complexes in $\mathbf{A}$,

$$\mathrm{Hom}^{\mathrm{L}}_{\mathscr{A}}(x,y) \overset{\sim}{\to} K(\mathrm{Hom}^*_{\mathbf{A}}(x,y))$$

(Proposition I-12.7). By Theorem I-11.13, the above calculation tells us that the mapping $\mathrm{Hom}^{\mathrm{L}}_{\mathscr{A}}(x,y)$ are contractible whenever the complex $\mathrm{Hom}^*_{\mathbf{A}}(x,y)$ is acyclic. So we observe the following.

lem:init_dg **Lemma 9.18.** *Let* $\mathbf{A}$ *be a dg category and take* $\mathscr{A} = \mathrm{N}^{\mathrm{dg}}(\mathbf{A})$*. If an object* x *is initial (resp. terminal) in* $\mathbf{A}$*, then the corresponding object* x *is initial (resp. terminal) in* $\mathscr{A}$*.*

Remark 9.19. The converse to Lemma 9.18 holds if we assume that our dg category $\mathbf{A}$ has a good shift functor (see Section 12.1).

For any abelian category $\mathbb{A}$, the object 0 is both initial and terminal in the dg category $\mathrm{Ch}(\mathbb{A})$ of cochains over $\mathbb{A}$, and hence also in the subcategories of K-projective and K-injective complexes. We recall that the derived ∞-category $\mathscr{D}(\mathbb{A})$ is defined by taking the dg nerve of the dg category of K-injective objects in $\mathrm{Ch}(\mathbb{A})$ when we have enough such objects, or K-projectives when we have enough such objects (see Section I-13).

Corollary 9.20. *For any Grothendieck abelian category* $\mathbb{A}$*, the zero complex* 0 *is both initial and terminal in the derived* ∞*-category* $\mathscr{D}(\mathbb{A})$*.*

9.7. **Initial objects and weak contractibility.** We phrase all results below in terms of initial objects. The corresponding results hold for terminal objects via duality.

Lemma 9.21. *A Kan complex* $\mathscr{X}$ *admits an initial object if and only if* $\mathscr{X}$ *is contractible.*

Proof. If x is initial in $\mathscr{X}$, then every object in $\mathscr{X}$ admits a morphism from $\mathscr{X}$, and hence is isomorphic to x (since $\mathscr{X}$ is a Kan complex). Since any object which is isomorphic to an initial object is initial, we conclude that $\mathscr{X}$ consists entirely of initial objects. We conclude that $\mathscr{X}$ is contractible by Lemma 9.2. $\square$

In the case of an ∞-category $\mathscr{C}$ we do not gain such a precise understanding of $\mathscr{C}$ via the existence of an initial object. This is clear from the examples discussed above. We can, however, constrain certain relative phenomena between ∞-categories via the preservation of initial objects. The remainder of this section is dedicated to an elaboration on this, somewhat criptic, point.

`lem:1936` **Lemma 9.22.** *An object x in $\mathscr{C}$ is initial if and only if the forgetful functor $\mathscr{C}_{x/} \to \mathscr{C}$ admits a section $F : \mathscr{C} \to \mathscr{C}_{x/}$ with $F(x) = id_x$.*

Proof. If x is initial then the forgetful functor is a trivial Kan fibration, by Proposition 9.9. It follows that the lifting problem

$$\begin{tikzcd} * \arrow[r, "id_x"] \arrow[d, "x"'] & \mathscr{C}_{x/} \arrow[d] \\ \mathscr{C} \arrow[r, "id"'] \arrow[ur, dotted] & \mathscr{C} \end{tikzcd}$$

admits a solution $s : \mathscr{C} \to \mathscr{C}_{x/}$. This solution provides the desired section. Conversely, if we have such a section F then for each y in $\mathscr{C}$ we can split the identity on the mapping space as

$$\mathrm{Hom}_{\mathscr{C}}(x, y) \xrightarrow{F} \mathrm{Hom}_{\mathscr{C}_{x/}}(id_x, F(y)) \xrightarrow{forget} \mathrm{Hom}_{\mathscr{C}}(x, y).$$

Since id_x is initial in $\mathscr{C}_{x/}$, by Proposition 9.11, each mapping space $\mathrm{Hom}_{\mathscr{C}_{x/}}(id_x, F(y))$ is contractible. Thus each mapping space $\mathrm{Hom}_{\mathscr{C}}(x, y)$ is a retract of a contractible space, and hence contractible itself. $\square$

We record a little lemma.

`lem:1957` **Lemma 9.23** ([5, 0196])**.** *If $i : A \to B$ is an inclusion of simplicial sets, then the induced map*

$$\{*\} \star i : \{*\} \star A \to \{*\} \star B$$

is left anodyne.

Proof. The class of i at which $\{*\} \star i$ is left anodyne is saturated. We we need only show that it contains the inclusions $\partial\Delta^n \to \Delta^n$. But in this case the inclusion in question is identified with the left anodyne map $\Lambda_0^{n+1} \to \Delta^{n+1}$. $\square$

`prop:init_lanodyne` **Proposition 9.24.** *An object x in $\mathscr{C}$ is initial if and only if the inclusion $x : * \to \mathscr{C}$ is left anodyne. If y is terminal in $\mathscr{C}$, then the inclusion $y : * \to \mathscr{C}$ is right anodyne.*

Proof. We deal with the initial claim. If $x : * \to \mathscr{C}$ is left anodyne then we can solve the lifting problem

$$\begin{tikzcd} * \arrow[r, "id_x"] \arrow[d, "x"'] & \mathscr{C}_{x/} \arrow[d] \\ \mathscr{C} \arrow[r, "id"'] \arrow[ur, dotted] & \mathscr{C} \end{tikzcd}$$

and hence obtain a section $F : \mathscr{C} \to \mathscr{C}_{x/}$ as in Lemma 9.22. We conclude that x is initial in $\mathscr{C}$.

Conversely, if x is initial then the section $F : \mathscr{C} \to \mathscr{C}_{x/}$ of Lemma 9.22 provides a map $F' : \{*\} \star \mathscr{C} \to \mathscr{C}$ with $F'|_{\mathscr{C}} = id_{\mathscr{C}}$, $F'(*) = x$, and $F'(* \to x) = id_x$. In particular, F' is defined on each simplex outside of $\mathscr{C}$ by taking

$$F'(\{*\} \star \Delta^m) = F(\Delta^m).$$

This map F' gives a diagram

$$\begin{tikzcd} * \arrow[r, "x"] \arrow[d, "x"'] & \{*\} \star \{x\} \arrow[r] \arrow[d] & * \arrow[d, "x"] \\ \mathscr{C} \arrow[r] & \{*\} \star \mathscr{C} \arrow[r, "F"] & \mathscr{C} \end{tikzcd}$$

so that the inclusion $\{x\} \to \mathscr{C}$ is a retract of the inclusion $\{*\} \star \{x\} \to \{*\} \star \mathscr{C}$. Since this latter inclusion is left anodyne, by Lemma 9.23, we conclude that the inclusion $\{x\} \to \mathscr{C}$ is left anodyne as well. $\square$

As a consequence of Proposition 9.24 we observe a kind of relative triviality for $\mathscr{C}$.

cor:initial_eval **Corollary 9.25.** *Suppose $f : X \to S$ is a left fibration, and that $\mathscr{C}$ admits an initial object $x : * \to \mathscr{C}$. Then the map*

$$\mathrm{Fun}(\mathscr{C}, X) \to X \times_S \mathrm{Fun}(\mathscr{C}, S), \quad F \mapsto (F|_x, fF). \tag{22}$$ eq:2002

is a trivial Kan fibration.

Proof. Immediate from Propositions 9.24 and 4.3. $\square$

Let's consider what Corollary 9.25 is telling, in semi-human terms. In the extreme case where $\mathscr{X} \to *$ is a Kan complex, the right hand side of (22) is just $\mathscr{X}$ and we obtain a trivial Kan fibration

$$ev_x : \mathrm{Fun}(\mathscr{C}, \mathscr{X}) \to \mathscr{X}$$

which just evaluates a functor F at $x : * \to \mathscr{C}$. This says that for any choice of a point $z : * \to \mathscr{X}$ there is a unique functor $F_z : \mathscr{C} \to \mathscr{X}$ which evaluates as $F_z(x) = z$. Indeed, we can just take the fiber of ev_x at z to obtain a space $\mathrm{Fun}(\mathscr{C}, \mathscr{X})_z$ which parametrizes such functors, and observe that this space is contractible. In this way $\mathscr{C}$ "looks like a point" relative to any Kan complex.

In the relative setting, we consider a left fibration $f : X \to S$ and see that for any choice of a functor $\bar{F} : \mathscr{C} \to S$, and a point z in X which lifts $F(x)$, there is a unique lift of $\bar{F}$ to a functor $F : \mathscr{C} \to X$ with $F(x) = z$. Rather, we observe that any lifting problem of the form

$$\begin{tikzcd} * \arrow[r] \arrow[d, "x"'] & X \arrow[d, "f"] \\ \mathscr{C} \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

admits a unique solution.

9.8. Equivalences of fibrations via initial objects.

prop:equiv_initial

Proposition 9.26. *Let*

$$\begin{tikzcd} \mathscr{C} \arrow[rr, "F"] \arrow[dr] & & \mathscr{D} \arrow[dl] \\ & \mathscr{T} & \end{tikzcd}$$

be a diagram of ∞-categories in which both of the maps to $\mathscr{T}$ are left fibrations. If $\mathscr{C}$ admits an initial object x, then F is an equivalence if and only if $F(x)$ is initial in $\mathscr{D}$.

Proof. If F is an equivalence then it preserves initial objects, by Lemma 9.4. Suppose conversely that F is such a map, that x is initial in $\mathscr{C}$, and that the image $F(x)$ is initial in $\mathscr{D}$. Consider the lifting problem

$$\begin{tikzcd} * \arrow[r, "x"] \arrow[d, "F(x)"'] & \mathscr{C} \arrow[d] \\ \mathscr{D} \arrow[r] \arrow[ur, dotted] & \mathscr{T}. \end{tikzcd}$$

By Corollary 9.25 there exists a solution to this problem, and hence there exists a functor $G : \mathscr{D} \to \mathscr{C}$ over $\mathscr{T}$ with $GF(x) = x$. By Corollary 9.25 again the composition GF is also seen to be isomorphic to the identity in $\mathrm{Fun}_{\mathscr{T}}(\mathscr{C}, \mathscr{C})$. We also have $FG(Fx) = F(x)$ so that FG is isomorphic to the identity on $\mathscr{D}$. Thus F is an equivalence. □

Again, one observes a corresponding statement for right fibrations and terminal objects, via the opposite duality.

At first glance this proposition seems ridiculous. Indeed, it suggests that if $f : \mathscr{E} \to \mathscr{C}$ is a left fibration of ∞-categories, and e is an object in $\mathscr{E}$ with image x in $\mathscr{C}$, then the induced map on undercategories $F : \mathscr{E}_{e/} \to \mathscr{C}_{x/}$ is an equivalence. This is because F fits into a diagram

$$\begin{tikzcd} \mathscr{E}_{e/} \arrow[rr] \arrow[dr] & & \mathscr{C}_{x/} \arrow[dl] \\ & \mathscr{C} & \end{tikzcd}$$

and which both maps to $\mathscr{E}$ are left fibrations, and F is seen to send the initial object id_e to the initial object id_x. However, one sees that this is as bad as it gets.

cor:2491

Corollary 9.27. *Let $\mathscr{C} \to \mathscr{T}$ be a left fibration, suppose that $\mathscr{C}$ admits an initial object x, and let t denote the image of x in $\mathscr{T}$. Then there is an equivalence $F : \mathscr{C} \to \mathscr{T}_{t/}$ of left fibrations over $\mathscr{T}$ which sends x to id_t.*

So Proposition 9.26, said another way, classifies left fibrations up to equivalence via isoclasses of objects in $\mathscr{T}$.

Proof. By Proposition 9.26 the map $\mathscr{C}_{x/} \to \mathscr{T}_{t/}$ is an equivalence of left fibrations which sends id_x to id_t. The proposed equivalence $\mathscr{C} \to \mathscr{T}_{t/}$ is obtained by composing the equivalence $\mathscr{C}_{x/} \to \mathscr{T}_{t/}$ with a section $F : \mathscr{C} \to \mathscr{C}_{x/}$ as in Lemma 9.22. □

10. Aside: Simplicialification for dg categories

10.1. **Simplicial categories for dg categories.** Let k be a commutative ring and consider the category sSet_k of simplicial k-modules. This category is symmetric monoidal with the expected product $M \otimes_k N$, where

$$(M \otimes_k N)[r] := M[r] \otimes_k N[r].$$

This product admits inner-Homs, so that sSet_k is naturally enriched in itself, and we obtain the corresponding simplicial category $\underline{\mathrm{sSet}_k}$. Explicitly the morphism complexes $\mathrm{Fun}_k(M, N)$ are the complexes with n-simplices

$$\mathrm{Fun}_k(M, N)[n] = \mathrm{Hom}_{\mathrm{sSet}_k}(k\Delta^n \otimes_k M, N).$$

Since all simplicial k-modules are Kan complexes, the simplicial category $\underline{\mathrm{sSet}_k}$ is Kan-enriched and we have the forgetful simplicial functor $\underline{\mathrm{sSet}_k} \to \underline{Kan}$.

Definition 10.1. We take $\mathscr{K}an_k := \mathrm{N}^{\mathrm{hc}}(\underline{\mathrm{sSet}_k})$.

Via faithfulness of the functor $\underline{\mathrm{sSet}_k} \to \underline{Kan}$ we observe that the induced functor on ∞-categories $\mathscr{K}an_k \to \mathscr{K}an$ is an inclusion of simplicial sets.

Recall that we have the Dold-Kan functor

$$K : \mathrm{Ch}(k) \to \mathrm{sSet}_k$$

which restricts to an equivalence $K^{\leq 0}$ from connective cochains. In particular, K factors through the truncation

$$\mathrm{Ch}(k) \to \mathrm{Ch}(k)^{\leq 0}, \quad X \mapsto (\cdots \to X^{-1} \to Z^0 X \to 0)$$

and is identified with the composite of this truncation and the equivalence $K^{\leq 0}$. We let $\mathbf{Ch}(k)$ denote the usual dg category of k-cochains.

prop:K_lax **Proposition 10.2** ([5, 00SD])**.** *The functor K admits a lax-monoidal structure $m_{V,W} : K(V) \otimes_k K(W) \to K(V \otimes_k W)$. Furthermore, at each pair of objects this morphism $m_{V,W}$ is a (non-linear) homotopy equivalence.*

Proof. This lax monoidal structure is adjoint to the monoidal structure on the normalized cochain functor provided by the Alexander-Whitney map [5, 00S6]. Since the Alexander-Whitney maps are quasi-isomorphism [5, 00SB], we conclude that each $m_{V,W}$ is a homotopy equivalence. In particular, $m_{V,W}$ is obtained as the composite

$$K(V) \otimes_k K(W) \overset{\sim}{\to} KN(K(V) \otimes_k K(W)) \overset{K(\mathrm{AW})}{\to} K(NK(V) \otimes_k NK(W)) \overset{\sim}{\to} K(V \otimes_k W).$$

□

In the case that one of V or W is concentrated in degree 0 one observes natural identifications $K(V) \otimes_k K(W) \cong K(V \otimes W)$ and the aforementioned lax monoidal structure extends these identifications to arbitrary complexes. In the case where V is concentrated in degree 0, for example, we have $K(V)[n] = V$ at all n and the aforementioned identification is explicitly the map

$$V \otimes_k \mathrm{Hom}^*_k(N\Delta^n, W) \to \mathrm{Hom}^*_k(N\Delta^n, V \otimes_k W), \quad v \otimes f \mapsto (x \mapsto v \otimes f(x)).$$

Corollary 10.3. *For any dg category $\mathbf{A}$ there is an associated simplicial category $K\mathbf{A}$ obtained by applying the lax monoidal functor K to the morphism complexes.*

The identifications $K(V)[0] = V^0$ induce an identification of homotopy categories

$$\mathrm{h}\,\mathrm{N}^{\mathrm{hc}}(K\mathbf{A}) = \mathrm{h}\,K\mathbf{A} = H^0\mathbf{A} = \mathrm{h}\,\mathrm{N}^{\mathrm{dg}}(\mathbf{A}).$$

It's shown in [4, 5] that this equivalence lifts to the ∞-categorical level.

thm:dk_compare **Theorem 10.4** ([5, 00SV])**.** *For any dg category $\mathbf{A}$, there is a natural equivalence of ∞-categories*

$$\mathfrak{Z}_{\mathbf{A}} : \mathrm{N}^{\mathrm{hc}}(K\mathbf{A}) \overset{\sim}{\to} \mathrm{N}^{\mathrm{dg}}(\mathbf{A})$$

which is furthermore a trivial Kan fibration and fits into a diagram over the associated discrete category

$$\begin{tikzcd} \mathrm{N}^{\mathrm{hc}}(K\mathbf{A}) \arrow[rr, "\mathfrak{Z}_{\mathbf{A}}"] & & \mathrm{N}^{\mathrm{dg}}(\mathbf{A}) \\ & Z^0\mathbf{A} \arrow[ul] \arrow[ur] & \end{tikzcd} .$$

By naturality, we mean that any dg functor F fits into a diagram

$$\begin{tikzcd} \mathrm{N}^{\mathrm{dg}}(\mathbf{A}) \arrow[r, "\mathrm{N}^{\mathrm{dg}} F"] & \mathrm{N}^{\mathrm{dg}}(B) \\ \mathrm{N}^{\mathrm{hc}}(K\mathbf{A}) \arrow[u, "\mathfrak{z}"] \arrow[r, "N^{\mathrm{hc}}KF"] & \mathrm{N}^{\mathrm{hc}}(K\mathbf{B}). \arrow[u, "\mathfrak{z}"'] \end{tikzcd}$$

10.2. **Derived Dold-Kan for vector spaces.** If one considers dg categories as, loosely, categories enriched in the ∞-category of vector

Lax monoidality tells us that the Dold-Kan functor K enriches to a simplicial functor $K\,\mathrm{Ch}(k)^* \to \underline{\mathrm{sSet}}_k$ which just sends a cochain V to the object KV and which is defined on morphisms via the unique map

$$K\,\mathrm{Hom}_k(V, W) \to \underline{\mathrm{Hom}}_{\mathrm{sSet}_k}(KV, KW)$$

which is compatible with evaluation.

thm:enriched_dk **Theorem 10.5.** *Let $\mathbf{Ch}(k)$ denote the dg category of cochains. If k is a field, the simplicial functor*

$$K : K\mathbf{Ch}(k)^* \to \underline{\mathrm{sSet}}_k$$

restricts to an equivalence on $K\mathbf{Ch}(k)^{\leq 0}$.

Proof. In this case a map in $\mathrm{Ch}(k)^{\leq 0}$ is a homotopy equivalence if and only if it is a quasi-isomorphism. It follows via the Dold-Kan equivalence, Theorems I-11.12 & I-11.13 and Proposition I-11.16, that a map in sSet_k is is linear homotopy equivalence if and only if it is a homotopy equivalence. In particular, the comparison map $m_{V,W} : K(V) \otimes_k K(W) \to K(V \otimes_k W)$ is a linear homotopy equivalence. It follows that K induces a monoidal euqivalence on homotopy categories

$$\mathrm{h}\,K : D(k)^{\leq 0} \overset{\sim}{\to} \mathrm{h}\,\mathscr{K}an_k.$$

Now, both of the categories $D(k)$ and $\mathrm{h}\,\mathscr{K}an_k$ admit inner-Homs, which are just given by the inner-Homs at the pre-homotopical level Hom^*_k and $\underline{\mathrm{Hom}}_{\mathrm{sSet}_k}$. Since $\mathrm{h}\,K$ is an equivalence we now obtain a unique isomorphism of inner-Homs

$$K\,\mathrm{Hom}^*_k(V, W) \overset{\sim}{\to} \underline{\mathrm{Hom}}_{\mathrm{sSet}_k}(KV, KW)$$

in $\mathrm{h}\,\mathscr{K}an_k$ which is compatible with evaluation. This unique isomorphism is the image of the correpsonding map at the pre-homotopical level, from which we conclude that the original morphism is a homotopy equivalence. This implies that K is fully faithful, and essential surjectivity just follows form the fact that the non-enriched Dold-Kan functor is essentially surjective. $\square$

We refer to the enriched equivalence of Theorem 10.5 as the enriched Dold-Kan equivalence.

cor:infty_dk

Corollary 10.6. *Suppose k is a field. Then enriched Dold-Kan provides a functor between ∞-categories $K : \mathscr{V}ect_k \to \mathscr{K}an_k$ which restricts to an equivalence of ∞-categories*

$$K : \mathscr{V}ect_k^{\leq 0} \overset{\sim}{\to} \mathscr{K}an_k.$$

Here $\mathscr{V}ect_k := \mathrm{N}^{\mathrm{dg}}(\mathbf{Ch}(k))$ denotes the ∞-category of connective cochains, and we've written simply K for the composite of equivalences

$$\mathscr{V}ect_k \overset{\sim}{\to} \mathrm{N}^{\mathrm{hc}}(K\mathbf{Ch}(k)) \overset{\mathrm{N}^{\mathrm{hc}}K}{\to} \mathscr{K}an_k$$

by an abuse of notaiton.

10.3. **Derived Dold-Kan for abelian categories with projectives.** Let $\mathbb{A}$ be an abelian category with enough projectives. We recall the following strengthening of the usual Dold-Kan equivalence.

Theorem 10.7 (Dold-Kan, [4, Theorem 1.2.3.7])**.** *Let $\mathbb{A}$ be any abelian category and $\mathbb{A}'$ be any additive subcategory in $\mathbb{A}$ which is closed under taking summands. Then the Dold-Kan functor $K : \mathrm{Ch}(\mathbb{A}') \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbb{A}')$ restricts to an equivalence*

$$K : \mathrm{Ch}(\mathbb{A}')^{\leq 0} \overset{\sim}{\to} \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbb{A}')$$

with inverse given by the normalized cochains functor.

Though our notation is a bit cumbersome, we recall that the category $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbb{A}')$ is simply the category of simplicial "sets" T whose simplicices $T[n]$ have the structure of objects in $\mathbb{A}'$, and whose structure maps are all maps in $\mathbb{A}'$. In the case where $\mathbb{A}'$ is the full subcategory $\mathrm{Proj}_{\mathbb{A}}$ of projectives in $\mathbb{A}$ we obtain an equivalence

$$\mathrm{Proj}_{\mathbb{A}}^{\leq 0} \overset{\sim}{\to} \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Proj}_{\mathbb{A}}).$$

Definition 10.8. Let k be a field and Vect be the category of finite-dimensional vector spaces. A k-linear category $\mathbb{A}$ is a Vect-module category for which the action map $\mathrm{Vect} \times \mathbb{A} \to \mathbb{A}$ commutes with all colimits which exist in either factor.

Remark 10.9. When $\mathbb{A}$ is locally finite, i.e. has all objects of finite length and finite dimensional morphisms, one should replace Vect with the subcategory Vect_{fin} of finite dimensional vector spaces in the above definition. Equivalently, one can consider such $\mathbb{A}$ along with a Vect-action on its Ind-category.

Note that when $\mathbb{A}$ is k-linear then the category sSet_k acts on simplicial objects in $\mathbb{A}$ via the expected formula

$$V \otimes_k M[n] := V[n] \otimes_k M[n].$$

Via the same arguments as employed in [5, 00RF, kerodon] one produces the Eilenberg-Zilber morphisms, Alexander-Whitney morphisms, and (associative and unital) adjoint morphism

$$m_{V,M} : K(V) \otimes_k K(M) \to K(V \otimes_k M)$$

which compares the Vect-action on $\mathrm{Ch}(\mathbb{A})$ with the sSet_k-action on $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbb{A})$.

Lemma 10.10. *Suppose $\mathbb{A}$ is a k-linear abelian category. Then for an arbitrary complex of vector spaces V and complex of objects M in $\mathbb{A}$, the map $m_{V,M} : K(V) \otimes_k K(M) \to K(V \otimes_k M)$ is a quasi-isomorphism.*

Proof. One argues exactly as in [5] and Proposition 10.2. However, when necessary, on should replace the simplicial abelian group $\mathbb{Z}[\Delta^n]$ with $T[\Delta^n]$ in arguments from [5], where T is an arbitrary object in $\mathbb{A}$. □

Using the lax structure maps m we can enrich the Dold-Kan equivalence to a simplicial functor

$$K : K\mathbf{Ch}(\mathbb{A}) \to \underline{\mathrm{Fun}}(\Delta^{\mathrm{op}}, \mathbb{A}),$$

where the k-linear simplicial mapping complexes on $\underline{\mathrm{Fun}}(\Delta^{\mathrm{op}}, \mathbb{A})$ are the inner-Homs relative to the sSet_k-action. On objects we simply M take the associated space KM and the maps

$$K\,\mathrm{Hom}^*_{\mathbb{A}}(M, N) \to \underline{\mathrm{Hom}}_{\mathrm{Fun}}(KM, KN)$$

are adjoint to the evaluation morphisms

$$K\,\mathrm{Hom}^*_{\mathbb{A}}(M, N) \otimes_k KM \overset{m}{\to} K(\mathrm{Hom}^*_{\mathbb{A}}(M, N) \otimes_k M) \overset{Kev}{\to} KN.$$

Definition 10.11. For a k-linear abelian category $\mathbb{A}$ with enough projectives, take

$$\mathscr{K}an_{\mathbb{A}} := \mathrm{N}^{\mathrm{hc}}\,\underline{\mathrm{Fun}}(\Delta^{\mathrm{op}}, \mathrm{Proj}_{\mathbb{A}}).$$

thm:enriched_d2

Theorem 10.12. *Let $\mathbb{A}$ be a k-linear abelian category with enough projectives, and take $\mathbf{Proj}^{\leq 0}_{\mathbb{A}}$ the dg category of connective cochains of projectives. The enriched Dold-Kan functor*

$$K : K\mathbf{Proj}^{\leq 0}_{\mathbb{A}} \to \underline{\mathrm{Fun}}(\Delta^{\mathrm{op}}, \mathrm{Proj}_{\mathbb{A}})$$

is an equivalence of simplicial categories.

Proof. Same as the proof of Theorem 10.5. □

cor:derived_dk_A

Corollary 10.13. *For any k-linear abelian category $\mathbb{A}$ which has enough projectives, there is an equivalence of ∞-categories*

$$K : \mathscr{D}^{\leq 0}(\mathbb{A}) \overset{\sim}{\to} \mathscr{K}an_{\mathbb{A}}.$$

11. Representable and corepresentable functors

Definition 11.1. Let $\mathscr{C}$ be an ∞-category. A functor $F : \mathscr{C} \to \mathscr{K}an$ is corepresented by an object x in $\mathscr{C}$ if F is a transport functor for the left fibration $\mathscr{C}_{x/} \to \mathscr{C}$. We say F is corepresentable if it is corepresented by some object in $\mathscr{C}$.

We say a functor $G : \mathscr{C}^{\mathrm{op}} \to \mathscr{K}an$ is represented by an object y in $\mathscr{C}$ if it is corepresented by y when considered as an object in $\mathscr{C}^{\mathrm{op}}$, i.e. if it is a contravariant transport functor for the right fibration $\mathscr{C}_{/y} \to \mathscr{C}$. A corepresentable functor is a fnuctor which is corepresented by some object in $\mathscr{C}$.

We note that if F and F' are corepresented by an object x in $\mathscr{C}$, then F and F' are isomorphic, via the uniqueness of transport functors. We also see, by Corollary 9.27, that a functor $F : \mathscr{C} \to \mathscr{K}an$ is representable if the corresponding ∞-category $\mathrm{Un}(F) \cong \int_{\mathscr{C}} F$ has an initial object. It is clear from Corollary 9.27 that h is representable if, in some sense, it has an "initial object" in some fiber $F(x)$.

def:initial_fun **Definition 11.2.** Given a functor $F : \mathscr{C} \to \mathscr{K}an$, we say an object $1_x : * \to F(x)$ is an initial object for F, over x, if at each y in $\mathscr{C}$ the composite

$$\mathrm{Hom}_{\mathscr{C}}(x, y) \overset{F}{\to} \mathrm{Hom}_{\mathscr{K}an}(F(x), F(y)) \overset{\theta^{-1}}{\to} \mathrm{Fun}(F(x), F(y)) \overset{1_x^*}{\to} F(y) \qquad (23)$$ eq:initial_fun

is an isomorphism in $\mathrm{h}\,\mathscr{K}an$.

Note that this condition is really a restriction on the induced functor on enriched categories $\pi F : \pi\mathscr{C} \to \pi\mathscr{K}an$. Since the isomorpihsms θ of Theorem 5.27 is seen to preserve identity morphisms, we see that the above composite at x,

$$\mathrm{Hom}_{\mathscr{C}}(x, x) \to F(x)$$

sends the identity $id_x : x \to x$ to 1_x. Since the θ^{-1} assemble into an equivalence of $\mathrm{H}\,\mathscr{K}an$-enriched categories $\pi\mathscr{K}an \overset{\sim}{\to} \pi\underline{\mathrm{Kan}}$ (Proposition 7.6), we also observes at any choice of $1_x : * \to F(x)$ a diagram

$$\begin{tikzcd}
F(x) \arrow[r, "F(\alpha)"] & F(y) \\
\mathrm{Hom}_{\mathscr{C}}(x, x) \arrow[u] \arrow[r, "\alpha_*"'] & \mathrm{Hom}_{\mathscr{C}}(x, y). \arrow[u]
\end{tikzcd}$$

Our first aim is to prove the following.

thm:rep_funs **Theorem 11.3.** *A functor $F : \mathscr{C} \to \mathscr{K}an$ is corepresented by an object x in $\mathscr{C}$ if and only if F admits an initial object which lies over x, $1_x : * \to F(x)$.*

11.1. **Left fibrations with initial transport.** Consider a left fibration $q : \mathscr{E} \to \mathscr{C}$ with transport functor $F : \mathscr{C} \to \mathscr{K}an$. The induced functor $\pi F : \pi\mathscr{C} \to \pi\mathscr{K}an$ is determined by paramtrized homotopy transport, according to Corollary 7.12. Hence the composite of (23) at any object $1_x : * \to F(x) \cong \mathscr{E}_x$ is identified with the composite

$$\mathrm{Hom}_{\mathscr{C}}(x, y) \times \{1_x\} \to \mathrm{Hom}_{\mathscr{C}}(x, y) \times \mathscr{E}_x \to \mathscr{E}_y, \qquad (24)$$ eq:2540

where the final map is given by parametrized homotopy transport (Definition 7.7). More precisely, the adjunction

$$\mathrm{Adj} : \mathrm{Hom}_{\mathrm{Kan}}(\mathrm{Hom}(x, y), \mathrm{Fun}(*, \mathscr{E}_y)) \overset{\sim}{\to} \mathrm{Hom}_{\mathrm{Kan}}(\mathrm{Hom}(x, y) \times \{*\}, \mathscr{E}_y)$$

the map (23) is identified with the map (24). One sees directly that any map $\gamma : \mathrm{Hom}_{\mathscr{C}}(x, y) \to \mathrm{Fun}(*, \mathscr{E}_y) \cong \mathscr{E}_y$ fits into a diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(x, y) \times \{*\} \arrow[r, "\mathrm{Adj}(\gamma)"] & \mathscr{E}_y \\
\mathrm{Hom}_{\mathscr{C}}(x, y) \arrow[u, "\cong"] \arrow[r, "\gamma"'] & \mathrm{Fun}(*, \mathscr{E}_y), \arrow[u, "\cong"']
\end{tikzcd}$$

so that γ is an euqivalence if and only if $\mathrm{Adj}(\gamma)$ is an equivalence. So we see that an object $1_x : * \to F(x)$ is initial if and only if the maps (24) are all equivalences.

prop:2653 **Proposition 11.4.** *Let $F : \mathscr{C} \to \mathscr{K}an$ be any functor and take $q : \mathscr{E} = \int_{\mathscr{C}} F \to \mathscr{C}$ the corresponding left fibration. For any choice object x in $\mathscr{C}$, any object $1_x : * \to F(x)$, and let $\widetilde{x}$ be the image of 1_x in $\mathscr{E}_x$ under the equivalence $\theta : F(x) \to \mathscr{E}_x$. Then $\widetilde{x}$ is initial in $\mathscr{E}$ if and only if 1_x is an initial object for F.*

Proof. Via the equivalence $\mathscr{C}_{x/} \xrightarrow{\sim} \{x\}\widetilde{\times}_{\mathscr{C}}\mathscr{C}$ we see that the oriented fiber product admits an initial object. By the specific expression of this equivalence give in Section I-(10.5) one sees that this equivalence sends id_x to id_x, so that the identity specifically is seen to be initial in $\{x\} \times^{\mathrm{or}}_{\mathscr{C}} \mathscr{C}$.

By Corollary 9.25 there is a unique map of left fibrations $t : \{x\} \times^{\mathrm{or}}_{\mathscr{C}} \mathscr{C} \to \mathscr{E}$ which sends id_x to $\widetilde{x}$. This map is an equivalence if and only if all of the induced maps on the fibers

$$t : \mathrm{Hom}_{\mathscr{C}}(x, y) \to \mathscr{E}_y$$

are isomorphisms.

Since enriched homotopy transport for the oriented fiber product is given by composition on $\mathrm{Hom}_{\mathscr{C}}$ (Proposition **??**), the diagrams

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(x,y) \times \mathscr{E}_x \arrow[rr] & & \mathscr{E}_y \\
\mathrm{Hom}_{\mathscr{C}}(x,y) \times \mathrm{Hom}_{\mathscr{C}}(x,x) \arrow[u, "id\times t"] \arrow[rr, "\circ"] & & \mathrm{Hom}_{\mathscr{C}}(x,y) \arrow[u, "t"']
\end{tikzcd}$$

commute at all y in $\mathscr{C}$, where the top map is enriched homotopy transport for $\mathscr{E}$. Restricting to id_x in the second factor produces a diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(x,y) \times \{\widetilde{x}\} \arrow[r] & \mathscr{E}_y \\
\mathrm{Hom}_{\mathscr{C}}(x,y) \times \{id_x\} \arrow[u, "id"] \arrow[r, "id"'] & \mathrm{Hom}_{\mathscr{C}}(x,y) \arrow[u, "t_y"']
\end{tikzcd}$$

from which we see that t_y is an equivalence at all y if and only if the maps (24) are all isomorphisms in $\mathrm{h}\,\mathscr{K}an$. Via the equivalence of enriched functors $\pi F \xrightarrow{\sim} q_!$ we observe that these maps are all equivalences if and only if 1_x is an initial object for F. $\square$

Now, for an arbitrary left fibration $q : \mathscr{E} \to \mathscr{C}$, with transport functor $F : \mathscr{C} \to \mathscr{K}an$, we have the equivalence of left fibrations $\mathscr{E} \xrightarrow{\sim} \int_{\mathscr{C}} F$ which is implicit in the assertion that F is a transport functor. We therefore see that $\mathscr{E}$ admits an initial object if and only if $\int_{\mathscr{C}} F$ admits an initial object. So Proposition 11.4 implies the following.

Corollary 11.5. *Let $q : \mathscr{E} \to \mathscr{C}$ be a left fibration between ∞-categories. Then $\mathscr{E}$ admits an initial object $\widetilde{x}$ over a point x in $\mathscr{C}$ if and only if the corresponding transport functor F admits an initial object $1_x : * \to F(x)$.*

We now observe the proof of Theorem 11.3.

Proof of Theorem 11.3. By Proposition 11.4 a functor F admits an initial object over x if and only if the corresponding left fibration $\int_{\mathscr{C}} F$ admits an initial object in the fiber over x. This occurs if and only if there is an equivalence of left fibrations

$$\mathscr{C}_{x/} \xrightarrow{\sim} \int_{\mathscr{C}} F$$

over $\mathscr{C}$. The existence of such an equivalence, by definition, characterizes F as a transport functor for the fibration $\mathscr{C}_{x/} \to \mathscr{C}$. $\square$

11.2. **Corepresentable functors for simplicial categories.** Let $\underline{A}$ be a simplicial category which is enriched in Kan complexes. At any choice of an object x in $\underline{A}$ we have the simplicial functor

$$\underline{\mathrm{Hom}}_{\underline{A}}(x,-) : \underline{A} \to \underline{\mathrm{Kan}}.$$

Taking homotopy coherent nerves then provides a functor

$$\underline{\mathrm{Hom}}_{\mathrm{N}^{\mathrm{hc}}(\underline{A})}(x,-) : \mathrm{N}^{\mathrm{hc}}(\underline{A}) \to \mathscr{K}an.$$

prop:simplicial_corep

Proposition 11.6. *Let $\underline{A}$ be a* Kan*-enriched simplicial category and take $\mathscr{A} = \mathrm{N}^{\mathrm{hc}}(\underline{A})$. At any object x in $\mathscr{A}$ the functor $\underline{\mathrm{Hom}}_{\mathscr{A}}(x,-) : \mathscr{A} \to \mathscr{K}an$ is corepresented by x.*

Proof. Since the map $\pi\underline{A} \to \pi\mathscr{A}$ induced by the equivalences of Theorem 5.27 is an equivalence of ∞-categories, we have an identification of $\mathrm{h}\,\mathscr{K}an$-enriched functors

$$\underline{\mathrm{Hom}}_{\mathscr{A}}(x,-) \cong \mathrm{Hom}_{\mathscr{A}}(x,-).$$

Since the functor $\mathrm{Hom}_{\mathscr{A}}(x,-)$ admits an initial object over x, so does $\underline{\mathrm{Hom}}_{\mathscr{A}}(x,-)$. It follows that $\underline{\mathrm{Hom}}_{\mathscr{A}}(x,-)$ is corepresented by x. $\square$

Corollary 11.7. *For $\underline{A}$ and $\mathscr{A}$ as in Proposition 11.6, a functor $F : \mathscr{A} \to \mathscr{K}an$ is corepresentable if and only if F admits an isomorphism*

$$\underline{\mathrm{Hom}}_{\mathscr{A}}(x,-) \overset{\sim}{\to} F$$

at some x in $\underline{A}$.

sect:corep_dg

11.3. **Corepresentable functors for dg categories.** Let $\mathbf{A}$ be a dg category, and $\mathscr{A} = \mathrm{N}^{\mathrm{dg}}(\mathbf{A})$. At any object x in $\mathbf{A}$ we have the dg functor $\mathrm{Hom}^*_{\mathbf{A}}(x,-) : \mathbf{A} \to \mathrm{Ch}(k)$.

lem:2851

Lemma 11.8. *Let $\mathbf{A}$ be a dg category. At any object V in $\mathbf{A}$ the simplicial functor*

$$K \circ \mathrm{Hom}^*_{\mathbf{A}}(V,-) : K\mathbf{A} \to K\mathbf{Ch}(k) \overset{K}{\to} \underline{\mathrm{sSet}}_k$$

is equal to the functor $\underline{\mathrm{Hom}}_{KA}(V,-) : K\mathbf{A} \to \underline{\mathrm{sSet}}_k$.

Proof. Take $h_V = \mathrm{Hom}^*_{\mathbf{A}}(V,-)$ and $\underline{h}_V = \underline{\mathrm{Hom}}_{K\mathbf{A}}(V,-)$. On objects these functors are the same. For the composite, the original map

$$h_V : \mathrm{Hom}^*_{\mathbf{A}}(W,W') \to \mathrm{Hom}^*_k(h_V W, h_V W')$$

fits into, and is specified by, a diagram

$$\begin{tikzcd}
\mathrm{Hom}^*_{\mathbf{A}}(W,W') \otimes_k h_V(W) \arrow[d, "h_V \otimes_k id"'] \arrow[dr, "\circ"] & \\
\mathrm{Hom}^*_k(h_V W, h_V W') \otimes h_V(W) \arrow[r, "ev"'] & h_V(W').
\end{tikzcd}$$

Hence Kh_V fits into a diagram

$$\begin{tikzcd}
K\,\mathrm{Hom}^*_{\mathbf{A}}(W,W') \otimes_k Kh_V(W) \arrow[d, "Kh_V \otimes_k id"'] \arrow[dr, "K\circ"] & \\
K\,\mathrm{Hom}^*_k(h_V W, h_V W') \otimes_k Kh_V(W) \arrow[r, "ev"'] & Kh_V(W').
\end{tikzcd}$$

Taking inner-Homs for sSet_k now gives a diagram

$$\begin{tikzcd} K\operatorname{Hom}^*_{\mathbf{A}}(W,W') \arrow[d, "Kh_V"'] \arrow[dr, "\underline{h}_V"] & \\ K\operatorname{Hom}^*_k(h_VW,h_VW') \arrow[r, "K"'] & \underline{\operatorname{Hom}}_{\mathrm{sSet}_k}(Kh_VW,Kh_VW'). \end{tikzcd}$$

This implies an equality between these two functors on morphism complexes as well. $\square$

lem:2889

Lemma 11.9. *Given a diagram in $\mathscr{C}at_\infty$*

$$\begin{tikzcd} \mathscr{C} \arrow[d, "\xi"'] \arrow[dr, "F"] & \\ \mathscr{D} \arrow[r, "G"'] & \mathscr{K}an \end{tikzcd}$$

in which ξ is an equivalence, then F is corepresentable by an object x if and only if G is corepresentable by $\xi(x)$.

Proof. We have an isomorphism of functors $F \cong G\circ\xi$. Since representability is stable under isomorphism, we may assume $F = G\circ\xi$. Fix arbitrary points $x, y : * \to \mathscr{C}$ and take $x' = \xi(x)$, $y' = \xi(y)$.

We consider the corresponding maps on π-enriched categories to observe a diagram

$$\begin{tikzcd} \operatorname{Hom}_{\mathscr{C}}(x,y)\times F(x) \arrow[r] \arrow[d, "\xi\times id"'] & \operatorname{Hom}_{\mathscr{K}an}(F(x),F(y))\times F(x) \arrow[r, "ev"] \arrow[d, "="] & F(y) \arrow[d, "="] \\ \operatorname{Hom}_{\mathscr{D}}(x',y')\times G(\xi x) \arrow[r] & \operatorname{Hom}_{\mathscr{K}an}(G(x'),G(y'))\times G(x') \arrow[r, "ev"'] & G(y') \end{tikzcd}$$

in $\mathrm{h}\,\mathscr{K}an$ in which all vertical maps are isomorphisms. From this diagram we see that F admits an initial object if and only if G admits an initial object. Hence F is representable if and only if G is representable. $\square$

We understand that, at any dg category, the functor

$$\underline{\operatorname{Hom}}_{K\mathbf{A}}(V,-) : \mathrm{N}^{\mathrm{hc}}(K\mathbf{A}) \to \mathscr{K}an_k \subseteq \mathscr{K}an$$

is corepresented by the given object V. This follows by Proposition 11.6. Naturality of the equivalence $\mathfrak{Z}$ from Theorem 10.4, in conjunction with Lemmas 11.8 and 11.9 above, tell us that the Hom-complexes for dg categories also provide representable functors, in the only way that makes sense.

prop:dg_corep

Proposition 11.10. *Let $\mathbf{A}$ be any dg category with associated ∞-category $\mathscr{A} = N^{\mathrm{dg}}(A)$. At any object V in $\mathbf{A}$ the functor*

$$K\operatorname{Hom}^*_{\mathbf{A}}(V,-) : \mathscr{A} \to \mathscr{V}ect \to \mathscr{K}an$$

is corepresented by V.

Corollary 11.11. *Let $\mathbf{A}$ be any dg category with associated ∞-category $\mathscr{A} = N^{\mathrm{dg}}(A)$. A functor $F : \mathscr{A} \to \mathscr{K}an$ is corepresented by an object V in $\mathscr{A}$ if and only if F is isomorphic to the functor $K\operatorname{Hom}^*_{\mathbf{A}}(V,-)$.*

11.4. **Representable functors for simplicial and dg categories.**

Lemma 11.12. *Let $\underline{A}$ be a simplicial category which is enriched in Kan complexes. Then we have* $\mathrm{N}^{\mathrm{hc}}(\underline{A}^{\mathrm{op}}) = \mathrm{N}^{\mathrm{hc}}(\underline{A})^{\mathrm{op}}$.

Here the opposite category $\underline{A}^{\mathrm{op}}$ is simply the category obtained by applying the symmetry on sSet to the morphisms. The identification of opposites follows from the identification

$$\mathrm{Fun}_{\mathrm{sCat}}(\mathrm{Path}\,\Delta^n, \underline{A}^{\mathrm{op}}) = \mathrm{Fun}((\mathrm{Path}\,\Delta^n)^{\mathrm{op}}, \underline{A}) = \mathrm{Fun}(\mathrm{Path}((\Delta^n)^{op}), \underline{A})$$

We now consider the functor $\underline{\mathrm{Hom}}_{\underline{A}}(-,y)$, for y in $\underline{A}$ as a functor from the opposite category

$$\underline{\mathrm{Hom}}_{\underline{A}}(-,y) : \underline{A}^{\mathrm{op}} \to \underline{\mathrm{sSet}}.$$

As a corollary to Proposition 11.6 we observe the following.

cor:simp_rep **Corollary 11.13.** *Let $\underline{A}$ be a simplicial category which is enriched in Kan complexes. Then at any object y in $\underline{A}$, the functor*

$$\underline{\mathrm{Hom}}_{\underline{A}}(-,y) : \mathrm{N}^{\mathrm{hc}}(\underline{A})^{\mathrm{op}} \to \mathscr{K}an$$

is a representable functor which is represented by y.

In the dg setting we also have the opposite category $\mathbf{A}^{\mathrm{op}}$. Here we employ the Koszul sign rule in the symmetric on $\mathrm{Ch}(k)$, so that composition for $\mathbf{A}^{\mathrm{op}}$ inherits a sign

$$f \cdot_{\mathrm{op}} g := (-1)^{\deg(f)\deg(g)} gf.$$

One can check the following.

lem:2965 **Lemma 11.14.** *For any dg category $\mathbf{A}$, the maps on n-simplices*

$$\mathrm{N}^{dg}(\mathbf{A})^{\mathrm{op}}[n] \to \mathrm{N}^{\mathrm{dg}}(\mathbf{A}^{\mathrm{op}})[n], \quad \{f_I : I \subseteq [n]\} \mapsto \{(-1)^{|I|(|I|-1)/2} f_I : I \subseteq [n]\},$$

define an isomorphism of ∞-categories $\mathrm{N}^{dg}(\mathbf{A})^{\mathrm{op}} \overset{\sim}{\to} \mathrm{N}^{\mathrm{dg}}(\mathbf{A}^{\mathrm{op}})$.

From Proposition 11.10 and Lemma 11.9 we now observe the following.

Proposition 11.15. *For any dg category $\mathbf{A}$, and any object W in $\mathbf{A}$, the functor*

$$K\,\mathrm{Hom}^*_{\mathbf{A}}(-,W) : \mathrm{N}^{\mathrm{dg}}(\mathbf{A})^{\mathrm{op}} \to \mathscr{K}an$$

is a representable functor which is represented by W.

Here of course we have abused notation to view the functor $\mathrm{Hom}^*_{\mathbf{A}}(-,W)$: $\mathrm{N}^{\mathrm{hc}}(\mathbf{A}^{\mathrm{op}}) \to \mathscr{V}ect$ as a functor from $\mathrm{N}^{\mathrm{dg}}(\mathbf{A})^{\mathrm{op}}$, via the identification of Lemma 11.14.

sect:twarrows

12. Twisted arrows and bifunctorial Homs

12.1. **The twisted arrows category.**

Definition 12.1. Given a simplicial set $\mathscr{C}$, we define the twisted arrow category $\mathscr{T}w(\mathscr{C})$ as the simplicial set whose n-simplices are

$$\mathscr{T}w(\mathscr{C})[n] := \mathrm{Hom}_{\mathrm{sSet}}((\Delta^n)^{\mathrm{op}} \star \Delta^n, \mathscr{C}).$$

Restricting along the inclusions

$$(\Delta^n)^{\mathrm{op}} \to (\Delta^n)^{\mathrm{op}} \star \Delta^n \quad \text{and} \quad \Delta^n \to (\Delta^n)^{\mathrm{op}} \star \Delta^n$$

provides a natural map to the product

$$\lambda : \mathscr{T}w(\mathscr{C}) \to \mathscr{C}^{\mathrm{op}} \times \mathscr{C}. \tag{25}$$ eq:lambda

To get our heads on straight here, let's observe directly that an object in $\mathscr{T}w(\mathscr{C})$ is a choice of a morphism $\alpha : x \to y$ in $\mathscr{C}$. A morphism from an objects $\alpha : x \to y$ to some other $\alpha' : x' \to y'$ is a diagram of the form

$$\begin{tikzcd} x \arrow[r, "\alpha"] \arrow[dr] & y \arrow[d] \\ x' \arrow[u] \arrow[ur] \arrow[r, "\alpha'"'] & y'. \end{tikzcd}$$

If we consider the fiber $\{(x,y)\} \times_{(\mathscr{C}^{\mathrm{op}} \times \mathscr{C})} \mathscr{T}w(\mathscr{C})$, a simplex in this fiber can be visualized as some directed diagram from x to y which is "completely filled in",

$$\begin{tikzcd} x \arrow[r, bend left, dotted] \arrow[r, dotted] \arrow[r, bend right, dotted] & y. \end{tikzcd}$$

We prove below that these fibers are a type of bifunctorial Hom space for $\mathscr{C}$, where bifunctoriality simply refers to the fact that one has two variables to tune in the base.

We note that the join $(\Delta^n)^{\mathrm{op}} \star \Delta^n$ is identified with Δ^{2n+1} via the bijection

$$[2n+1] \to [n] \amalg [n], \quad i \mapsto \begin{cases} n-i \text{ in the first set if } i \le n \\ i-n \text{ in the second set if } i \ge n. \end{cases}$$

prop:tw_inner **Proposition 12.2** ([5, 03JT])**.** *The restriction map $\lambda : \mathscr{T}w(\mathscr{C}) \to \mathscr{C}^{\mathrm{op}} \times \mathscr{C}$ is a left fibration. More generally, if $\mathscr{C} \to S$ is an inner fibration of simplicial sets, then the restriction map $\mathscr{T}w(\mathscr{C}) \to (\mathscr{C}^{\mathrm{op}} \times \mathscr{C}) \times_{(S^{\mathrm{op}} \times S)} \mathscr{T}w(S)$ is a left fibration.*

We only outline the main points of the proof. The reader can find details in the cited text.

Proof outline. We wish to show that any lifting problem of the form

$$\begin{tikzcd} \Lambda^n_{n-i} \arrow[r] \arrow[d] & \mathscr{T}w(\mathscr{C}) \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & (\mathscr{C}^{\mathrm{op}} \times \mathscr{C}) \times_{(S^{\mathrm{op}} \times S)} \mathscr{T}w(S), \end{tikzcd}$$

with $i \le n$, admits a solution. Such a lifting problem admits a solution if an only if the corresponding problem

$$\begin{tikzcd} K_0 \arrow[r] \arrow[d] & \mathscr{C} \arrow[d] \\ \Delta^{2n+1} \arrow[r] \arrow[ur, dotted] & S \end{tikzcd} \tag{26}$$ eq:1641

admits a solution, where $K_0 \subseteq \Delta^{2n+1}$ is some subcomplex which we descirbe below.

Given a subset $J \subseteq [2n+1]$, the non-degenerate simplex $\Delta^J \subseteq \Delta^{2n+1}$ lies in K_0 if and only if J is contained in one of $[n]$ or $[2n+1] - [n]$, or J is contained in a subset $[2n+1]-\{j,2n+1-j\}$ with $j\neq i$. It is argued in [5] that the inclusion $K_0\to\Delta^{2n+1}$ is in fact anodyne, by factoring this map into a sequence of inclusions

$$K_0\to K_1\to\cdots\to K_m=\Delta^{2n+1}$$

in which each K_{i+1} is obtained from K_i by attaching a single non-degenerate simplex. Each such inclusion $K_i\to K_{i+1}$ is shown to be anodyne, so that the composition $K_0\to\Delta^{2n+1}$ is in fact anodyne, and we find that the problem (26) admits a solution, as desired. □

Since $\mathscr{C}^{\mathrm{op}}\times\mathscr{C}$ is itself an ∞-category whenever $\mathscr{C}$ is an ∞-category, we find that the twisted arrow category $\mathscr{T}w(\mathscr{C})$ is also an ∞-category in this case.

Corollary 12.3. *If $\mathscr{C}$ is an ∞-category, the twisted arrow category $\mathscr{T}w(\mathscr{C})$ is also an ∞-category.*

Via Proposition 12.2, and the general phenomena of transport for left fibrations (Proposition 6.17), we understand that the left fibration $\lambda:\mathscr{T}w(\mathscr{C})\to\mathscr{C}^{\mathrm{op}}\times\mathscr{C}$ identifies a associated transport functor

$$H:\mathscr{C}^{\mathrm{op}}\times\mathscr{C}\to\mathscr{K}an.$$

This transport functor is uniquely determined, up to a contractible space of choices, by the assertion that H fits into a categorical pullback diagram

$$\begin{tikzcd}
\mathscr{T}w(\mathscr{C}) \arrow[r] \arrow[d] & \mathscr{K}an_{*/} \arrow[d] \\
\mathscr{C}^{\mathrm{op}}\times\mathscr{C} \arrow[r, "H"'] & \mathscr{K}an.
\end{tikzcd}$$

def:hom_functor **Definition 12.4.** A Hom-functor for and ∞-category $\mathscr{C}$ is a transport functor

$$H:\mathscr{C}^{\mathrm{op}}\times\mathscr{C}\to\mathscr{K}an$$

for the left fibration $\lambda:\mathscr{T}w(\mathscr{C})\to\mathscr{C}^{\mathrm{op}}\times\mathscr{C}$.

The first aims of this section are to provide a calculation of the fibers of the twisted arrow fibration sufficient conditions which allow us to identify a Hom functor when we see one. Of interest are Hom functors for nerves of dg and simplicial categories (e.g. Hom functors for derived categories).

Let us note, as a bit of foreshadowing, that any Hom functor determines maps into the functor categories

$$H_*:\mathscr{C}\to\mathrm{Fun}(\mathscr{C}^{\mathrm{op}},\mathscr{K}an)\ \text{ and }\ H^*:\mathscr{C}^{\mathrm{op}}\to\mathrm{Fun}(\mathscr{C},\mathscr{K}an).$$

We will eventually find that these functors are both fully faithful embeddings.

12.2. **Fibers of the twisted arrows fibration.** At any points in an ∞-category $x:*\to\mathscr{C}$ we can restrict along the projection $(\Delta^n)^{\mathrm{op}}\to *$ to obtain an inclusion into the twisted arrows category

$$\mathscr{C}_{x/}\to\{x\}\times_{\mathscr{C}^{\mathrm{op}}}\mathscr{T}w(\mathscr{C}). \tag{27}$$ eq:2199

This map fits into a diagram over $\mathscr{C}$.

lem:initial_tw **Lemma 12.5** ([5, 03JW])**.** *At any object $x:*\to\mathscr{C}^{\mathrm{op}}$, and isomorphism $\alpha:x\to x'$, the map α is an initial object in the fiber $\{x\}\times_{\mathscr{C}^{\mathrm{op}}}\mathscr{T}w(\mathscr{C})$.*

We note that we can take $x' = x$ and $\alpha = id_x$. In particular, we observe that the fiber of the twisted arrow category over any point in $\mathscr{C}^{\mathrm{op}}$ admits an initial object.

Outline of proof. We want to show that the forgetful map

$$(\{x\} \times_{\mathscr{C}^{\mathrm{op}}} \mathscr{T}w(\mathscr{C}))_{\alpha/} \to \{x\} \times_{\mathscr{C}^{\mathrm{op}}} \mathscr{T}w(\mathscr{C})$$

is a trivial Kan fibration. We note that solving the relevant lifting problem along an inclusion $\partial\Delta^{n-1} \to \Delta^{n-1}$ is equivalent to extending the boundary of an n-simplex $\bar{\sigma} : \partial\Delta^n \to \mathscr{T}w(\mathscr{C})$, with $n > 0$ and $\bar{\sigma}|_{\{0\}} = \alpha$, to an n-simplex $\sigma : \Delta^n \to \mathscr{T}w(\mathscr{C})$. This problem, in turn, is equivalent to solving a lifting problem of the form

$$\begin{tikzcd} K \arrow[r, "\tau_0"] \arrow[d] & \mathscr{C} \arrow[d] \\ \Delta^{2n+1} \arrow[r] \arrow[ur, dotted] & * \end{tikzcd} \tag{28}$$

eq:2209

with K the subcomplex in Δ^{2n+1} which is the union of the J-simpleces $\Delta^J \to \Delta^{2n+1}$, where $J \subseteq [2n+1]$ is any subset which is either contained in $[n]$ or $[2n+1] - [n]$ or $[2n+1] - \{i, 2n+1-i\}$ for some i. The assumption that $\bar{\sigma}$ lands in the fiber $\{x\} \times_{\mathscr{C}^{\mathrm{op}}} \mathscr{T}w(\mathscr{C})$ forces $\tau_0|_{\Delta^n}$ to be of constant value x, and the assmuption that $\bar{\sigma}|_{\{0\}} = \alpha$ forces $\tau_0|\Delta^{\{n,n+1\}} = \alpha$

Now, one argues that there is a factoring of the inclusion $K \to \Delta^{2n+1}$ into a sequence of inclusions

$$K = K_0 \to K_1 \to \cdots \to K_m = \Delta^{2n+1}$$

with each K_{i+1} fitting into a pushout square

$$\begin{tikzcd} \Lambda^{d_i}_{k_i} \arrow[r] \arrow[d] & K_i \arrow[d] \\ \Delta^{d_i} \arrow[r] & K_{i+1} \end{tikzcd}$$

with each $k_i < d_i$, or $k_i = 0$, $d_i > 1$, and $\Delta^{\{0,1\}} \to \Lambda^{d_i}_0 \to \Delta^{2n+1}$ landing in the 1-skeleton $\mathrm{Sk}_1\, \Delta^{n+1} \subseteq K$. This final condition implies that, in the case $k_i = 0$, and map $\tau_i : K_i \to \mathscr{C}$ extending $\tau_0 : K \to \mathscr{C}$ sends the initial vertex

$$\Delta^{\{0,1\}} \to \Lambda^{d_i}_0 \to K_i \xrightarrow{\tau_i} \mathscr{C}$$

to an isomorphism in $\mathscr{C}$.

Using the above information, and Proposition I-5.33, we can produce sequential solutions to the lifting problems

$$\begin{tikzcd} K_i \arrow[r, "\tau_i"] \arrow[d] & \mathscr{C} \arrow[d] \\ K_{i+1} \arrow[r] \arrow[ur, dotted, "\tau_{i+1}"] & * \end{tikzcd}$$

in order to produce the desired solution $\tau = \tau_m : \Delta^{2n+1} \to \mathscr{C}$ to the problem (28). $\square$

prop:2273 **Proposition 12.6.** *The maps* (27) *define equivalences of left and right fibrations*

$$\begin{tikzcd} \mathscr{C}_{x/} \arrow[rr, "\sim"] \arrow[dr] & & \{x\} \times_{\mathscr{C}^{\mathrm{op}}} \mathscr{T}w(\mathscr{C}) \arrow[dl] \\ & \mathscr{C} & \end{tikzcd}.$$

Proof. The map $F : \mathscr{C}_{x/} \to \{x\} \times_{\mathscr{C}^{\mathrm{op}}} \mathscr{T}w(\mathscr{C})$ preserves the initial object id_x, and so is an equivalence of ∞-categories by Proposition 9.26. It follows that F is an equivalence of cocartesian fibrations, by Proposition 3.5. □

We recall that the fibers of any equivalence of isofibrations are again equivalences (Corollary I-6.24). So Proposition 12.6 tells us that the fibers of the twisted arrows category $\mathscr{T}w(\mathscr{C})$ are identified with the mapping spaces for $\mathscr{C}$.

cor:tw_fibers **Corollary 12.7.** *Let $\mathscr{C}$ be an ∞-category. At any pair of points $x, y : * \to \mathscr{C}$ the natural map*

$$\mathrm{Hom}^{\mathrm{L}}_{\mathscr{C}}(x,y) = \mathscr{C}_{x/} \times_{\mathscr{C}} \{y\} \to \{x\} \times_{\mathscr{C}^{\mathrm{op}}} \mathscr{T}w(\mathscr{C}) \times_{\mathscr{C}} \{y\}$$

is an equivalence of Kan complexes.

Corollary 12.8. *For any Hom functor $H : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{K}an$, i.e. classifying functor for the twisted arrows fibration, the restriction*

$$\mathscr{C} \overset{x \times id}{\to} \mathscr{C} \times \mathscr{C} \overset{H}{\to} \mathscr{K}an$$

at any point $x : * \to \mathscr{C}$ is a classifying functor for the forgetful functor $\mathscr{C}_{x/} \to \mathscr{C}$.

12.3. **Recognition for Hom functors.**

sect:simp_hom

12.4. **Hom functors for simplicial and dg categories.**

sect:lim_colim

13. Limits and colimits

Let $\mathscr{C}$ be an ∞-category and K be a simplicial set. We consider the embedding

$$\mathscr{C} \cong \mathrm{Fun}(*, \mathscr{C}) \to \mathrm{Fun}(K, \mathscr{C})$$

induced by the terminal map $K \to *$ and, for any object x in $\mathscr{C}$, let $\underline{x}$ denote the corresponding image in $\mathrm{Fun}(K, \mathscr{C})$. This map $\underline{x}$ is just the constant function at x, i.e. the composite

$$K \to * \overset{x}{\to} \mathscr{C}.$$

def:lim_colim **Definition 13.1.** Let $\mathscr{C}$ be an ∞-category and $p : K \to \mathscr{C}$ be an arbitrary map from a simplicial set. A transformation $l : \underline{y} \to p$ is said to exhibit y as a limit of p if, for each object x in $\mathscr{C}$, the composite

$$\mathrm{Hom}_{\mathscr{C}}(z,y) \to \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{z}, \underline{y}) \overset{l_*}{\to} \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{z}, p)$$

is an isomorphism in $\mathrm{h}\,\mathscr{K}an$. A transformation $c : p \to \underline{x}$ is said to exhibit x as a colimit of p if at each z in $\mathscr{C}$ the composite

$$\mathrm{Hom}_{\mathscr{C}}(x,z) \to \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{x}, \underline{z}) \overset{c^*}{\to} \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(p, \underline{z})$$

is an isomorphism in $\mathrm{h}\,\mathscr{K}an$.

Definition 13.2. Let $p : K \to \mathscr{C}$ be a diagram in an ∞-category $\mathscr{C}$. We say an object y is a limit (resp. colimit) for p if there is a transformation $l : \underline{y} \to p$ (resp. $c : p \to \underline{y}$) which exhibits y as a limit (resp. colimit) for p.

sect:char_lim

13.1. **Characterizations of limits.** Having fixed diagram $p : K \to \mathscr{C}$, we recall the oriented fiber product

$$\mathscr{C} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \{p\} := \mathrm{Fun}(\Delta^1 \times K, \mathscr{C}) \times_{\mathrm{Fun}(\partial\Delta^1 \times K,\mathscr{C})} (\mathscr{C} \times \{p\}).$$

Here $\mathscr{C}$ maps to $\mathrm{Fun}(\{0\}\times K, \mathscr{C})$ via the constant functions and p maps to $\mathrm{Fun}(\{1\}\times K, \mathscr{C})$ as the given diagram. The objects in this oriented product are transformations $\underline{z} \to p$ in $\mathrm{Fun}(K,\mathscr{C})$.

prop:3309

Proposition 13.3. *Let $\mathscr{C}$ be an ∞-category and $p : K \to \mathscr{C}$ be an arbitrary diagram. For a transformation $l : \underline{y} \to p$ in $\mathrm{Fun}(K,\mathscr{C})$, the following are equivalent:*

(a) *l exhibits y as a limit of p in $\mathscr{C}$.*
(b) *l is terminal in the category $\mathscr{C} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \{p\}$.*

Proof. We have the pullback diagram of right fibrations

$$\begin{tikzcd}
\mathscr{C} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \{p\} \arrow[r] \arrow[d, "f'"'] & \mathrm{Fun}(K,\mathscr{C}) \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \{p\} \arrow[d, "f"] \\
\mathscr{C} \arrow[r, "\mathrm{const}"'] & \mathrm{Fun}(K,\mathscr{C})
\end{tikzcd}$$

so that enriched homotopy transport for the fibration f' is the composite of the constant functor with the transport functor for f,

$$f'_! = f_! \,\mathrm{const} : \pi\mathscr{C}^{\mathrm{op}} \to \pi\mathscr{K}an,$$

by Lemma 7.9. Additionally the fiber of $\mathscr{C} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \{p\}$ over a given point z is the fiber of the fibration $\mathrm{Fun}(K,\mathscr{C}) \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \{p\}$ at the constant diagram $\underline{z}$, i.e. the space $\mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{z}, p)$. Hence enriched transport for f' is the composite

$$\mathrm{Hom}_{\mathscr{C}}(z,y) \times \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{y},p) \overset{\mathrm{const}\times 1}{\to} \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{z},\underline{y}) \times \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{y},p)$$
$$\overset{\circ^{\mathrm{op}}}{\to} \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{z},p),$$

by Proposition 7.13. Hence a transformation l in $\mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{y},p)$ exhibits y as a limit for p if and only if l provides a terminal object for the transport functor associated to the right fibration $\mathscr{C} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \{p\} \to \mathscr{C}$. Here by a terminal object for a "contravariant" functor to $\mathscr{K}an$ we simply mean an initial object for the functor $\mathscr{C}^{\mathrm{op}} \to \mathscr{K}an$, in the sense of Definition 11.2.

As in the proof of Proposition 11.4, we see that l is terminal for the transport functor associated to the given fibration if and only if l is terminal as an object in the oriented fiber product $\mathscr{C} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \{p\}$. □

As in the case of a single object $K = *$, we have at general K the equivalence of right fibrations

$$\begin{tikzcd}
\mathscr{C}_{/p} \arrow[rr, "\sim"] \arrow[dr] & & \mathscr{C} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \{p\} \arrow[dl] \\
& \mathscr{C} &
\end{tikzcd}$$

from Theorem I-10.15. On objects this equivalence sends a diagram $t : \Delta^0 \star K \to \mathscr{C}$ to the diagram $\Delta^1 \times K \to \mathscr{C}$ obtained by restricting along the comparison map

$$\Delta^1 \times K \to \Delta^0 \star K$$

of Section I-10.3, which sends the the subcomplex $\{0\} \times K$ to the cone point $\{0\}$ and sends $\{1\} \times K$ identically to the subcomplex K.

cor:lim_cone **Corollary 13.4.** *There exists a transformation $l : y \to p$ which exhibits an object y as a limit of a diagram $p : K \to \mathscr{C}$ if and only if there is a diagram $l' : \Delta^0 \star K \to \mathscr{C}$ for which the corresponding object $l' : * \to \mathscr{C}_{/p}$ is terminal and which has $l(0) = y$.*

Furthermore, if l exhibits y as a limit of p, then a diagram $l' : \Delta^0 \star K \to \mathscr{C}$ is terminal in $\mathscr{C}_{/p}$ if and only if its image in $\mathscr{C} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \{p\}$ is isomorphic to l.

Proof. Follows from Proposition 13.3, the fact that terminal objects are preserved under equivalence (Lemma 9.4), and the fact that terminal objects are unique up to isomorphism (Lemma 9.2). □

cor:isom_lim **Corollary 13.5.** *Consider any diagram $p : K \to \mathscr{C}$, and suppose there is an isomorphism $\alpha : y \overset{\sim}{\to} y'$ in $\mathscr{C}$. Then y is a limit for p if and only if y' is a limit for p.*

Proof. The forgetful functor $\mathscr{C}_{/p} \to \mathscr{C}$ is a right fibration, and hence an isofibration by Lemma I-5.31. So if $\widetilde{y} : * \to \mathscr{C}_{/p}$ is any lift of y to the overcategory, there is an isomorphism $\widetilde{\alpha} : \widetilde{y} \to \widetilde{y}'$ in $\mathscr{C}_{/p}$ which lifts the given isomorphism α. Hence $\widetilde{y}$ is initial if and only if $\widetilde{y}'$ is initial, by Lemma 9.3. It follows by Corollary 13.4 that y is a limit of p if and only if y' is a limit of p. □

cor:lim_isom **Corollary 13.6.** *If y and y' are limits for a diagram $p : K \to \mathscr{C}$, then y and y' are isomorphic in $\mathscr{C}$.*

Proof. By Lemma 9.2 the space of terminal objects in $\mathscr{C}_{/p}$ is contractible. In particular, any two initial lifts $\widetilde{y}$ and $\widetilde{y}'$ of y and y' are necessarily isomorphic. □

sect:char_colim

13.2. **Characterizations of colimits.** The obvious analogs of the arguments provided in Section 13.1 provide the following characterization of colimits in a given ∞-category $\mathscr{C}$.

prop:char_colim **Proposition 13.7.** *Let $p : K \to \mathscr{C}$ be an arbitrary diagram in an ∞-category $\mathscr{C}$. The following are equivalent:*

- (a) *There exists a morphism $c : p \to \underline{x}$ in $\mathrm{Fun}(K,\mathscr{C})$ which exhibits x as a colimit of p.*
- (b) *There is an initial object c in $\{p\} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \mathscr{C}$ which lies over x in $\mathscr{C}$.*
- (c) *There exists an initial object $\widetilde{x}$ in $\mathscr{C}_{p/}$ which maps to x in $\mathscr{C}$.*

We can furthermore compare objects in the undercategory $\mathscr{C}_{p/}$ and tranformations in $\mathrm{Fun}(K,\mathscr{C})$ via the equivalence of left fibrations

$$\begin{tikzcd} \mathscr{C}_{p/} \arrow[rr, "\sim"] \arrow[dr] & & \{p\} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \mathscr{C} \arrow[dl] \\ & \mathscr{C} & \end{tikzcd}$$

from Theorem I-10.15. In particular, an object $\widetilde{x}$ in $\mathscr{C}_{p/}$ is initial if and only if the image $c : p \to \underline{x}$ in $\{p\} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \mathscr{C}$ is initial, by Lemma 9.4. Finally c is initial in the oriented fiber product if and only if c exhibits x as a colimit of p, in the sense of Definition 13.1, by similar arguments to those employed in the proof of Proposition 13.3.

Corollary 13.8. *Given a digram $p : K \to \mathscr{C}$, and isomorphic objects $x, x' : * \to \mathscr{C}$, x is a colimit of p if and only if x' is a colimit of p. Furthermore, any two colimits of p are isomorphic in $\mathscr{C}$.*

Proof. Follows from the fact that the forgetful functor $\mathscr{C}_{p/} \to \mathscr{C}$ is an isofibration and stability of initial objects under isomorphism. $\square$

13.3. **Limit and colimit diagrams.** Given the conclusions of Corollary 13.4 and Proposition 13.7, the following definition now makes sense.

Definition 13.9. Let $p : K \to \mathscr{C}$ be a diagram in an ∞-category. We call a diagram $l' : \Delta^0 \star K \to \mathscr{C}$ a limit diagram for p if $l'|_K = p$ and l' is terminal when considered as an object in the overcategory $\mathscr{C}_{/p}$. We call a diagram $c' : K \star \Delta^0 \to \mathscr{C}$ a colimit diagram if $c'|_K = p$ and c' is initial when considered as an object in the undercategory $\mathscr{C}_{p/}$

We say a diagram $l : * \to \mathscr{C}_{/p}$ exhibits an object y as a limit of p if l is terminal and $l(0) = y$. Similarly, we say $c : * \to \mathscr{C}_{p/}$ exhibits an object x as a colimit of p if c is initial and $c(0) = x$.

13.4. **Limits and colimts under equivalence.** We record an expected result.

:lim_diagram_equivalence

Proposition 13.10. *Let $F : \mathscr{C} \to \mathscr{D}$ be an equivalence of ∞-categories.*

1. *A diagram $\widetilde{p} : \{0\} \star K \to \mathscr{C}$ is a limit diagram in $\mathscr{C}$ if and only if the diagram $F\widetilde{p} : \{0\} \star K \to \mathscr{D}$ is a limit diagram in $\mathscr{D}$.*
2. *A diagram $\widetilde{p}' : K \star \{1\} \to \mathscr{C}$ is a colimit diagram in $\mathscr{C}$ if and only if the diagram $F\widetilde{p}' : K \star \{1\} \to \mathscr{D}$ is a colimit diagram in $\mathscr{D}$.*

Proof. (1) Take $p = \widetilde{p}|_K$. By Corollary I-6.22 and Proposition I-6.23 the induced map on oriented fiber products

$$\mathscr{C} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \{p\} \to \mathscr{D} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{D})} \{Fp\}$$

is an equivalence. By naturality of the slice diagonal equivalence (Theorem I-10.15) we find that the induced map $\mathscr{C}_{/p} \to \mathscr{D}_{/Fp}$ is an equivalence as well. Since equivalences preserve and detect terminal objects, by Lemma 9.4, we see that $\widetilde{p} : * \to \mathscr{C}_{/p}$ is terminal if and only if $F\widetilde{p} : * \to \mathscr{D}_{/Fp}$ is terminal. (2) Follows by taking opposites. $\square$

As one sees from the proof, the analogous result holds for limit and colimit transformations as well.

p:lim_transf_equivalence

Proposition 13.11. *Let $F : \mathscr{C} \to \mathscr{D}$ be an equivalence of ∞-categories and let $p : K \to \mathscr{C}$ be any diagram.*

1. *A transformation $l : \underline{x} \to p$ exhibits an object x in $\mathscr{C}$ as a limit of p if and only if $Fl : \underline{F(x)} \to Fp$ exhibits the object Fx in $\mathscr{D}$ as a limit of Fp.*
2. *A transformation $c : p \to \underline{y}$ exhibits and object y in $\mathscr{C}$ as a colimit of p if and only if $Fc : Fp \to \underline{F(y)}$ exhibits the object Fy in $\mathscr{D}$ as a colimit of Fp.*

Proof. Refer to Propositions 13.3 and 13.7 and proceed as in the proof of Proposition 13.10. $\square$

13.5. **Some results for change of diagrams.** We recall that a map of simplicial sets $i : K \to L$ is called a *categorical equivalence* if, for each ∞-category $\mathscr{C}$, the induced map

$$i_* : \pi_0(\mathrm{Fun}(L, \mathscr{C})^{\mathrm{Kan}}) \to \pi_0(\mathrm{Fun}(K, \mathscr{C})^{\mathrm{Kan}})$$

is an equivalence of ∞-categories (Definition I-10.11 and Lemma I-10.12).

Example 13.12. If $i : K \to L$ is inner anodyne, then i is a categorical equivalence by Corollary I-5.8.

Example 13.13. If $F : \mathbb{A} \to \mathbb{B}$ is an equivalence of discrete categories–or more generally ∞-categories–then F is a categorical equivalence.

prop:lim_catequiv

Proposition 13.14. *Suppose that a map $i : L \to K$ is a categorical equivalence, and let $p : K \to \mathscr{C}$ be a diagram in an ∞-category. A transformation $l : \underline{y} \to p$ exhibits an object y as a limit if p if and only if the corresponding transformation $l|_L : \underline{y} \to p|_L$ exhibits y as a limit of $p|_L$. Dually, a transformation $c : p \to \underline{x}$ exhibits x as a colimit of p if and only if $c|_L : p|_L \to \underline{x}$ exhibits x as a colimit of $p|_L$.*

Proof. We have the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(z, y) \arrow[r] \arrow[dr] & \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{z}, \underline{y}) \arrow[r, "l^*"] \arrow[d, "\sim"] & \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{z}, p) \arrow[d, "\sim"] \\
 & \mathrm{Hom}_{\mathrm{Fun}(L,\mathscr{C})}(\underline{z}, \underline{y}) \arrow[r, "(li)^*"] & \mathrm{Hom}_{\mathrm{Fun}(L,\mathscr{C})}(\underline{z}, p)
\end{tikzcd}$$

in $\mathrm{h}\mathscr{K}an$, from which we conclude that the top composite is an equivalence if and only if the bottom composite is an equivalence. Hence l is a limit transformation if and only if li is a limit transformation. The case of c versus ci is similar. $\square$

Corollary 13.15. *Let K be a category, or more generally an ∞-category. If K has an initial object $a : * \to K$, then for any diagram $p : K \to \mathscr{C}$ the value $p(a)$ is a limit for p. Similarly, if K has a terminal object $z : * \to K$, then for any diagram $p : K \to \mathscr{C}$ the value $p(z)$ is a colimit for p.*

Proof. If a is initial or terminal then the inclusion $a : * \to K$ is inner anodyne, by Proposition 9.24. $\square$

Corollary 13.16. *Suppose $i : L \to K$ is a categorical equivalence and let $p : K \to \mathscr{C}$ be a diagram. Take $p' = pi : L \to \mathscr{C}$.*

1. *For any object $l : * \to \mathscr{C}_{/p}$, l is a limit diagram if and only if the image of l under the forgetful functor $\mathscr{C}_{/p} \to \mathscr{C}_{/p'}$ is a limit diagram.*
2. *For an object $c : * \to \mathscr{C}_{p/}$, c is a colimit diagram if and only if the image of c under the forgetful functor $\mathscr{C}_{p/} \to \mathscr{C}_{p'/}$ is a colimit diagram.*

Proof. We prove (1). Consider the diagram

$$\begin{tikzcd}
\mathscr{C}_{/p} \arrow[r, "\sim"] \arrow[d] & \mathscr{C} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \{p\} \arrow[d] \\
\mathscr{C}_{/p'} \arrow[r, "\sim"] & \mathscr{C} \times^{\mathrm{or}}_{\mathrm{Fun}(L,\mathscr{C})} \{p'\}.
\end{tikzcd}$$

By Propositions 13.3 and 13.14 we see that an object in $\mathscr{C} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \{p\}$ is terminal if and only if its image in $\mathscr{C} \times^{\mathrm{or}}_{\mathrm{Fun}(L,\mathscr{C})} \{p'\}$. By preservation of terminal objects under equivalence, Lemma 9.4, we therefore find that a diagram l is initial, i.e. a limit diagram, if and only if its image in $\mathscr{C}_{/p'}$ is initial. $\square$

`prop:isom_replace`

Proposition 13.17. *Suppose that $p, p' : K \to \mathscr{C}$ are diagrams in an ∞-category, and that $\xi : p \to p'$ is a morphism in $\mathrm{Fun}(K, \mathscr{C})$ for which, at each vertex $x : * \to K$, the map $\xi(x) : p(x) \to p'(x)$ is an isomorphism in $\mathscr{C}$.*

(1) *A transformation $l : \underline{y} \to p$ exhibits y as a limit of p if and only if the transformation ξl exhibits y as a limit of p'.*
(2) *A transformation $c : p' \to \underline{x}$ exhibits x as a colimit of p' if and only if $c\xi : p \to \underline{x}$ exhibits x as a colimit of p.*

Proof. By Proposition I-7.9 the map ξ is an isomorphism in the ∞-category $\mathrm{Fun}(K, \mathscr{C})$. So (1) follows from a consideration of the diagram

$$\begin{tikzcd} & \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{z}, \underline{y}) \arrow[dl, "l_*"'] \arrow[dr, "\xi_* l_*"] & \\ \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{z}, p) \arrow[rr, "\xi_*"] & & \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{z}, p') \end{tikzcd}$$

in $\mathrm{h}\mathscr{K}an$. One similarly observes (2). $\square$

`prop:lim_isom_diagram`

Proposition 13.18. *Let $\mathscr{C}$ be an ∞-category.*

(1) *If $F : \Delta^1 \to \mathrm{Fun}(\{0\} \star K, \mathscr{C})$ is an isomorphism of diagrams, then the restriction $F|_{\{0\}}$ is a limit diagram in $\mathscr{C}$ if and only if $F|_{\{1\}}$ is a limit diagram in $\mathscr{C}$.*
(2) *If $F' : \Delta^1 \to \mathrm{Fun}(K \star \{1\}, \mathscr{C})$ is an isomorphism of diagrams, then $F'|_{\{0\}}$ is a colimit diagram in $\mathscr{C}$ if and only if $F'|_{\{1\}}$ is a colimit diagram in $\mathscr{C}$.*

Proof. We deal with the case of a limit, the case of a colimit being similar. Take

$$p = F|_{\{0\}\times K}, \quad p' = F|_{\{1\}\times K}, \quad x = F|_{(0,0)}, \quad x' = F|_{(1,0)}$$

and

$$\xi = F|_{\Delta^1 \times K} : p \to p', \quad \alpha = F|_{\Delta^1 \times \{0\}} : x \to x'.$$

We have the unique map $t : \Delta^1 \times K \to \{0\} \star K$ which is of constant value 0 on $\{0\} \times K$ and the identity $K \to K \subseteq \{0\} \star K$ on $\{1\} \times K$. The transformation F therefore determines a map

$$\Lambda : \Delta^1 \times \Delta^1 \times K \overset{id \times t}{\to} \Delta^1 \times (\{0\} \star K) \overset{F}{\to} \mathscr{C},$$

which we might interpret at a morphism $\Lambda : \Delta^1 \times \Delta^1 \to \mathrm{Fun}(K, \mathscr{C})$.

Let $l : \underline{x} \to p$ and $l' : \underline{x}' \to p'$ denote the transformation associated to $F|_{\{0\}}$ and $F_{\{1\}}$, under the slice diagonal equivalences $\mathscr{C}_{/q} \to \mathscr{C} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \{q\}$. This morphism Λ has restrictions

$$\Lambda|_{(0,0)} = \underline{x}, \quad \Lambda|_{(1,0)} = \underline{x}', \quad \Lambda|_{(0,1)} = p, \quad \Lambda|_{(1,1)} = p'$$

and

$$\Lambda|_{\{0\}\times\Delta^1} = l, \quad \Lambda|_{\{1\}\times\Delta^1} = l', \quad \Lambda|_{\Delta^1\times\{0\}} = \underline{\alpha}, \quad \Lambda|_{\Delta^1\times\{1\}} = \xi.$$

Hence the non-degenerate 2-simplices in the square $\Delta^1 \times \Delta^1$ exhibit the diagonal as the simultaneous composites

$$\xi l = l'\underline{\alpha} : \underline{x} \to p'$$

in the enriched category $\pi\,\mathrm{Fun}(K, \mathscr{C})$.

We recall that, by our assumption that F is a natural isomorphism, and Proposition I-7.9, the maps α and ξ are isomorphisms. Hence $\underline{\alpha}$ provides an isomorphism between $l'\underline{\alpha}$ and l' in $\mathscr{C} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \{p'\}$. Specifically, the diagram

$$\begin{tikzcd} & p' \\ \underline{x} \arrow[ur, "\mathrm{comp}"] \arrow[r, "\underline{\alpha}"'] & \underline{x}' \arrow[u, "l'"'] \end{tikzcd}$$

expands to a degenerate diagram

$$\begin{tikzcd} p' \arrow[r, "id_{p'}"] & p' \\ \underline{x} \arrow[u, "\mathrm{comp}"] \arrow[ur, "\mathrm{comp}"] \arrow[r, "\underline{\alpha}"'] & \underline{x}' \arrow[u, "l'"'] \end{tikzcd}$$

which provides the claimed isomorphism. Hence l' is terminal in the oriented fiber product if and only if $l'\underline{\alpha}$ is terminal in the oriented fiber product. Consequently, l' exhibits x' as a limit of p' if and only if $l'\underline{\alpha}$ exhibits x as a limit of p, by Proposition 13.3. So we see finally

$$\begin{aligned} F|_{\{0\}} \text{ is a limit diagram } &\Leftrightarrow\ l \text{ exhibits } x \text{ as a limit of } p && \text{(Proposition 13.3)} \\ &\Leftrightarrow\ \xi l (= l'\alpha) \text{ exhibits } x \text{ as a limit of } p' && \text{(Proposition 13.17)} \\ &\Leftrightarrow\ l' \text{ exhibits } x' \text{ as a limit of } p' \\ &\Leftrightarrow\ F|_{\{1\}} \text{ is a limit diagram} && \text{(Proposition 13.3).} \end{aligned}$$

□

We leave the details for the following related result to the reader.

prop:isom_lim_diagram

Proposition 13.19. *Let $\mathscr{C}$ be an ∞-category.*

1. *Let $F : \Delta^1 \to \mathrm{Fun}(\{*\} \star K, \mathscr{C})$ be a transformation of diagrams which restricts to a natural isomorphism on K, and suppose that $F|_{\{0\}}$ is a limit diagram. Then $F|_{\{1\}}$ is a limit diagram if and only if F restricts to an isomorphism at the cone point in $\{*\} \star K$.*
2. *Let $F' : \Delta^1 \to \mathrm{Fun}(K \star \{*\}, \mathscr{C})$ be a transformation of diagrams which restricts to a natural isomorphism on K, and suppose that $F'|_{\{0\}}$ is a colimit diagram. Then $F'|_{\{1\}}$ is a colimit diagram in $\mathscr{C}$ if and only if F' restricts to an isomorphism at the cone point in $K \star \{*\}$.*

Idea of proof. For (1), for example, one argues as in the proof of 13.19 and refers to uniqueness of terminal objects in the oriented product $\mathscr{C} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \{p'\}$ (Lemmas 9.2 and 9.3). □

prop:lift_isom_lim

Proposition 13.20. *Suppose we have diagrams $p_0, p'_0 : K \to \mathscr{C}$, an isomorphism $\zeta_0 : p_0 \overset{\sim}{\to} p'_0$, and limit diagrams $p, p' : \{0\} \star K \to \mathscr{C}$. The following hold.*

(1) *There is an isomorphism* $\zeta : p \xrightarrow{\sim} p'$ *with* $\zeta|_K = \zeta_0$.

(2) *The full subcategory* $Y \subseteq \mathrm{Fun}(\Delta^1 \times \{0\} \star K, \mathscr{C})$ *spanned by such isomorphisms* ζ *lifting* ζ_0 *is a contractible Kan complex.*

Furthermore, the analogous result holds for colimit diagrams in $\mathscr{C}$.

Proof. The result for colimits follows by taking opposites. (1) In the case of limits, we consider the isofibration $\mathrm{Fun}(\{0\} \star K, \mathscr{C}) \to \mathrm{Fun}(K, \mathscr{C})$ provided by restriction (Corollary I-6.14). Hence we can lift p'_0 to an object $p'' : \{0\} \star K \to \mathscr{C}$ and ζ_0 to an isomorphism $\zeta' : p \xrightarrow{\sim} p''$. From Proposition 13.18 we conclude that p'' is a limit diagram. Thus p'' and p' are terminal in $\mathscr{C}_{/p'_0}$ and we have an isomorphism $\eta : p'' \to p'$ in $\mathscr{C}_{/p'_0}$.

Translating the information above, we have a diagram $\eta : \Delta^1 \star K \to \mathscr{C}$ and we have the unique map

$$\pi : \Delta^1 \times (\{0\} \star K) \to \Delta^1 \star K$$

which, when interpreted as a map $\Delta^1 \to \mathrm{Fun}(\{0\} \star K, \Delta^1 \star K)$, restricts to the inclusion $\{0\} \star K \to \Delta^1 \star K$ at 0 and the inclusion $\{0\} \star K \cong \{1\} \star K \to \Delta^1 \star K$ at 1. Restricting along π,

$$\pi^* : \mathrm{Fun}(\Delta^1 \star K, \mathscr{C}) \to \mathrm{Fun}(\Delta^1 \times \{0\} \star K, \mathscr{C}),$$

we obtain an transformation $\pi^*(\eta) : p'' \to p'$ in $\mathrm{Fun}(\{0\} \star K, \mathscr{C})$ which evaluates at each object as

$$\pi^*(\eta)(x) = \begin{cases} \eta|_{\Delta^1} : p''(0) \to p'(0) & \text{when } x = 0 \\ id_x & \text{when } x \in K[0]. \end{cases}$$

In particular, $\pi^*(\eta)$ evaluates to an isomorphism at all vertices, and is therefore a natural isomorphism by Theorem I-7.6. We obtain the claimed isomorphism ζ as the composite $\pi^*(\eta)\zeta' : p \to p'$.

(2) By Proposition I-5.33 the map ζ is a cocartesian lift of the map ζ along the isofibration $\mathrm{Fun}(\{0\} \star K, \mathscr{C}) \to \mathrm{Fun}(K, \mathscr{C})$. By Proposition 2.4, the space $Y' \subseteq \mathrm{Fun}(\Delta^1 \times \{0\} \star K, \mathscr{C})$ spanned by cocartesian lifts $\zeta' : p \to p''$ of ζ_0 is contractible. Since all isomorphisms lifting ζ_0 are cocartesian, by Proposition I-5.33, this subspace Y' is precisely the full subcategory of isomorphisms over ζ_0.

We now consider the isofibration

$$Y' \subseteq \mathrm{Fun}(\Delta^1 \times \{0\} \star K, \mathscr{C}) \to \mathrm{Fun}(\{1\} \times \{0\} \star K, \mathscr{C})$$

and take the fiber $Y = Y' \times_{\mathrm{Fun}(\{1\} \times \{0\} \star K, \mathscr{C})} \{p'\}$. This fiber is the full subcategory in Y' spanned by the objects p and p'. Contractibility of Y' implies contractibility of Y. Hence the isomorphism $\zeta : p \to p'$ is seen to be unique up to a contractible space of choices. □

13.6. **Limits and colimits under adjunctions.** As in the discrete setting, one can show that any left adjoint respects colimits and any right adjoint respects limits. We begin this discussion with a result concerning exponentiation of natural transformations. For each functor $F : \mathscr{C} \to \mathscr{D}$ between ∞-categories, and simplicial set K, we have the induced functor

$$F_* : \mathrm{Fun}(K, \mathscr{C}) \to \mathrm{Fun}(K, \mathscr{D}).$$

This assigment $F \mapsto F_*$ extends to a map of simplicial sets.

prop:exp_nat

Proposition 13.21. *Let $\mathscr{C}_i$ and $\mathscr{D}$ be ∞-categories, and K be a simplicial set. Take $\mathscr{E}^K = \mathrm{Fun}(K, \mathscr{E})$ for any ∞-category $\mathscr{E}$. There is a map of ∞-categories*

$$(-)_* : \mathrm{Fun}(\mathscr{C}, \mathscr{D}) \to \mathrm{Fun}(\mathscr{C}^K, \mathscr{D}^K)$$

which is natural in K, which is the apparent isomorphism when $K = *$, and which on objects sends a functor F to the induced map F_*. Furthermore, for any triple of ∞-categories the diagram

$$\begin{tikzcd}
\mathrm{Fun}(\mathscr{C}_1, \mathscr{C}_2) \times \mathrm{Fun}(\mathscr{C}_0, \mathscr{C}_1) \arrow[r, "(-)_* \times (-)_*"] \arrow[d, "\circ"'] & \mathrm{Fun}(\mathscr{C}_1^K, \mathscr{C}_2^K) \times \mathrm{Fun}(\mathscr{C}_0^K, \mathscr{C}_1^K) \arrow[d, "\circ"] \\
\mathrm{Fun}(\mathscr{C}_0, \mathscr{C}_2) \arrow[r, "(-)_*"'] & \mathrm{Fun}(\mathscr{C}_0^K, \mathscr{C}_2^K)
\end{tikzcd}$$

commutes.

Construction 13.21. This is a general fact about simplicial categories. For any n-simplex $F : \Delta^n \to \mathrm{Fun}(\mathscr{C}, \mathscr{D})$ we define F_* via composition in $\underline{\mathrm{Cat}}_\infty$,

$$F_* := \left(\Delta^n \times \mathscr{C}^K \overset{F \times 1}{\to} \mathrm{Fun}(\mathscr{C}, \mathscr{D}) \times \mathscr{C}^K \overset{\circ}{\to} \mathscr{D}^K \right).$$

At each vertex $\{i\}$ and $F_i = F|_{\{i\}}$ we have directly $F_*|_{\{i\}} = (F_i)_*$.

As for composition, following around the left of the proposed diagram produces from a pair of simplices $(G, F) : \Delta^n \to \mathrm{Fun}(\mathscr{C}_1, \mathscr{C}_2) \times \mathrm{Fun}(\mathscr{C}_0, \mathscr{C}_1)$ the simplex

$$\Delta^n \times \mathscr{C}^K \overset{(G,F)\times 1}{\to} \mathrm{Fun}(\mathscr{C}_1, \mathscr{C}_2) \times \mathrm{Fun}(\mathscr{C}_0, \mathscr{C}_1) \times \mathscr{C}_0^K \overset{\circ(\circ \times 1)}{\to} \mathscr{C}_2^K.$$

Proceeding along the right of the proposed diagram produces the n-simplex

$$\Delta^n \times \mathscr{C}^K \overset{(G,F)\times 1}{\to} \mathrm{Fun}(\mathscr{C}_1, \mathscr{C}_2) \times \mathrm{Fun}(\mathscr{C}_0, \mathscr{C}_1) \times \mathscr{C}_0^K \overset{\circ(1 \times \circ)}{\to} \mathscr{C}_2^K.$$

By associativity of composition these simplices are equal. □

We note that the composite of a natural transformation $\zeta : \Delta^1 \times \mathscr{C}_0 \to \mathscr{C}_1$ with a functor $F : \mathscr{C}_1 \to \mathscr{C}_2$ is recovered as the composite of 1-simplices

$$F\zeta = (s_0^* F)\eta$$

where $s_0 : \Delta^1 \to \Delta^0$ is the degeneracy map. Similarly the composition of ζ with a functor $G : \mathscr{C}_{-1} \to \mathscr{C}_0$ is identified with a composite of 1-simplices

$$\zeta(id_{\Delta^1} \times G) = \zeta(s_0^* G).$$

It follows that the functor from Proposition 13.21 respects these compositions between natural transformations and functors. We therefore observe preservation of adjoints under exponentiation.

cor:exp_adj

Corollary 13.22. *Suppose a functor $F : \mathscr{C} \to \mathscr{D}$ is left adjoint to a functor $G : \mathscr{D} \to \mathscr{C}$. Let $\eta : id_{\mathscr{C}} \to GF$ and $\epsilon : FG \to id_{\mathscr{D}}$ be the associated unit and counit transformations. Then for any simplicial set K, the transformations η_* and ϵ_* exhibit the functor $F_* : \mathrm{Fun}(K, \mathscr{C}) \to \mathrm{Fun}(K, \mathscr{D})$ as left adjoint to the functor $G_* : \mathrm{Fun}(K, \mathscr{D}) \to \mathrm{Fun}(K, \mathscr{C})$.*

We use the above information to prove that left adjoints preserve colimits.

prop:left_cocont

Proposition 13.23. *Suppose a functor $F : \mathscr{C} \to \mathscr{D}$ admits a right adjoint, and let $p : K \to \mathscr{C}$ be a diagram in $\mathscr{C}$. If a given transformation $c : p \to \underline{x}$ exhibits an object x as a colimit to p in $\mathscr{C}$, then the transformation $Fc : Fp \to \underline{F(x)}$ exhibits $F(x)$ as a colimit to the diagram $Fp : K \to \mathscr{D}$.*

Proof. By Corollary 13.22 the functor $F_* : \mathrm{Fun}(K, \mathscr{C}) \to \mathrm{Fun}(K, \mathscr{D})$ admits some right adjoint G_* which is induced by a right adjoint $G : \mathscr{D} \to \mathscr{C}$ to F. We therefore have, at each z in $\mathscr{D}$, a diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{D}}(F(x), z) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{D})}(F_*\underline{x}, \underline{z}) \arrow[r, "(F_*c)^*"] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{D})}(F_*p, \underline{z}) \arrow[d, "\cong"'] \\
\mathrm{Hom}_{\mathscr{C}}(x, G(z)) \arrow[r] & \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{x}, G_*\underline{z}) \arrow[r, "c^*"'] & \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(p, G_*\underline{z}),
\end{tikzcd}$$

in $\mathrm{h}\mathscr{K}an$ by Corollary I-14.4. It follows that the top row is an isomorphism in $\mathrm{h}\mathscr{K}an$ since the bottom row is an isomorphism in $\mathrm{h}\mathscr{K}an$ by hypothesis. $\square$

Similar arguments establish the analogous result for limits and right adjoints.

prop:right_cont

Proposition 13.24. *Suppose a functor $G : \mathscr{D} \to \mathscr{C}$ admits a left adjoint, and let $q : K \to \mathscr{D}$ be a diagram in $\mathscr{D}$. If a given transformation $l : \underline{y} \to q$ exhibits an object y as a limit to q in $\mathscr{D}$, then the transformation $Gl : \underline{G(y)} \to Gq$ exhibits $G(y)$ as a limit to the diagram $Gq : K \to \mathscr{C}$.*

13.7. Co/completeness and co/continuity.

def:fin_small

Definition 13.25. A simplicial set K is called finite if K has only finitely many non-degenerate simplices. We call K small if, for each $n \geq 0$, the set of n-simplices $K[n]$ is small.

Definition 13.26. We say an ∞-category $\mathscr{C}$ is complete (resp. cocomplete) if, for each diagram $p : K \to \mathscr{C}$ from a small simplicial set, p admits a limit (resp. colimit) in $\mathscr{C}$. A functor $F : \mathscr{C} \to \mathscr{D}$ from a complete (resp. cocomplete) category is called continuous (resp. cocontinuous) if F commutes with limits (resp. colimits).

From Propositions 13.23 and 13.24 we observe the following.

Proposition 13.27. *Suppose $\mathscr{C}$ and $\mathscr{D}$ are both complete and cocomplete, and let $F : \mathscr{C} \to \mathscr{D}$ be any functor. If F admits a right adjoint then F is cocontinuous. If F admits a left adjoint then F is continuous.*

13.8. Limits and colimits in functor categories.

prop:fun_complete

Proposition 13.28. *Let $\mathscr{C}$ be a complete ∞-category and L be any simplicial set. Then the ∞-category $\mathrm{Fun}(L, \mathscr{C})$ is complete and for any diagram $p : K \to \mathrm{Fun}(L, \mathscr{C})$ an extension $\{0\} \star K \to \mathrm{Fun}(L, \mathscr{C})$ is a limit diagram if and only if, at each x in L, evaluation*

$$\{0\} \star K \to \mathrm{Fun}(L, \mathscr{C}) \xrightarrow{x^*} \mathscr{C}$$

produces a limit diagram in $\mathscr{C}$.

prop:fun_cocomplete

Proposition 13.29. *Let $\mathscr{C}$ be a cocomplete ∞-category and L be any simplicial set. Then the ∞-category $\mathrm{Fun}(L, \mathscr{C})$ is cocomplete and for any diagram $p : K \to$*

$\mathrm{Fun}(L, \mathscr{C})$ *an extension* $K \star \{1\} \to \mathrm{Fun}(L, \mathscr{C})$ *is a colimit diagram if and only if, at each* x *in* L*, evaluation*

$$\{0\} \star K \to \mathrm{Fun}(L, \mathscr{C}) \xrightarrow{x^*} \mathscr{C}$$

produces a colimit diagram in $\mathscr{C}$.

14. Limits and colimits in $\mathscr{C}at_\infty$ and $\mathscr{K}an$

14.1. **Limits in ∞-categories.** Consider an arbitrary diagram $p : K \to \mathscr{C}at_\infty$ and the corresponding cocartesian fibration $\mathscr{E} \to K$. One can take explicitly $\mathscr{E} = \int_K p$ here. A limit diagram for p is a particular diagram from the join $\widetilde{p} : \Delta^0 \star K \to \mathscr{C}at_\infty$, which is then specified by the corresponding fibration $\mathscr{E}' \to \Delta^0 \star K$, which fits into a pullback diagram

$$\begin{tikzcd} \mathscr{E} \arrow[r] \arrow[d] & \mathscr{E}' \arrow[d] \\ \mathscr{C} \arrow[r] & \Delta^0 \star K. \end{tikzcd}$$

Definition 14.1. For any cocartesian fibrations $X \to S$ and $Y \to S$, take $\mathrm{Fun}_S^{\mathrm{CCart}}(X, Y)$ to be the full ∞-subcategory in $\mathrm{Fun}_S(X, Y)$ spanned by those functors which preserve cocartesian edges.

Notation 14.2. For any simplicial set K, we take $K^< := \Delta^0 \star K$. We refer to the vertex $\{0\}$ in $K^<$ as the cone point in $K^<$.

We have the following general result.

Lemma 14.3 ([5, 018Q])**.** *Consider inclusions of simplicial sets* $f : A \to A'$ *and* $g : B \to B'$*, and the corresponding inclusion*

$$\mu : (A \star B') \coprod_{(A \star B)} (A' \star B) \to A' \star B'.$$

If f *is anodyne, then* μ *is left anodyne. If* g *is anodyne, then* μ *is right anodyne.*

In considering the extreme cases where $f = id_A$ and $B = \emptyset$ we find the following.

cor:cone_lano **Corollary 14.4.** *The inclusions* $\{0\} \to K^<$ *is left anodyne.*

We also have the following basic result, whose proof we omit.

prop:3558 **Proposition 14.5** ([5, 035S])**.** *For any cocartesian fibration* $q : X \to S$*, left anodyne morphisms of simplicial sets* $S_0 \to S$*, and* $X_0 = X \times_S S_0$*, the restriction functor*

$$\mathrm{Fun}_S^{\mathrm{CCart}}(S, X) \to \mathrm{Fun}_{S_0}^{\mathrm{CCart}}(S_0, X_0)$$

is a trivial Kan fibration.

Note that in the case of a cocartesian fibration over a point $\mathscr{C} \to *$, i.e. an ∞-category, we have $\mathrm{Fun}_*^{\mathrm{CCart}}(*, \mathscr{C}) = \mathrm{Fun}(*, \mathscr{C}) \cong \mathscr{C}$.

cor:conical_sections **Corollary 14.6.** *Let* K *be any simplicial set. For any cocartesian fibration* $\mathscr{E} \to K^<$ *the restriction functor*

$$\mathrm{Fun}_{K^<}^{\mathrm{CCart}}(K^<, \mathscr{E}) \to \mathrm{Fun}_{\{0\}}^{\mathrm{CCart}}(\{0\}, \mathscr{E}_0) = \mathscr{E}_0$$

is a trivial Kan fibration.

Proof. By Corollary 14.4 the inclusion $\{0\} \to K^{<}$ is left anodyne, and hence the result follows by Proposition 14.5. $\square$

lem:extend_transp

Lemma 14.7. *Suppose that we have a pullback diagram of cocartesian fibrations*

$$\begin{tikzcd} \mathscr{E} \arrow[r] \arrow[d, "q"'] & \mathscr{E}' \arrow[d, "q'"] \\ K \arrow[r] & K' \end{tikzcd}$$

where $K \to K'$ is a monomorphism and let $F : K \to \mathscr{C}at_\infty$ be the transport functor for q. There exists a transport functor $F' : K' \to \mathscr{C}at_\infty$ for q' with $F'|_K = F$.

Proof. Let $G : K' \to \mathscr{C}at_\infty$ be any transport functor for q'. Then there is an isomorphism $\xi : G|_K \xrightarrow{\sim} F$ by uniqueness of transport and Lemma 6.12. We have now the restriction functor

$$\mathrm{Fun}(K', \mathscr{C}at_\infty) \to \mathrm{Fun}(K, \mathscr{C}at_\infty)$$

which is an isofibration by Proposition I-**??**. Hence there exists a functor $F' : K \to \mathscr{C}at_\infty$ which extends F and an isomorphism $\xi' : G \xrightarrow{\sim} F'$ which lifts the isomorphism ξ. It follows by unstraightening, Section **??**, that there is an equivalence $\mathscr{E}' \xrightarrow{\sim} \int_{K'} F'$ of cocartesian fibrations over K and hence that F' is a transport functor for q'. $\square$

We have the following characterization of colimit diagrams in $\mathscr{C}at_\infty$.

thm:diff_crit

Theorem 14.8 (Diffraction criterion, [5, 02T8])**.** *Given any diagram $p : K \to \mathscr{C}at_\infty$, an extension $p' : K^{<} \to \mathscr{C}at_\infty$ is a limit diagram if and only if, for the corresponding cocartesian fibrations $\mathscr{E} = \int_K p$ and $\mathscr{E}' = \int_{K^<} p'$, the restriction functor*

$$\mathrm{Fun}^{\mathrm{CCart}}_{K^<}(K^<, \mathscr{E}') \to \mathrm{Fun}^{\mathrm{CCart}}_K(K, \mathscr{E})$$

is an equivalence of ∞-categories.

Lurie then shows that such extensions p' always exist [5, 02TG], so that we obtain completeness of the ∞-category $\mathscr{C}at_\infty$.

thm:infty_lim

Theorem 14.9. *Any diagram $p : K \to \mathscr{C}at_\infty$ admits a limit. Furthermore, for the associated cocartesian fibration $\mathscr{E} = \int_K p$, the ∞-category $\mathrm{Fun}^{\mathrm{CCart}}_K(K, \mathscr{E})$ is a limit for the diagram p.*

Proof. Existence of the limit follows by the Diffraction Criterion of Theorem 14.8, and for any limit diagram $p' : K^{<} \to \mathscr{C}at_\infty$ with corresponding cocartesian fibration $\mathscr{E}'$ we have equivalences

$$p'(0) \xrightarrow{\sim} \mathscr{E}'_0 \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{CCart}}_{K^<}(K^<, \mathscr{E}') \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{CCart}}_K(K, \mathscr{E}).$$

Here the first equivalence follows by the fiber calculation of Theorem 5.27, the second equivalence is from Corollary 14.6, and the third equivalence follows by the Diffraction Criterion. By definition $p'(0)$ is a limit for p, and by Corollary 13.5 we find that the ∞-category $\mathrm{Fun}^{\mathrm{CCart}}_K(K, \mathscr{E})$ is also a limit for p. $\square$

To clean the situation up slightly, in the setting of Theorem 14.9 we know that there exists a colimit diagram $p' : K^{<} \to \mathscr{C}at_\infty$ and an equivalence $F_0 : \mathrm{Fun}^{\mathrm{CCart}}_K(K, \mathscr{E}) \to p'(0)$ which is uniquely determined up to a contractible space. We have the right fibration $\mathscr{C}_{/p} \to \mathscr{C}$ along which we can lift the equivalence F_0 to a uniquely associated diagram $p'' : K^{<} \to \mathscr{C}at_\infty$ with $p''(0) = \mathrm{Fun}_K^{\mathrm{CCart}}(K, \mathscr{E})$ and an equivalence $p'' \xrightarrow{\sim} p'$. In particular, p'' is also a limit diagram for p'. So for any diagram $p : K \to \mathscr{C}at_\infty$ we can find, specifically, a limit diagram $p' : K^{<} \to \mathscr{C}at_\infty$ with cone point $p'(0) = \mathrm{Fun}_K^{\mathrm{CCart}}(K, \mathscr{E})$ for $\mathscr{E} = \int_K p$.

We now restrict our attention to small diagrams $K \to \mathscr{C}at_\infty$ to observe completeness.

cor:infty_complete **Corollary 14.10.** *The ∞-category of ∞-categories $\mathscr{C}at_\infty$ is complete.*

sect:lim_description

14.2. **Describing limits in $\mathscr{C}at_\infty$.** From the description of the cocartesian fibration $\int_K F \to K$ provided in Section 6.2 we can understand objects in the ∞-category $\mathrm{Fun}_K^{\mathrm{CCart}}(K, \int_K F)$, to some minimal extent. A section $t : K \to \int_K F$ specifies, at least, a choice of an object $t_x : * \to F(x)$ over each vertex x in K, and over each edge $\alpha : x \to y$ we have a morphism $t_\alpha : F(\alpha)(t_x) \to t_y$. Since every edge in K is cocartesian, the section t lies in $\mathrm{Fun}_K^{\mathrm{CCart}}(K, \int_F K)$ if and only if each map t_α is an isomorphism in $F(y)$. Apply t to 2-simplices in K provide compatibilities for the t_α.

In the special case where K is a discrete category, this relatively shallow description of objects in the space of sections can be filled out completely. First, by Lemma 6.29 we may assume that $F : K \to \mathscr{C}at_\infty$ is a strictly commuting diagram of ∞-categories, i.e. is the nerve of a simplicial functor from the underlying plain category. The ∞-category $\int_K F$ can then be replaced with the weighted nerve $\mathrm{N}^F(K)$ by Theorem 6.28. A section $t : K \to \mathrm{N}^F(K)$ in $\mathrm{Fun}_K^{\mathrm{CCart}}(K, \mathrm{N}^F(K))$ specifies for each compatible collection of morphisms $\alpha_{ij} : x_i \to x_j$ in K for $0 \le i < j \le n$, i.e. each n-simplex $\alpha : \Delta^n \to K$, an expanding sequence of simplices

$$\begin{tikzcd} \Delta^i \arrow[r] \arrow[d, "\tau_i(\alpha)"'] & \Delta^j \arrow[d, "\tau_j(\alpha)"] \\ F(x_i) \arrow[r, "F(\alpha_{ij})"'] & F(x_j) \end{tikzcd}$$

in which each constituent morphism $\tau_i(\alpha)|_{\Delta^{\{a,b\}}} : \Delta^1 \to F(x_i)$ is an isomorphism in the ∞-category $F(x_i)$. Said imprecisely, an object in $\mathrm{Fun}_K^{\mathrm{CCart}}(K, \mathrm{N}^F(K))$ is just a choice of objects $t_x : * \to F(x)$ for each x in K, a choice of an isomorphism $t_\alpha : F(\alpha)(t_x) \to t_y$ over each morphism $\alpha : x \to y$ in K which enjoy arbitrarily high levels of compatibility under composition. The easiest way for this compatibility to occur is for each compatibility to be degenerate. We discuss this case below.

14.3. **Comparing limits in $\mathscr{C}at_\infty$ to discrete limits.** Let us take, for any strictly commuting diagram $p : K \to \mathscr{C}at_\infty$ indexed by a discrete category K. Take

$$\lim{}^0(p) := \text{the limit of } p \text{ in the discrete category } \mathrm{sSet}.$$

We make no claim that this limit is an ∞-category at general p, however we do claim that there is a comparison map

$$\lambda : \lim{}^0(p) \to \mathrm{Fun}_K^{\mathrm{CCart}}(K, \mathscr{E}) \tag{29}$$ eq:comp_lambda

where $\mathscr{E}$ is the weighted nerve.

For each object x let $\pi_x : \lim^0(p) \to p(x)$ be the corresponding projection. For an n-simplex $\sigma : \Delta^n \to \lim^0(p)$ define the simplex

$$\lambda(\sigma) : \Delta^n \times K \to \mathscr{E}$$

as follows: Consider an m-simplex $\tau = (\tau_1, \tau_2) : \Delta^m \to \Delta^n \times K$ and take

$$x_i = \tau_2(i), \quad \alpha_{ij} = \tau_2(i \leq j) : x_i \to x_j, \quad \mathscr{E}_i = p(x_i), \quad \pi_i = \pi_{x_i}.$$

We define $\lambda(\sigma)(\tau) : \Delta^m \to \mathscr{E} = \mathrm{N}^p(K)$ to be the sequence of simplices $\{\sigma_i : \Delta^i \to \mathscr{E}_i : 0 \leq i \leq m\}$ with each σ_i equal to the composite

$$\sigma_i = (\Delta^m \to \Delta^n \times K \to \Delta^n \overset{\pi_i \sigma}{\to} \mathscr{E}_i).$$

The assignment $\tau \mapsto \lambda(\sigma)(\tau)$ is compatible with restriction and so defines a map of simplicial sets $\lambda(\sigma) : \Delta^n \times K \to \mathscr{E}$. These $\lambda(\sigma)$ are furthermore compatible with restriction in the Δ^n factor, so that we obtain the proposed comparison map $\lambda : \lim^0(p) \to \mathrm{Fun}_K^{\mathrm{CCart}}(K, \mathscr{E})$ of (29).

def:comp_lambda

Definition 14.11. For any discrete category K and strictly commuting diagram $p : K \to \mathscr{C}at_\infty$ we let $\lim^0(p)$ denote the corresponding limit in the discrete category sSet and

$$\lambda : \lim^0(p) \to \mathrm{Fun}_K^{\mathrm{CCart}}(K, \mathscr{E})$$

denote the comparison morphism constructed above.

There are some obvious cases where the discrete limit $\lim^0(p)$ is an ∞-category. For example, one might consider the limit of a sequence of inner fibrations $\cdots \to \mathscr{C}_2 \to \mathscr{C}_1 \to \mathscr{C}_0$, or the pullback of a pair of inner fibrations. In this case we obtain an easy extension of the given diagram $p : K \to \mathscr{C}at_\infty$ to a strictly commuting diagram over the cone of K.

Definition 14.12. Suppose $p : K \to \mathscr{C}at_\infty$ is a strictly commuting diagram of ∞-categories, and suppose the discrete limit $\lim^0(p)$ in sSet is an ∞-category. We define

$$p^0 : K^< \to \mathrm{Cat}_\infty \subseteq \mathscr{C}at_\infty$$

to be the discrete limit diagram associated to $\lim^0(p)$ in Cat_∞.

Remark 14.13. The limit $\lim^0(p)$ is an actual limit for the corresponding map of discrete categories $\mathrm{Plain}(p) : K \to \mathrm{sSet}$, in both the "classical" and ∞-categorical sense. So we are interested in probing continuity of the inclusion $\mathrm{Cat}_\infty \to \mathscr{C}at_\infty$, and the failures of continuity. This $\lim^0$ notation is just used to indicate where a specific limit occurs.

Proposition 14.14. *Let K be a discrete category and $p : K \to \mathscr{C}at_\infty$ is a strictly commuting diagram of ∞-categories. Suppose that the discrete limit $\lim^0(p)$ is an ∞-category, and take $\mathscr{E} = \mathrm{N}^p(K)$ and $\mathscr{E}^0 = \mathrm{N}^{p^0}(K^<)$. Then the trivial Kan fibration*

$$\mathrm{Fun}_{K^<}^{\mathrm{CCart}}(K^<, \mathscr{E}^0) \to \mathrm{Fun}_{K^<}(\{0\}, \mathscr{E}^0) \cong \lim^0(p)$$

admits a section $\lambda^0 : \lim^0(p) \to \mathrm{Fun}_{K^<}^{\mathrm{CCart}}(K^<, \mathscr{E}^0)$. This section is an equivalence of ∞-categories and fits into the diagram

$$\begin{tikzcd}
\mathrm{Fun}_{K^<}^{\mathrm{CCart}}(K^<, \mathscr{E}^0) \arrow[r] & \mathrm{Fun}_K^{\mathrm{CCart}}(K, \mathscr{E}) \\
\lim^0(p) \arrow[u, "\lambda^0"] \arrow[ur, "\lambda"'] &
\end{tikzcd}.$$

Proof. The discrete limit $\lim^0(p)$ is also the limit for the diagram p^0. So we can take λ^0 to be the comparison map for the strictly commuting diagram p^0. $\square$

The diffraction criterion, Theorem 14.8, now provides a necessary and sufficient criterion for determining when the discrete limit of a strictly commuting diagram in $\mathscr{C}at_\infty$ is an ∞-categorical limit.

`cor:disc_v_infty_lim` **Corollary 14.15.** *Let K be a discrete category and $p : K \to \mathscr{C}at_\infty$ is a strictly commuting diagram of ∞-categories. Suppose that the discrete limit $\lim^0(p)$ is an ∞-category. Then the corresponding strictly commuting diagram $p^0 : K^{<} \to \mathscr{C}at_\infty$ with cone point $\lim^0(p)$ is a limit diagram in $\mathscr{C}at_\infty$ if and only if the comparison map*

$$\lambda : \lim{}^0(p) \to \mathrm{Fun}_K^{\mathrm{CCart}}(K, \mathscr{E})$$

of Definition 14.11 is an equivalence of ∞-categories.

`ex:products` **Example 14.16** (Products)**.** Let K be a discrete set, $K = \coprod_{x \in K}\{x\}$. For any diagram $p : K \to \mathscr{C}at_\infty$ the weighted nerve $\mathrm{N}^p(K)$ is simply the coproduct $\amalg_{x\in K}\mathscr{E}_x$, where $\mathscr{E}_x = p(x)$. This is because every n-simplex in K is constant, so that every n-simplex in the weighted nerve factors through a unique fiber $\Delta^n \to \mathscr{E}_x \to \mathrm{N}^p(K)$. Consequently,

$$\mathrm{Fun}_K^{\mathrm{CCart}}(K, \mathrm{N}^p(K)) = \mathrm{Fun}_K(\amalg_x\{x\}, \amalg_x \mathscr{E}_x) = \prod_{x\in K} \mathscr{E}_x.$$

This limit agrees with the usual limit, the map λ is an isomorphism, and we see that products in $\mathscr{C}at_\infty$ are just the usual products from Cat_∞.

14.4. **Colimits in ∞-categories.** We again outline a construction of colimits in the ∞-category of ∞-categories. We provide the basic rationalle, but omit arguments for some of the technical pinch points. Of course, one can find all details in the original text [5]. We begin with a brisk discussion of localization. Recall that a marked simplicial set (E, W) is a simplicial set E with a prescribed collection $W \subseteq E[1]$ which contains all degenerate edges.

`def:localize` **Definition 14.17.** For a marked simplicial set (E, W) a (Dwyer-Kan) localization of E relative to W is a map of simplicial sets $F : E \to \mathscr{D}$ into an ∞-category $\mathscr{D}$ for which, at each ∞-category $\mathscr{C}$, the restriction functor

$$F^* : \mathrm{Fun}(\mathscr{D}, \mathscr{C}) \to \mathrm{Fun}(E, \mathscr{C})$$

is an equivalence onto the full subcategory of all maps $E \to \mathscr{C}$ which send all edges in W to equivalences in $\mathscr{C}$.

It is the case that such localizations always exist and are unique.

Lemma 14.18. *For any marked simplicial set (E, W), a localization functor $\theta : E \to \mathscr{D}$ exists. Furthermore, any two localizations $F : E \to \mathscr{D}$ and $F' : E \to \mathscr{D}'$ admit an equivalence $\vartheta : \mathscr{D} \to \mathscr{D}'$ which fits into homotopy commuting diagram*

$$\begin{tikzcd}
\mathscr{D} \arrow[rr, "\vartheta"] & & \mathscr{D}' \\
& E \arrow[ul, "F"] \arrow[ur, "F'"'] &
\end{tikzcd}.$$

Proof. Take $\mathscr{D}$ a fibrant replacement for (E, W) in the model category of marked simplicial sets, with respect to the cartesian model structure [3]. Uniqueness follows by the expected nonsense argument. $\square$

In the event that we do not have a specific realization of the lecalization in mind, we write $\mathscr{E}[W^{-1}]$ for the localization of a simplicial set E relative to a collection of edges W.

Given a pullback diagram of cocartesian fibrations

$$\begin{tikzcd} \mathscr{E} \arrow[r, "i"] \arrow[d, "q"'] & \widetilde{\mathscr{E}} \arrow[d, "\widetilde{q}"] \\ K \arrow[r] & K \star \{1\} \end{tikzcd} \tag{30}$$

eq:3939

we have the unique solution the lifting problem

$$\begin{tikzcd} \{0\} \times \mathscr{E} \arrow[r, "i"] \arrow[d] & \widetilde{\mathscr{E}} \arrow[d, "\widetilde{q}"] \\ \Delta^1 \times \mathscr{E} \arrow[r] \arrow[ur, dotted] & K \star \{1\} \end{tikzcd} \tag{31}$$

eq:3946

where the bottom map is the unique map of simplicial sets which is q on $\{0\} \times \mathscr{E}$ and the terminal map $\mathscr{E} \to \{1\}$ on $\{1\} \times \mathscr{E}$ (see Section 6.8). We recall that there is a unique cocartesian solution $s : \Delta^1 \times \mathscr{E} \to \widetilde{\mathscr{E}}$ to this problem by Theorem 2.7, i.e. one which sends each edge $\Delta^1 \times \{e\}$ to a cocartesian morphism in $\widetilde{\mathscr{E}}$.

Definition 14.19. Given a pullback diagram of cocartesian fibrations (30), and an equivalence $\widetilde{\mathscr{E}} \times_{(K\star\{1\})} \{1\} \to \mathscr{E}_1$, a refraction diagram for $\widetilde{\mathscr{E}}$ is a map of simplicial sets $\mathrm{Rf} : \mathscr{E} \to \mathscr{E}_1$ which appears as the restriction $\mathrm{Rf} = s|_{\{1\}\times\mathscr{E}}$ of a cocartesian solution $s : \Delta^1 \times \mathscr{E} \to \widetilde{\mathscr{E}}$ to the lifting problem (32).

By uniqueness of s, we see that the refraction diagram $\mathrm{Rf} : \mathscr{E} \to \mathscr{E}_1$ is uniquely specified by the fibration $\widetilde{q} : \widetilde{\mathscr{E}} \to K \star \{1\}$ and the choice of identification $\widetilde{\mathscr{E}}_1 \cong \mathscr{E}_1$.

thm:refract

Theorem 14.20 (Refraction Criterion, [5, 02UU])**.** *Let $F : K \to \mathscr{C}at_\infty$ be a diagram of ∞-categories, and $q : \mathscr{E} = \int_K F \to K$ the corresponding cocartesian fibration. An ∞-category $\mathscr{E}_1$ is a colimit for the diagram F if and only if there is a pullback diagram of cocartesian fibrations*

$$\begin{tikzcd} \mathscr{E} \arrow[r] \arrow[d, "q"'] & \widetilde{\mathscr{E}} \arrow[d, "\widetilde{q}"] \\ K \arrow[r] & K \star \{1\} \end{tikzcd}$$

for which the fiber $\widetilde{\mathscr{E}}_1$ admits an equivalence $\widetilde{\mathscr{E}}_q \xrightarrow{\sim} \mathscr{E}_1$, and for which the refraction diagram $\mathrm{Rf} : \mathscr{E} \to \mathscr{E}_1$ exhibits $\mathscr{E}_1$ as a localization of $\mathscr{E}$ relative to the class W of q-cocartesian morphisms.

We note that the transport functor $\widetilde{F} : K \star \{1\} \to \mathscr{C}at_\infty$ associated to such a cocartesian fibration $\widetilde{\mathscr{E}} \to K\star\{1\}$ has $\widetilde{F}|_K \cong F$ and has $\widetilde{F}(1)$ equivalent to $\mathscr{E}_1$. Since the forgetful functor $(\mathscr{C}at_\infty)_{\widetilde{F}|_K/} \to \mathscr{C}at_\infty$ we see that the diagram $\widetilde{F}$ determines, up to isomorphism, an object $e_1 : * \to \mathscr{C}_{\widetilde{F}|_K/}$ over $\mathscr{E}_1$. Now, by Proposition 13.17, e_1 realizes $\mathscr{E}_1$ as a colimit of $\widetilde{F}|_K$ if and only if $\mathscr{E}_1$ is a colimit of F. So the assertion of Theorem 14.20 at least makes sense. Furthermore, by considering refraction diagrams one can intuit the nature of the refraction criterion and the necessity of the proposed localization.

We refer the reader directly to [5] for a precise accounting of Theorem 14.20 and its proof.

prop:refract_exist

Proposition 14.21. *Given any cocartesian fibration $q : \mathscr{E} \to K$ over a simplicial set K, there exists a cocartesian fibration $\widetilde{q} : \widetilde{\mathscr{E}} \to K \star \{1\}$ for which the refraction diagram $\mathrm{Rf} : \mathscr{E} \to \mathscr{E}_1$ exhibits $\mathscr{E}_1$ as a localization of $\mathscr{E}$.*

Proof. Take any localization $F : \mathscr{E} \to \mathscr{E}_1$ and corresponding cocartesian fibration $\mathscr{E}[W^{-1}] \to \{1\}$. Then we have the diagram

$$\begin{tikzcd} \mathscr{E} \arrow[r, "F"] \arrow[d, "q"'] & \mathscr{E}[W^{-1}] \arrow[d] \\ K \arrow[r] & \{1\} \end{tikzcd}$$

where the top map F preserves cocartesian edges. By Lemma 6.27 we then obtain a cocartesian fibration

$$\widetilde{q} : \widetilde{\mathscr{E}} := \mathscr{E} \star_{\mathscr{E}_1} \mathscr{E}_1 \to K \star_{\{1\}} \{1\}.$$

We have $K \star_{\{1\}} \{1\} \cong K \star \{1\}$, and identifications of the fibers

$$\mathscr{E} \overset{\cong}{\to} \widetilde{\mathscr{E}} \times_{(K\star\{1\})} K, \quad \mathscr{E}_1 \overset{\cong}{\to} \widetilde{\mathscr{E}} \times_{(K\star\{1\})} \{1\}.$$

We have the unique map $\Delta^1 \times \mathscr{E} \to \mathscr{E} \star \mathscr{E}_1$ which is the identity $\mathscr{E} \to \mathscr{E} \subseteq \mathscr{E} \star \mathscr{E}_1$ at $\{0\}$ and the localization $F : \mathscr{E} \to \mathscr{E}_1 \subseteq \mathscr{E} \star \mathscr{E}_1$ at $\{1\}$. We also have the map $id \times F : \Delta^1 \times \mathscr{E} \to \Delta^1 \times \mathscr{E}_1$. These two morphisms fit into the appropriate diagram

$$\begin{tikzcd} \Delta^1 \times \mathscr{E} \arrow[r] \arrow[d] & \Delta^1 \times \mathscr{E}_1 \arrow[d] \\ \mathscr{E} \star \mathscr{E}_1 \arrow[r] & \mathscr{E}_1 \star \mathscr{E}_1 \end{tikzcd}$$

and so define a morphism to the relative join $s : \Delta^1 \times \mathscr{E} \to \widetilde{\mathscr{E}}$ which sends each edge $\Delta^1 \times \{e\}$ to the pairing of the unique edge $e \to F(e)$ in $\mathscr{E} \star \mathscr{E}_1$ with the edge $\Delta^1 \times \{F(e)\}$ in $\mathscr{E}_1$. By the characterization of $\widetilde{q}$-cocartesian edges in $\widetilde{\mathscr{E}}$ provided in Lemma **??** we see that s provides a cocartesian solution to the lifting problem

$$\begin{tikzcd} \{0\} \times \mathscr{E} \arrow[r, "i"] \arrow[d] & \widetilde{\mathscr{E}} \arrow[d, "\widetilde{q}"] \\ \Delta^1 \times \mathscr{E} \arrow[r] \arrow[ur, dotted, "s"] & K \star \{1\}. \end{tikzcd} \tag{32}$$

eq:3946

Since $s|_{\{1\}\times\mathscr{E}} = F$, we see that F is realized as the refraction diagram for $\widetilde{q}$. $\square$

We combine Theorem 14.20 with Proposition 14.21 to see that $\mathscr{C}at_\infty$ arbitrary colimits.

cor:infty_co_complete

Corollary 14.22. *The ∞-category $\mathscr{C}at_\infty$ is both complete and cocomplete.*

Proof. Follows by Corollary 14.10, Theorem 14.20, and Proposition 14.21. $\square$

ex:coprod

Example 14.23 (Coproducts)**.** Let K be a discrete set, $K = \coprod_{x \in K}\{x\}$. We have already argued in Example 14.23 that for any diagram $p : K \to \mathscr{C}at_\infty$ the corresponding fibration $q : \mathscr{E} = \mathrm{N}^p(K) \to K$ is just the coprodct

$$\mathscr{E} = \coprod_{x \in K} \mathscr{E}_x,$$

where $\mathscr{E}_x = p(x)$ in the above expression.

Since K is a discrete set, and hence has only identity morphisms, the q-cocartesian morphisms in $\mathscr{E}$ are precisely equivalences, so that the localization with respect to cocartesian edges simply returns $\mathscr{E}$ itself, $\mathscr{E}[W^{-1}] = \mathscr{E}$. Hence the discrete colimit diagram for $\mathscr{E}$ in $\mathrm{Cat}_\infty \subseteq \mathscr{C}at_\infty$ is a colimit diagram, by Theorem 14.20, and we find that coproducts in $\mathscr{C}at_\infty$ are just the usual coproducts of ∞-categories.

Remark 14.24. It is a bit odd that we've, apparently, shown that $\mathscr{C}at_\infty$ admits limits and colimits indexed by *arbitrary* simplicial sets, rather than all small sets. However we recall our sizing conventions. All simplicial sets, ∞-categories, and Kan complexes, are in our universe of "medium sized" sets, while we make special exemptions for the ∞-categories $\mathscr{C}at_\infty$ and $\mathscr{K}an$, which lift in our universe of "large" sets. So all simplicial sets are small relative to our large categories $\mathscr{C}at_\infty$ and $\mathscr{K}an$, and the claim that $\mathscr{C}at_\infty$ admits all colimits indexed by medium sized simplicial sets poses no philosophical or material error. In any case, we are only concerned with the existence of small limits and colimits in $\mathscr{C}at_\infty$.

sect:lim_spaces

14.5. **Limits and colimits in spaces.** Our ultimate conclusion here is that $\mathscr{K}an$ is complete and cocomplete, and is in fact closed under both limits and colimits in the ambient category $\mathscr{C}at_\infty$.

thm:kan_complete

Theorem 14.25. *The ∞-category $\mathscr{K}an$ is complete and the inclusion $\mathscr{K}an \to \mathscr{C}at_\infty$ is continuous.*

Proof. Given a diagram $p : K \to \mathscr{K}an$ we have the inclusion $\mathscr{K}an_{/p} \to (\mathscr{C}at_\infty)_{/p}$ which identifies $\mathscr{K}an_{/p}$ with a full ∞-subcategory in $(\mathscr{C}at_\infty)_{/p}$. Furthermore, $\mathscr{K}an_{/p}$ is seen to fit into a pullback diagram

$$\begin{tikzcd} \mathscr{K}an_{/p} \arrow[r] \arrow[d] & (\mathscr{C}at_\infty)_{/p} \arrow[d] \\ \mathscr{K}an \arrow[r] & \mathscr{C}at_\infty. \end{tikzcd}$$

It follows that the limit diagram $l : * \to (\mathscr{C}at_\infty)_{/p}$ is a limit in $\mathscr{K}an_{/p}$ if and only if the cone point $\lim(p)$ in $\mathscr{C}at_\infty$ is a Kan complex.

Take $\mathscr{E} = \int_K p \to K$ the left fibration associated to p. By the calculation of Theorem 14.9 we must show that the functor category $\mathrm{Fun}_K^{\mathrm{CCart}}(K, \mathscr{E})$ is a Kan complex, and for this it suffices to show that the ∞-category $\mathrm{Fun}_K(K, \mathscr{E})$ is a Kan complex. Since the fibers of $\mathscr{E}_x$ over K are all Kan complexes, it follows that for any natural transformation $\zeta : \Delta^1 \times K \to \mathscr{E}$ between functors F and G the maps $\zeta(x) : F(x) \to G(x)$ are isomorphisms in $\mathscr{E}_x$. Hence each transformation in $\mathrm{Fun}_K(K, \mathscr{E})$ is an isomorphism by Proposition I-7.9, we see that $\mathrm{Fun}_K(K, \mathscr{E})$ is a Kan complex, and since $\mathrm{Fun}_K^{\mathrm{CCart}}(K, \mathscr{E})$ is a full ∞-subcategory in $\mathrm{Fun}_K(K, \mathscr{E})$ we conclude that $\mathrm{Fun}_K^{\mathrm{CCart}}(K, \mathscr{E})$ is a Kan complex. $\square$

thm:kan_cocomplete

Theorem 14.26. *The ∞-category $\mathscr{K}an$ is cocomplete and the inclusion $\mathscr{K}an \to \mathscr{C}at_\infty$ is cocontinuous.*

Proof. Fix a diagram $p : K \to \mathscr{K}an$. Since $\mathscr{C}at_\infty$ is cocomplete by Corollary 14.22, it suffices to show that the colimit $\mathscr{C} = \mathrm{colim}(p)$ is a Kan complex.

Let $i : \mathscr{K}an \to \mathscr{C}at_\infty$ be the inclusion, and recall from Proposition I-14.15 that this functor is left adjoint to the associated Kan complex functor. The counit of this adjunction is provided by a transformation $id_{\mathscr{C}at_\infty} \to i(-)^{\mathrm{Kan}}$ which is just the inclusion $\mathscr{C}^{\mathrm{Kan}} \to \mathscr{C}$ on objects. This counit transformation realizes these functors as adjoints at the level of the enriched homotopy categories as well, by Corollary **??**, and we have an induced adjunction for the functors

$$i_* : \mathrm{Fun}(K, \mathscr{K}an) \to \mathrm{Fun}(K, \mathscr{C}at_\infty) \quad \text{and} \quad (-)^{\mathrm{Kan}}_* : \mathrm{Fun}(K, \mathscr{C}at_\infty) \to \mathrm{Fun}(K, \mathscr{K}an)$$

by Proposition 13.21. Here i_* is just the inclusion.

Take $c : i_*p \to \underline{\mathscr{C}}$ a transformation which exhibits an ∞-category $\mathscr{C}$ as a colimit for i_*p in $\mathscr{C}at_\infty$ and take $\mathscr{X} = \mathscr{C}^{\mathrm{Kan}}$. Let $\epsilon : \mathscr{X} \to \mathscr{C}$ be the inclusion. Then we have the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}at_\infty}(\mathscr{C}, \mathscr{X}) \arrow[r] \arrow[d, "\epsilon_*"] & \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C}at_\infty)}(\underline{\mathscr{C}}, \underline{\mathscr{X}}) \arrow[r, "c^*"] \arrow[d, "\underline{\epsilon}_*"] & \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{K}an)}(p, \underline{\mathscr{X}}) \arrow[d, "\underline{\epsilon}_*", "\cong"'] \\
\mathrm{Hom}_{\mathscr{C}at_\infty}(\mathscr{C}, \mathscr{C}) \arrow[r] & \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C}at_\infty)}(\underline{\mathscr{C}}, \underline{\mathscr{C}}) \arrow[r, "c^*"'] & \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C}at_\infty)}(i_*p, \underline{\mathscr{C}})
\end{tikzcd}$$

Since c exhibits $\mathscr{C}$ as a colimit to p both the top and bottom composites are isomorphisms. It follows that the map

$$\epsilon_* : \mathrm{Hom}_{\mathscr{C}at_\infty}(\mathscr{C}, \mathscr{X}) \to \mathrm{Hom}_{\mathscr{C}at_\infty}(\mathscr{C}, \mathscr{C})$$

is an isomorphism. Take $f : \mathscr{C} \to \mathscr{X}$ a homotopy lift of the identity on $\mathscr{C}$, so that we have a 2-simplex

$$\begin{tikzcd}
 & \mathscr{X} \arrow[dr, "\epsilon"] & \\
\mathscr{C} \arrow[ur, "f"] \arrow[rr, "id"'] & & \mathscr{C}.
\end{tikzcd}$$

in $\mathscr{C}at_\infty$.

By definition, this 2-simplex is a simplicial map

$$\mathrm{Path}(\Delta^2) \to \underline{\mathrm{Cat}}_\infty$$

with the appropriate restrictions, which is then a choice of an isomorphism $\epsilon f \overset{\sim}{\to} id_{\mathscr{C}}$ in $\mathrm{Fun}(\mathscr{C}, \mathscr{C})$. Via the identification

$$\epsilon_* : \mathrm{Fun}(\mathscr{X}, \mathscr{X}) \overset{\sim}{\to} \mathrm{Fun}(\mathscr{X}, \mathscr{C})^{\mathrm{Kan}}$$

from Lemma I-10.10 this natural isomorphism restricts to a natural isomorphism

$$f\epsilon \overset{\sim}{\to} id_{\mathscr{X}}.$$

So, by definition, the inclusion $\mathscr{X} \to \mathscr{C}$ is an equivalence with inverse $f : \mathscr{C} \to \mathscr{X}$. Consequently, $\mathscr{C}$ is a Kan complex and so is in fact equal to $\mathscr{X}$. □

By Theorems 14.25 and 14.26 we can calculate limits in $\mathscr{K}an$ via the semi-explicit expressions from Theorems 14.9 and 14.20.

cor:lim_calckan

Corollary 14.27. *Let $p : K \to \mathscr{K}an$ be a diagram of Kan complexes and $\mathscr{E} = \int_K p \to K$ be the associated left fibration. Let W be the collection of all morphisms in $\mathscr{E}$. We have* $\lim(p) = \mathrm{Fun}_K(K, \mathscr{E})$ *and* $\mathrm{colim}(p) = \mathscr{E}[W^{-1}]$.

We note that it is not obvious that the localization $\mathscr{E}[W^{-1}]$ is a Kan complex. However, this is simply forced by the conclusion of Theorem 14.26.

Proof. Both formula follow from the fact that all morphisms in a left fibration are cocartesian. □

14.6. **∞-pullback vs. homotopy pullback vs. discrete pullback in spaces.** We deal with the case of Kan complexes first, then discuss the ∞-setting. Let

$$\begin{tikzcd} & \mathscr{X}_1 \arrow[d, "f_1"] \\ \mathscr{X}_0 \arrow[r, "f_0"'] & \mathscr{Y} \end{tikzcd} \tag{33}$$ eq:4226

be a diagram of Kan complexes, which is equivalently a functor $\mathrm{Plain}(\Lambda^2_2) \to \underline{\mathrm{Kan}}$. We can then calculate the limit of the corresponding diagram $p : \Lambda^2_2 \to \mathscr{K}an$ via the weighted nerve $\mathscr{E} = \mathrm{N}^p(\Lambda^2_2)$. We have the identifications of the fibers

$$\mathscr{E}_0 \cong \mathscr{X}_0, \quad \mathscr{E}_1 \cong \mathscr{X}_1, \quad \mathscr{E}_2 \cong \mathscr{Y}.$$

A section of the weighted nerve $\Lambda^2_2 \to \mathscr{E}$ is the information of a choice of objects x_0, x_1, and z in $\mathscr{X}_0$, $\mathscr{X}_1$ and $\mathscr{Y}$ respectively, and morphisms $\alpha_0 : x_0 \to z$ and $\alpha_1 : x_1 \to z$ in the weighted nerve. These morphisms are explicitly isomorphisms $\alpha_i : f_i(x_i) \to z$ in $\mathscr{Y}$, and since all morphisms in $\mathscr{Y}$ are isomorphisms we needn't concern ourselves with this restriction. So we find that objects in the functor category $\mathrm{Fun}_{\Lambda^2_2}(\Lambda^2_2, \mathscr{E})$ are identified with objects in the fiber product

$$\mathrm{Fun}(\Lambda^2_2, \mathscr{Y}) \times_{\mathrm{Fun}(\{0,1\}, \mathscr{Y})} (\mathscr{X}_0 \times \mathscr{X}_1).$$

Here we have $\{0,1\} = \partial\Lambda^2_2$ and the map $\mathrm{Fun}(\Lambda^2_2, \mathscr{Y}) \to \mathrm{Fun}(\{0,1\}, \mathscr{Y})$ is restriction to the boundary. This map is a Kan fibration by Corollary I-3.12. The map

$$(\mathscr{X}_0 \times \mathscr{X}_1) \to \mathrm{Fun}(\{0,1\}, \mathscr{Y}) \cong \mathscr{Y} \times \mathscr{Y}$$

is the product $f_0 \times f_1$.

Now, an n-simplex in $\mathrm{Fun}_{\Lambda^2_2}(\Lambda^2_2, \mathscr{E})$ is the data of a pair of n-simplices $\sigma_i : \Delta^n \to \mathscr{X}_i$, an n-simplex $\tau : \Delta^n \to \mathscr{Y}$, and transformations $\xi_i : f_i(\sigma_i) \to \tau$. This data specifies an n-simplex in the aforementioned fiber product, so that we obtain an identification

$$\lim(p) = \mathrm{Fun}_{\Lambda^2_2}(\Lambda^2_2, \mathscr{E}) = \mathrm{Fun}(\Lambda^2_2, \mathscr{Y}) \times_{\mathrm{Fun}(\{0,1\}, \mathscr{Y})} (\mathscr{X}_0 \times \mathscr{X}_1) \tag{34}$$ eq:4273

Though this description of the limit is explicit, it is somewhat inefficient. We can, in particular, replace the functor space $\mathrm{Fun}(\Lambda^2_2, \mathscr{Y})$ with the space $\mathrm{Fun}(\Delta^1, \mathscr{Y})$ via "composition". In particular, we have the anodyne maps

$$\begin{tikzcd} \Delta^1 \cong \Delta^{\{0,2\}} \arrow[dr] & & \Lambda^2_2 \arrow[dl] \\ & \Delta^2 & \end{tikzcd}$$

which induce trivial Kan fibrations

$$\begin{tikzcd}
 & \mathrm{Fun}(\Delta^2,\mathscr{Y}) \arrow[dl] \arrow[dr] & \\
\mathrm{Fun}(\Delta^1,\mathscr{Y}) & & \mathrm{Fun}(\Lambda^2_2,\mathscr{Y}),
\end{tikzcd}$$

by Corollary I-3.12. These trivial Kan fibrations fit into a diagram of Kan fibrations over $\mathrm{Fun}(\{0,1\},\mathscr{Y})$ so that we obtain, via Proposition I-6.3, an induced equivalence

$$\mathrm{Fun}(\Delta^1,\mathscr{Y})\times_{\mathrm{Fun}(\{0,1\},\mathscr{Y})}(\mathscr{X}_0\times\mathscr{X}_1)\overset{\sim}{\to}\mathrm{Fun}(\Lambda^2_2,\mathscr{Y})\times_{\mathrm{Fun}(\{0,1\},\mathscr{Y})}(\mathscr{X}_0\times\mathscr{X}_1).$$

We recall finally that the left-hand fiber product is the homotopy pullback from Section I-6.3,

$$\mathscr{X}_0\times^{\mathrm{htop}}_{\mathscr{Y}}\mathscr{X}_1=\mathrm{Fun}(\Delta^1,\mathscr{Y})\times_{\mathrm{Fun}(\{0,1\},\mathscr{Y})}(\mathscr{X}_0\times\mathscr{X}_1).$$

Proposition 14.28. *Given a diagram $p:\Lambda^2_2\to\mathscr{K}an$, which explicitly appears as (33), the homotopy pullback $\mathscr{X}_1\times^{\mathrm{htop}}_{\mathscr{Y}}\mathscr{X}_1$ is a limit of p.*

We recall that the homotopy pullback is isomorphic to the standard pullback in the event that one of the maps f_i is a Kan fibration by Proposition I-6.5.

`cor:4379` **Corollary 14.29.** *Consider a diagram $p:\Lambda^2_2\to\mathscr{K}an$, which explicitly appears as (33). If one of the maps f_i is a Kan fibration, the the usual fiber product $\mathscr{X}_1\times_{\mathscr{Y}}\mathscr{X}_2$ is a limit of p in $\mathscr{K}an$.*

We also have the comparison morphism

$$\mathscr{X}_0\times_{\mathscr{Y}}\mathscr{X}_1\cong\mathscr{Y}\times_{(\mathscr{Y}\times\mathscr{Y})}(\mathscr{X}_0\times\mathscr{X}_1)\to\mathrm{Fun}(\Lambda^2_2,\mathscr{Y})\times_{\mathrm{Fun}(\{0,1\},\mathscr{Y})}(\mathscr{X}_0\times\mathscr{X}_1)$$

given by restricting along the terminal map $\Lambda^2_2\to *$, and a similar restriction map for $\mathrm{Fun}(\Delta^2,\mathscr{Y})\times_{\mathrm{Fun}(\{0,1\},\mathscr{Y})}(\mathscr{X}_0\times\mathscr{X}_1)$. These maps fit into a diagram

$$\begin{tikzcd}
 & \mathrm{Fun}(\Delta^2,\mathscr{Y})\times_{\mathrm{Fun}(\{0,1\},\mathscr{Y})}(\mathscr{X}_0\times\mathscr{X}_1) \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\
\mathscr{X}_0\times^{\mathrm{htop}}_{\mathscr{Y}}\mathscr{X}_1 & & \mathrm{Fun}(\Lambda^2_2,\mathscr{Y})\times_{\mathrm{Fun}(\{0,1\},\mathscr{Y})}(\mathscr{X}_0\times\mathscr{X}_1) \\
 & \mathscr{X}_0\times_{\mathscr{Y}}\mathscr{X}_1 \arrow[ul] \arrow[uu] \arrow[ur] &
\end{tikzcd}.$$

from which we conclude the following.

Corollary 14.30. *If one of the maps f_i in a diagram (33) is a Kan fibration, then the comparison functor*

$$\mathscr{X}_0\times_{\mathscr{Y}}\mathscr{X}_1\to\mathrm{Fun}(\Lambda^2_2,\mathscr{Y})\times_{\mathrm{Fun}(\{0,1\},\mathscr{Y})}(\mathscr{X}_0\times\mathscr{X}_1)$$

is an equivalence.

One can check directly that, under the identification

$$\mathrm{Fun}_{\Lambda^2_2}(\Lambda^2_2,\mathscr{E})=\mathrm{Fun}(\Lambda^2_2,\mathscr{Y})\times_{\mathrm{Fun}(\{0,1\},\mathscr{Y})}(\mathscr{X}_0\times\mathscr{X}_1)$$

from (34), the comparison map employed in Corollary **??** is identified with the comparison map λ from Definition 14.11. So Corollary 14.15 provides a strong refinement of Corollary 14.29.

cor:kan_pullback

Corollary 14.31. *Consider a diagram $p : \Lambda^2_2 \to \mathscr{K}an$, which explicitly appears as (33). If one of the maps f_i is a Kan fibration then the (strictly commuting) pullback diagram*

$$\begin{tikzcd} \mathscr{X}_0 \times_{\mathscr{Y}} \mathscr{X}_1 \arrow[r] \arrow[d] \arrow[dr, dotted] & \mathscr{X}_1 \arrow[d, "f_1"] \\ \mathscr{X}_0 \arrow[r, "f_1"'] & \mathscr{Y} \end{tikzcd}$$

is a limit diagram in $\mathscr{K}an$.

One can use this explicit result to determine the precise ∞-categorical pullback diagram for $\mathrm{Fun}(\Lambda^2_2, \mathscr{Y}) \times_{\mathrm{Fun}(\{0,1\},\mathscr{Y})} (\mathscr{X}_0 \times \mathscr{X}_1)$.

Exercise 14.32. Consider arbitrary maps of Kan complexes $f_i : \mathscr{X}_i \to \mathscr{Y}$, $i = 0, 1$, and corresponding functor $p : \Lambda^2_2 \to \mathscr{K}an$. Prove that the "obvious diagram" in $\mathscr{K}an$,

$$\begin{tikzcd} \mathrm{Fun}(\Lambda^2_2, \mathscr{Y}) \times_{\mathrm{Fun}(\{0,1\},\mathscr{Y})} (\mathscr{X}_0 \times \mathscr{X}_1) \arrow[r] \arrow[d] \arrow[dr] & \mathscr{X}_1 \arrow[d] \\ \mathscr{X}_0 \arrow[r] & \mathscr{Y} \end{tikzcd}$$

is a limit diagram. Furthermore, determine what the "obvious diagram" is.

14.7. **Replacing limits with discrete limits.** We have just seen how one can compute an ∞-categorical pullback in $\mathscr{K}an$ by first replacing a diagram with an isomorphic one in which some maps are Kan fibrations. Specifically, for any diagram

$$\begin{tikzcd} & \mathscr{X}_1 \arrow[d] \\ \mathscr{X}_0 \arrow[r] & \mathscr{Y} \end{tikzcd}$$

we can factor the map $\mathscr{X}_0 \to \mathscr{Y}$ into a composite of an equivalence $\mathscr{X}_0 \to \mathscr{X}_0'$ and a Kan fibration $\mathscr{X}_0' \to \mathscr{Y}$ (Proposition I-6.2). The above diagram is then isomorphic to the diagram

$$\begin{tikzcd} & \mathscr{X}_1 \arrow[d] \\ \mathscr{X}_0' \arrow[r] & \mathscr{Y} \end{tikzcd}$$

in $\mathrm{Fun}(\Lambda^2_2, \mathscr{K}an)$, so that they have the same limits by Proposition 13.17. Corollary 14.31 now tells us that the limit for the latter diagram is computable via the discrete limit. This is not an isolated phenomena. 7.5.3.12

Definition 14.33. Let Λ be a plain category and let $p : K \to \mathrm{Cat}_\infty$ be a strictly commuting diagram. We say p is isofibrant if for each functor $F : K \to \mathrm{Cat}_\infty$ which is paired with a subfunctor $F' \subseteq F$ in which the inclusion $F'(x) \to F(x)$ is a equivalence at each x in K, and each natural transformation $\xi' : F' \to p$, the transformation ξ' admits an extension to a transformation $\xi : F \to p$.

We say a strictly commuting diagram $p : K \to \mathscr{C}at_\infty$ is isofibrant if it, in the implicit factorization $K \to \mathrm{Cat}_\infty \to \mathscr{C}at_\infty$, the corresponding diagram $K \to \mathrm{Cat}_\infty$ is isofibrant.

We have an alternate characterization of isofibrant diagram via functor categories. In the statement below, for functors $F_0, F_1 : \Lambda \to \mathrm{sSet}$ we employ the simplicial structure on the collection of natural transformations $\mathrm{Nat}(F_1, F_1)$ established in Section **??** above.

prop:isofibrant

Proposition 14.34 ([5, 034H]). *A diagram $p : K \to \mathrm{Cat}_\infty$ is isofibrant if and only if, for each functor $F : K \to \mathrm{Cat}_\infty$ and subfunctor $F' \subseteq F$ which is a pointwise categorical equivalence, the restriction map*

$$\mathrm{Nat}(F, p) \to \mathrm{Nat}(F', p)$$

is a trivial Kan fibration. Furthermore, in this case, for any choice of functor $G : K \to \mathrm{Cat}_i nfty$ and subfunctor $G' \subseteq G$ the restriction map

$$\mathrm{Nat}(G, p) \to \mathrm{Nat}(G', p)$$

is an isofibration.

cor:isofib_in_cat

Corollary 14.35. *For any isofibrant diagram $p : K \to \mathrm{Cat}_\infty$ the discrete limit $\lim^0(p)$ in sSet is an ∞-category.*

Proof. For the first claim take $G = *$ and $G' = \emptyset$ in Proposition 14.34. □

Lurie also establishes the following.

prop:isofib_in_kan

Proposition 14.36 ([5, 034S]). *Let $p : K \to \mathrm{Kan}$ be an isofibrant diagram. Then the discrete limit $\lim^0(p)$ in sSet is a Kan complex.*

Though it is not our primary point of view, one can also look at these issues from the perspective of the existence or non-existence of limits in the discrete categories Cat_∞ and Kan.

Corollary 14.37. *The discrete categories Cat_∞ and Kan admit all limits which are indexed by isofibrant diagrams.*

In considering diagrams indexed by partially ordered sets, we have the following example.

prop:po_isofib

Proposition 14.38 ([5, 034D]). *Let $K = (K_0, \leq)$ be a partially ordered set in which every non-empty subset $K' \subseteq K$ admits a (not necessarily unique) maximal element. Then a strictly commuting diagram $p : K \to \mathscr{C}at_\infty$ is isofibrant if and only if, at each x in K, the map to the discrete limit*

$$p(x) \to \lim^0(K_{>x} \to \mathrm{sSet})$$

is an isofibration of simplical sets.

ex:isofib_seq

Example 14.39. Any tower of isofibrations $\mathscr{C}_{i+1} \to \mathscr{C}_i$ which is indexed by the non-negative integers,

$$\cdots \to \mathscr{C}_2 \to \mathscr{C}_1 \to \mathscr{C}_0,$$

defines an isofibrant diagram $p : \mathbb{Z}_{\geq 0}^{op} \to \mathrm{Cat}_\infty$. Similarly, any tower of Kan fibrations $\cdots \to \mathscr{X}_2 \to \mathscr{X}_1 \to \mathscr{X}_0$ defines an isofibrant diagram of Kan complexes.

ex:4510

Example 14.40. Consider a diagram of ∞-categories

$$\begin{tikzcd} & \mathscr{C}_1 \arrow[d, "F_1"] \\ \mathscr{C}_0 \arrow[r, "F_0"'] & \mathscr{T} \end{tikzcd}$$

in which each of the F_i are isofibrations. The corresponding diagram in $\mathscr{C}at_\infty$ is isofibrant.

It is in fact the case that any strictly commuting diagram in $\mathscr{C}at_\infty$ admits an isofibrant replacement.

Theorem 14.41 ([5, 03AT]). *Let $p : K \to \mathscr{C}at_\infty$ be a strictly commuting diagram of ∞-categories. Then there is an isofibrant diagram $p' : K \to \mathscr{C}at_\infty$ which admits a natural isomorphism $p \xrightarrow{\sim} p'$.*

It's furthermore shown in [5, 03B1] that, when $p : K \to \mathscr{C}at_\infty$ is isofibrant, the comparison map from the discrete limit

$$\lambda : \lim^0(p) \to \mathrm{Fun}_K^{\mathrm{CCart}}(K, \int_K p)$$

of Definition 14.11 is an equivalence of ∞-categories. So Corollary 14.15 implies that the discrete limit is a limit of p in $\mathscr{C}at_\infty$, and that the discrete limit diagram provides a limit diagram in $\mathscr{C}at_\infty$.

thm:isofib_lim

Theorem 14.42 ([5, 03B1]). *If $p : K \to \mathscr{C}at_\infty$ be an isofibrant diagram then the discrete limit $\lim^0(p)$ is a limit for p, and the corresponding strictly commuting diagram $p^0 : K^{<} \to \mathscr{C}at_\infty$ with cone point $\lim^0(p)$ is a limit diagram in $\mathscr{C}at_\infty$.*

As an corollary we obtain an weak analog to Corollary 14.31 in the ∞-categorical context.

Corollary 14.43. *Consider a diagram of ∞-categories $\Lambda^2_2 \to \mathscr{C}at_\infty$ as in Example 14.40, in which each of $F_i : \mathscr{C}_i \to \mathscr{T}$ is an isofibration. Then the (strictly commuting) pullback diagram*

$$\begin{tikzcd}
\mathscr{C}_0 \times_{\mathscr{T}} \mathscr{C}_1 \arrow[r] \arrow[d] \arrow[dr, dotted] & \mathscr{C}_1 \arrow[d] \\
\mathscr{C}_0 \arrow[r] & \mathscr{T}
\end{tikzcd}$$

is a limit diagram in $\mathscr{C}at_\infty$.

As in the case of Kan complexes, one can argue directly that it suffices to have only one of the F_i an isofibration in order to obtain the above computation.

Example 14.44. Let $\mathscr{C}_0 \to \mathscr{C}_1 \to \mathscr{C}_2 \to \cdots$ be a sequence of injective maps of ∞-categories, $p : \mathbb{Z}_{\geq 0} \to \mathscr{C}at_\infty$ the corresponding strictly commuting diagram, and consider the discrete colimit

$$\mathscr{C} = \mathrm{colim}^0(p).$$

We see that $\mathscr{C}$ is an ∞-category by directly checking the lifting criterion.

For any other ∞-category $\mathscr{K}$, the functor $\mathrm{Fun}(-, \mathscr{K}) : \mathrm{Cat}_\infty \to \mathrm{Cat}_\infty$ produces a sequence of isofibrations

$$\cdots \to \mathrm{Fun}(\mathscr{C}_2, \mathscr{K}) \to \mathrm{Fun}(\mathscr{C}_1, \mathscr{K}) \to \mathrm{Fun}(\mathscr{C}_0, \mathscr{K})$$

by Proposition I-6.13, and hence has a limit in Cat_∞. The functor $\mathrm{Fun}(-, \mathscr{K})$ seen to commute with discrete colimits via the sequence of natural isomorphisms

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathscr{Z}, \mathrm{Fun}(\mathrm{colim}^0 -, \mathscr{K})) \cong \mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathrm{colim}^0 -, \mathrm{Fun}(\mathscr{Z}, \mathscr{K}) \\
&\cong \lim^0 \mathrm{Hom}_{\mathrm{Cat}_\infty}(-, \mathrm{Fun}(\mathscr{Z}, \mathscr{K})) \cong \lim^0 \mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathscr{Z}, \mathrm{Fun}(-, \mathscr{K})).
\end{aligned}$$

Hence $\mathrm{Fun}(\mathscr{C},\mathscr{K}) = \lim_i^0 \mathrm{Fun}(\mathscr{C}_i,\mathscr{K})$. By Theorem 14.42 we find that $\mathrm{Fun}(\mathscr{C},\mathscr{K})$ is a limit of the corresponding diagram $\mathrm{Fun}(p-,\mathscr{K}) : \mathbb{Z}_{\geq 0}^{\mathrm{op}} \to \mathscr{C}at_\infty$ at each $\mathscr{K}$ in $\mathscr{C}at_\infty$. We will see later that this implies $\mathscr{C} = \mathrm{colim}(p)$.

14.8. **Geometric realization of spaces.** We cover one particular instance of a colimit.

Proposition 14.45. *Let $\mathscr{X}$ be a Kan complex, and consider the corresponding simplicial object*

$$\mathscr{X}_\bullet : \Delta^{\mathrm{op}} \to \mathrm{Set} \subseteq \mathscr{K}an, \quad \mathscr{X}_\bullet([n]) := \mathscr{X}[n].$$

Then there is a transformation $\mathscr{X}_\bullet \to \underline{\mathscr{X}}$ which exhibits $\mathscr{X}$ as a colimit of the diagram $\mathscr{X}_\bullet$.

15. Kan extension

We give a bare bones presentation of Kan extensions, which we only provide as a foundation for our subsequent analysis of the Yoneda embedding. A significant amount of our textual landscape here is dedicated to a discussion of size constrains on ∞-categories and fibrations. The point being, a Kan extension might be thought of as a varying family of (co)limits. The existence of the requisite (co)limits, in advantageous settings, should depends only on vary course size restrictions imposed on the emergent diagrams in such a setting.

15.1. **Kan extensions.** Before beginning, let us establish some notations. Consider a map of simplicial sets $i : K \to \mathscr{C}$, where $\mathscr{C}$ is an ∞-category, and let $x, y : * \to \mathscr{C}$ be any objects in $\mathscr{C}$. We take

$$K_{/x} := K \times_{\mathscr{C}} \mathscr{C}_{/x} \ \text{ and } \ K_{y/} := K \times_{\mathscr{C}} \mathscr{C}_{y/},$$

These simplicial sets are, respectively, left and right fibrations over K. We have the slice diagonal morphisms $K_{/x} \to \mathscr{C} \times^{\mathrm{or}}_{\mathscr{C}} \{x\}$ and $K_{y/} \to \{y\} \times^{\mathrm{or}}_{\mathscr{C}} \mathscr{C}$ which define natural transformations

$$\gamma : i|_{K_{/x}} \to \underline{x} \ \text{ and } \ \gamma' : \underline{y} \to i|_{K_{y/}} \tag{35}$$ eq:kan_ext_transf

respectively.

Definition 15.1. Consider a simplicial set K and a pair of maps to ∞-categories $i : K \to \mathscr{C}$ and $\bar{F} : K \to \mathscr{D}$. A left Kan extension of $\bar{F} : K \to \mathscr{D}$ along i is the data of a functor $F' : \mathscr{C} \to \mathscr{D}$ and a transformation $\zeta : \bar{F} \to Fi$ for which, at each object x in $\mathscr{C}$, the transformation

$$\bar{F}|_{K_{/x}} \xrightarrow{\zeta} Fi|_{K_{/x}} \xrightarrow{F(\gamma)} \underline{F(x)}$$

exhibits $F(x)$ as a colimit of the diagram $\bar{F}|_{K_{/x}} : K_{/x} \to \mathscr{D}$.

We have the obvious dual notion.

Definition 15.2. Consider a simplicial set K and a pair of maps to ∞-categories $i : K \to \mathscr{C}$ and $\bar{F} : K \to \mathscr{D}$. A right Kan extension of $\bar{F} : K \to \mathscr{D}$ along i is the data of a functor $F' : \mathscr{C} \to \mathscr{D}$ and a transformation $\zeta' : F'i \to \bar{F}$ for which, at each object y in $\mathscr{C}$, the transformation

$$\underline{F'(y)} \xrightarrow{F(\gamma')} F'i|_{K_{y/}} \xrightarrow{\zeta'} \bar{F}|_{K_{y/}}$$

exhibits $F(x)$ as a limit of the diagram $\bar{F}|_{K_{y/}} : K_{y/} \to \mathscr{D}$.

To orient ourself, we can consider the case of a point $\mathscr{C} = *$. Then the functor $K \to *$ carries no information, we have $K_{/x} = K_{y/} = K$, and a left Kan extension for a functor $\bar{F} : K \to \mathscr{D}$ is simply the choice of a colimit $\mathrm{colim}(\bar{F}) : * \to \mathscr{D}$. Similarly, a right Kan extension is a choice of a limit. We recall that colimits and limits are unique up to a contractible space of choices, so that these choices of left and right extensions are uniquely determined.

In the case of a single edge $\mathscr{C} = \Delta^1$, the map $i : K \to \Delta^1$ splits K into subcomplexes $K_i = K \times_{\Delta^1} \{i\}$ in which all edges in K either occur in one of the K_i or move from K_0 to K_1. The important subcomplexes here, however, are K_0 and $K_{\leq 1} = K$ itself, as we have

$$K_0 = K_{/0} \quad \text{and} \quad K \cong K_{/1}$$

since 1 is terminal in Δ^1.

Consider a map $\bar{F} : K \to \mathscr{D}$ and let $\bar{F}_0$ denote the restriction $\bar{F}_0 = \bar{F}|_{K_0}$. A left Kan extension for $\bar{F}$ along i is then a choice of colimits $\mathrm{colim}(\bar{F}_0)$ and $\mathrm{colim}(F)$ and a choice of a particular morphism

$$f : \mathrm{colim}(\bar{F}_0) \to \mathrm{colim}(\bar{F}).$$

This morphism is the left Kan extension itself $f : \Delta^1 \to \mathscr{D}$, and the map f is the unique morphism obtained via the universal property of the colimit for $\bar{F}_0$. Namely, the implicit transformation $\bar{F}|_{K_{1/}} = \bar{F} \to \underline{\mathrm{colim}(\bar{F})}$ restricts to a tranformation $\bar{F}|_{K_0} \to \underline{\mathrm{colim}(\bar{F})}$ and this transformation determines such a map f. This situation can be depicted coarsely in a diagram

$$\begin{tikzcd}
\bar{F} \arrow[r] & \underline{\mathrm{colim}(\bar{F})} \\
\cup & \\
\bar{F}|_{K_0} \arrow[r] \arrow[uur] & \underline{\mathrm{colim}(\bar{F}_0)}. \arrow[uu, "f"']
\end{tikzcd}$$

Similarly, a left Kan extension of $\bar{F}$ is a choice of limits and corresponding morphism $f' : \lim(\bar{F}) \to \lim(\bar{F}_1)$ which we can place in a coarsely depicted diagram

$$\begin{tikzcd}
\underline{\lim(\bar{F})} \arrow[r] & \bar{F}|_{K_1} \\
 & \cup \\
\underline{\lim(\bar{F}_0)} \arrow[r] \arrow[uu, "f'"] \arrow[uur] & \bar{F}|_{K_0}.
\end{tikzcd}$$

A similar analysis holds when $\mathscr{C} = \Delta^1$ is replaced with any simplex Δ^n. In the case that the (∞-)category $\mathscr{C}$ admits more complicated collections of morphisms, the indexing category at each point, $K_{x/}$ or $K_{/y}$, expands to account for variances in morphisms and endmorphisms at the given vertex. However we can still think of the left Kan extension, say, as codifying relationships between a family of colimits in $\mathscr{D}$ which are parametrized, more-or-less, by the "bundle of fibrations" $K_{/x}$ over $\mathscr{C}$.

15.2. **Size constraints on ∞-categories and simplicial sets.** Recall, from Definition 13.25 above, that a simplicial set K is called small if, at each nonnegative integer n, the collection of n-simplices $K[n]$ is a small set.

Definition 15.3. An ∞-category $\mathscr{C}$ is called essentially small if $\mathscr{C}$ is equivalent to a small ∞-category, and $\mathscr{C}$ is called locally small if for each pair of objects $x, y : * \to \mathscr{C}$ the Kan complex $\mathrm{Hom}_{\mathscr{C}}(x, y)$ is essentially small.

Locally small ∞-categories are the "normal" ∞-categories which one expects to arise from everyday studies in algebra and representation theory. We record some apparent examples.

Example 15.4. By a dg category we mean a category enriched in dg vector spaces, and all dg vector spaces are small by definition. Hence, for any dg category $\mathbf{A}$, the dg nerve $\mathrm{N}^{\mathrm{dg}}(\mathbf{A})$ is a locally small ∞-category. This follows by the calculations of the pinched mapping spaces from Proposition I-**??**.

For specific examples, we find that the homotopy ∞-category $\mathscr{K}(\mathbb{A})$ of any additive category $\mathbb{A}$ is locally small, and the derived ∞-category $\mathscr{D}(\mathbb{A})$ of any Grothendieck abelian category is locally small.

`ex:infty_small` **Example 15.5.** Let $\underline{A}$ be a simplicial category which is enriched in small Kan complexes. Then, via the equivalence of Theorem 5.27, the homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}(\underline{A})$ is locally small.

For a specific example, let $\mathscr{C}$ and $\mathscr{D}$ be essentially small ∞-categories. Then we have an equivalence of Kan complexes between $\mathrm{Fun}(\mathscr{C}, \mathscr{D})^{\mathrm{Kan}}$ and $\mathrm{Fun}(\mathscr{C}', \mathscr{D}')^{\mathrm{Kan}}$ for some small ∞-categories $\mathscr{C}'$ and $\mathscr{D}'$. One sees directly that at each non-negative integer n the set

$$\mathrm{Hom}_{\mathrm{sSet}}(\Delta^n \times \mathscr{C}', \mathscr{D}') = \mathrm{Fun}(\mathscr{C}', \mathscr{D}')[n]$$

is small. Hence the functor category $\mathrm{Fun}(\mathscr{C}', \mathscr{D}')$ is small, as is its associated Kan complex, and it follows that the Kan complex $\mathrm{Fun}(\mathscr{C}, \mathscr{D})^{\mathrm{Kan}}$ is essentially small. So we see that the ∞-category $\mathscr{C}at_{\infty}^{\mathrm{sm}}$ of essentially small ∞-categories is locally small. Similarly, we find that the ∞-category $\mathscr{K}an^{\mathrm{sm}}$ of essentially small Kan complexes is locally small as well.

One can establish the expected relation between local smallness and essential smallness.

`prop:locsmall_essmall` **Proposition 15.6** ([5, 03TW])**.** *For an ∞-category $\mathscr{C}$, the following are equivalent:*

(a) *$\mathscr{C}$ is essentially small.*
(b) *$\mathscr{C}$ is locally small and the set $\pi_0(\mathscr{C}^{\mathrm{Kan}})$ is finite.*

One can employ this characterization to provide sufficient conditions for local smallness of functor categories.

`lem:locsmall_fun` **Lemma 15.7.** *If $\mathscr{C}$ is an essentially small ∞-category and $\mathscr{D}$ is a locally small ∞-category, then the functor category* $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$ *is locally small.*

Proof. Since the endofunctor $\mathrm{Fun}(-, \mathscr{D})$ preserves equivalences we can assume $\mathscr{C}$ is small. Then for any two functors $F, F' : \mathscr{C} \to \mathscr{D}$ there exists a small (full) ∞-subcategory $\mathscr{D}' \subseteq \mathscr{D}$ through which both F and F' factor. We can take in particular $\mathscr{D}'$ the full ∞-subcategory spanned by the small set of objects $F(\mathscr{C}[0]) \cup F'(\mathscr{C}[0])$, and observe essential smallness of $\mathscr{D}'$ as an application of Proposition 15.6. We now find that any two functors exist in the full ∞-subcategory $\mathrm{Fun}(\mathscr{C}, \mathscr{D}')$. It was argued in Example 15.5 that the ∞-category $\mathrm{Fun}(\mathscr{C}, \mathscr{D}')$ is essentially small, and in particular locally small. Since the embedding $\mathrm{Fun}(\mathscr{C}, \mathscr{D}') \to \mathrm{Fun}(\mathscr{C}, \mathscr{D})$ preserves mapping spaces, via full faithfulness, we conclude that the functor category $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$ is essentially small. □

It will be helpful to have a notion of essential smallness for simplicial sets. Recall that a map of simplicial sets $K \to L$ is called a categorical equivalence if, for each ∞-category $\mathscr{C}$, the induced map of ∞-categories $\mathrm{Fun}(L, \mathscr{C}) \to \mathrm{Fun}(K, \mathscr{C})$ is an equivalence.

Definition 15.8. An simplicial set K is called essentially small if there K admits a categorical equivalence $K \to \mathscr{K}$ to a small ∞-category.

It is in fact the case that any small simplicial set is essentially small.

prop:small_essmall

Proposition 15.9 ([5, 03SN]). *Any small simplicial set K admits an inner anodyne morphism $K \to \mathscr{K}$ to a small ∞-category $\mathscr{K}$. In particular, any small simplicial set is essentially small.*

We omit the proof. As in the case of ∞-categories, one can show that exponentials of locally small ∞-categories by essentially small simplicial sets are again locally small.

Lemma 15.10. *If K is an essentially small simplicial set, and $\mathscr{C}$ be a locally small ∞-category, then the functor category $\mathrm{Fun}(K, \mathscr{C})$ is locally small.*

Proof. Similar to the proof of Lemma 15.7. □

One also sees that completeness and cocompleteness for ∞-categories can be rephrased in terms of essentially small diagrams.

prop:ess_completeness

Proposition 15.11. *For any ∞-category $\mathscr{C}$, the following are equivalent:*

(a) *$\mathscr{C}$ admits all limits (resp. colimits) indexed by small simplicial sets.*
(b) *$\mathscr{C}$ admits all limits (resp. colimits) indexed by essentially small simplicial set.*

Proof. The implication (b) ⇒ (a) follows from essential smallness of any small simplicial set, by Proposition 15.9. Suppose now that (a) holds, and consider a diagram $p : K \to \mathscr{C}$ from an essentially small simplicial set. By definition we can find a categorical equivalence $i : K \to \mathscr{K}$ to a small ∞-category, and can therefore find a functor $P : \mathscr{K} \to \mathscr{C}$ whose restriction $P|_K$ is isomorphic to p. By Proposition 13.17, p admits a limit (resp. colimit) in $\mathscr{C}$ if and only if Pi admits a limit (resp. colimit) in $\mathscr{C}$. Furthermore, by Proposition 13.14 P admits a limit (resp. colimit) in $\mathscr{C}$ if and only if Pi admits a limit (resp. colimit) in $\mathscr{C}$. Since P admits a limit (resp. colimit) in $\mathscr{C}$ by assumption, it follows that p admits a limit (resp. colimit) in $\mathscr{C}$ as well. Hence $\mathscr{C}$ admits all limits (resp. colimits) indixed by essentially small simplicial sets. □

We also have a notion of smallness for inner fibrations.

Definition 15.12. An inner fibration of simplicial sets $X \to S$ is called essentially small if, for each n-simplex $\Delta^n \to S$, the fiber product $\Delta^n \times_S X$ is an essentially small ∞-category. Similarly, $X \to S$ is called locally small if each fiber $\Delta^n \times_S X$ is a locally small ∞-category.

We have the following fiberwise, and global characterizations of essentially small and locally small fibrations.

prop:essmall_fib

Proposition 15.13 ([5, 047P, 047Q]). *For an cartensian or cocartesian fibration $q : X \to S$, the following are equivalent:*

(a) *q is essentially small.*
(b) *For each vertex $s : * \to S$, the fiber X_s is essentially small.*

Furthermore, if the base S is an essentially small ∞-category, then these conditions are also equivalent to the following:

(c) *The ∞-category X is essentially small.*

The analogous results hold when one replaces essentially small with locally small in all of the above statements as well.

One can see [5] for a proof of Proposition 15.13.

Example 15.14. An ∞-category $\mathscr{C}$ is locally small if and only if the left fibration $\mathscr{C}_{x/} \to \mathscr{C}$ is essentially small at all x in $\mathscr{C}$, or equivalently if and only if the right fibration $\mathscr{C}_{/x} \to \mathscr{C}$ is essentially small at all x in $\mathscr{C}$.

Example 15.15. A cocartesian fibration $X \to S$ is essentially small if and only if the classifying functor $S \to \mathscr{C}at_\infty$ has image in the locally small subcategory of essentially small ∞-categories $\mathscr{C}at_\infty^{\mathrm{sm}}$. Conversely, for any functor $F : S \to \mathscr{C}at_\infty^{\mathrm{sm}} \subseteq \mathscr{C}at_\infty$ the corresponding cocartesian fibration $\int_S F \to S$ is essentially small. A similar analysis holds for left fibrations and functors in $\mathscr{K}an^{\mathrm{sm}} \subseteq \mathscr{K}an$.

Let us record a supporting lemma which will aid in our analysis of smallness for fibrations.

lem:025H

Lemma 15.16 ([5, 025H]). *Let $\mathscr{C} \to \mathscr{D}$ be a cartesian or cocartesian fibration, and consider a diagram of maps of simplicial sets*

$$\begin{tikzcd} E \arrow[r, "F"] \arrow[d] & E' \arrow[r] \arrow[d] & \mathscr{E} \arrow[d] \\ K \arrow[r, "f"] & K' \arrow[r] & \mathscr{C} \end{tikzcd}$$

in which each of the squares is a pullback square. If the map f is a categorical equivalence then the map F is a categorical equivalence as well.

Now, it is obvious, simply from the definition, that essential smallness of inner fibrations is stable under pullback. We prove a stronger version of such stability for cocartesian fibrations.

lem:4756

Lemma 15.17. *Let $\mathscr{E} \to \mathscr{C}$ be an essentially small cartesian or cocartesian fibration over a locally small ∞-category, and $F : K \to \mathscr{C}$ be a map of simplicial sets. Suppose that K is an essentially small simplicial set. Then the simplicial set $K \times_{\mathscr{C}} \mathscr{E}$ is essentially small.*

Sketch proof. Take a categorical equivalence $i : K \to \mathscr{K}$ to a small ∞-category and let $F' : \mathscr{K} \to \mathscr{C}$ be a functor with $F'i \cong F$. It follows, by Proposition **??**, that the two pullbacks for $F'i$ and F are categorically equivalent. So we can assume $F'i = F$. Take now $\mathscr{C}'$ the full ∞-subcategory whose vertices are those in the image of $F'[0] : \mathscr{K}[0] \to \mathscr{C}[0]$. Since $\mathscr{K}$ is small the set $\pi_0(\mathscr{C}')$ is small, and hence $\mathscr{C}'$ is essentially small by Proposition 15.6. By Proposition 15.13 there is a small

∞-category $\mathscr{E}'$ with an equivalence of fibrations $\mathscr{E} \times_{\mathscr{C}} \mathscr{C}' \xrightarrow{\sim} \mathscr{E}'$ over $\mathscr{C}'$. We then have an equivalence of fibrations

$$\mathscr{K} \times_{\mathscr{C}} \mathscr{E} = \mathscr{K} \times_{\mathscr{C}'} (\mathscr{E} \times_{\mathscr{C}} \mathscr{C}') \xrightarrow{\sim} \mathscr{K} \times_{\mathscr{C}'} \mathscr{E}'. \tag{36}$$ eq:4712

Note that the simplicial set $\mathscr{K} \times_{\mathscr{C}'} \mathscr{E}'$ is a subset of the small simplicial set $\mathscr{K} \times \mathscr{E}'$, and hence is small. By Proposition **??** he equivalence (36) now pulls back to an equivalence of fibrations

$$K \times_{\mathscr{C}} \mathscr{E} \xrightarrow{\sim} K \times_{\mathscr{C}'} \mathscr{E}',$$

and the induced map $K \times_{\mathscr{C}'} \mathscr{E}' \to \mathscr{K} \times_{\mathscr{C}'} \mathscr{E}'$ is a categorical equivalence by Lemma 15.16. Hence we obtain a categorical equivalence to a small ∞-category $K \times_{\mathscr{C}} \mathscr{E} \to \mathscr{K} \times_{\mathscr{C}'} \mathscr{E}'$. □

cor:small_int

Corollary 15.18. *If K is an essentially small simplicial set and $p : K \to \mathscr{C}at_\infty$ is a functor with image in $\mathscr{C}at_\infty^{\mathrm{sm}}$, then the corresponding simplicial set $\int_K p$ is essentially small.*

Proof. Take

$$\mathscr{P}.\mathscr{C}at_\infty^{\mathrm{sm}} := \mathscr{C}at_\infty^{\mathrm{sm}} \times_{\mathscr{C}at_\infty} \mathscr{P}.\mathscr{C}at_\infty^{\mathrm{sm}}.$$

This ∞-category is, equivalently, the pith over the undercategory $(\mathfrak{Cat}_\infty^{\mathrm{sm}})_{*/}$ where $\mathfrak{Cat}_\infty^{\mathrm{sm}}$ is the homotopy coherent nerve of the simplicial category $\underline{\mathrm{Cat}}_\infty^{\mathrm{sm}}$ of essentially small ∞-categories. By the fiber calculation of Proposition 6.7 and Proposition 15.13 we see that the cocartesian fibration $\mathscr{P}.\mathscr{C}at_\infty^{\mathrm{sm}} \to \mathscr{C}at_\infty^{\mathrm{sm}}$ is essentially small. Hence the pullback $\int_K p \to K$ is an essentially small cocartesian fibration, and the simplicial set $\int_K p$ is essentially small by Lemma 15.17. □

15.3. Existence of Kan extensions.

prop:pre_kan_exists

Proposition 15.19 ([5, 0300])**.** *Consider a simplicial set K and a pair of maps to ∞-categories $i : K \to \mathscr{C}$ and $\bar{F} : K \to \mathscr{D}$. The following hold:*

(1) *$\bar{F}$ admits a left Kan extension along i if and only if, at each x in $\mathscr{C}$, the diagram $\bar{F}|_{K_{/x}} : K_{/x} \to \mathscr{D}$ admits a colimit in $\mathscr{D}$.*
(2) *$\bar{F}$ admits a right Kan extension along i if and only if, at each y in $\mathscr{C}$, the diagram $\bar{F}|_{K_{y/}} : K_{y/} \to \mathscr{D}$ admits a limit in $\mathscr{D}$.*

For details the reader should consult the original text [5, 02ZZ]. We simply take this result for granted. We are most interested in the following corollary.

cor:kan_exists

Corollary 15.20. *Let $i : K \to \mathscr{C}$ be a map from an essentially small simplicial set into a locally small ∞-category $\mathscr{C}$.*

(1) *If $\mathscr{D}$ is cocomplete, then any map $\bar{F} : K \to \mathscr{D}$ admits a left Kan extension along i, $F : \mathscr{C} \to \mathscr{D}$.*
(2) *If $\mathscr{D}$ is complete, then any map $\bar{F} : K \to \mathscr{D}$ admits a right Kan extension along i, $F' : \mathscr{C} \to \mathscr{D}$.*

Proof. We prove (a), the proof of (b) being similar. Since $\mathscr{C}$ is locally small the left fibration $\mathscr{C}_{/x} \to \mathscr{C}$ is essentially small at all x in $\mathscr{C}$, and hence the simplicial set $K_{/x}$ is essentially small at all x by Lemma 15.17. By completeness of $\mathscr{D}$, and Proposition 15.11, we see that $\mathscr{D}$ admits all $K_{/x}$-colimits at all x in $\mathscr{C}$. It follows by Proposition 15.19 that a left Kan extension $F : \mathscr{C} \to \mathscr{D}$ exists. □

15.4. Uniqueness of Kan extensions.

prop:kan_univ

Proposition 15.21 (Universal property for LKE, [5, 0309]). *Suppose $i : K \to \mathscr{C}$ is a map from a simplicial set to an ∞-category, and let $\bar{F} : K \to \mathscr{D}$ be another map into an ∞-category. Suppose that we have a functor $F : \mathscr{C} \to \mathscr{D}$ and a transformation $\zeta : \bar{F} \to Fi$ which exhibits F as a left Kan extension of $\bar{F}$. Then for every functor $G : \mathscr{C} \to \mathscr{D}$ the composite*

$$\mathrm{Hom}_{\mathrm{Fun}(\mathscr{C},\mathscr{D})}(F, G) \to \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{D})}(Fi, Gi) \xrightarrow{\zeta^*} \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{D})}(\bar{F}, Gi)$$

is an isomorphism in $\mathrm{h}\,\mathscr{K}an$.

As with existence, we omit the proof and direct the interested reader to [5] for any details. We note that the identifications

$$\mathrm{Hom}_{\mathrm{sSet}}((\Delta^n)^{\mathrm{op}} \times K, \mathscr{D}) = \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n \times K^{\mathrm{op}}, \mathscr{D}^{\mathrm{op}})$$

provide an identification of ∞-categories $\mathrm{Fun}(K,\mathscr{D})^{\mathrm{op}} = \mathrm{Fun}(K^{\mathrm{op}}, \mathscr{D}^{\mathrm{op}})$. So by applying opposities we obtain the apparent analog of Proposition 15.21 for right Kan extensions.

Proposition 15.22 (Universal property for RKE). *Let $i : K \to \mathscr{C}$ and $\bar{F} : K \to \mathscr{D}$ be as in Proposition 15.21. Suppose that we have a functor $F' : \mathscr{C} \to \mathscr{D}$ and a transformation $\zeta' : F'i \to \bar{F}$ which exhibits F' as a right Kan extension of $\bar{F}$. Then for every functor $G : \mathscr{C} \to \mathscr{D}$ the composite*

$$\mathrm{Hom}_{\mathrm{Fun}(\mathscr{C},\mathscr{D})}(G, F') \to \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{D})}(Gi, F'i) \xrightarrow{\zeta'_*} \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{D})}(Gi, \bar{F})$$

is an isomorphism in $\mathrm{h}\,\mathscr{K}an$.

Corollary 15.23. *Take $i : K \to \mathscr{C}$ and $\bar{F} : K \to \mathscr{D}$ as in the statement of Proposition 15.21. If $\zeta : \bar{F} \to Fi$ and $\zeta' : \bar{F} \to F'i$ realize functors $F, F' : \mathscr{C} \to \mathscr{D}$ as left Kan extensions of $\bar{F}$, then there is an isomorphism $\alpha : F \xrightarrow{\sim} F'$ which fits into a diagram*

$$\begin{tikzcd} & \bar{F} \arrow[dl, "\zeta"'] \arrow[dr, "\zeta'"] & \\ Fi \arrow[rr, "i_*(\alpha)"'] & & F'i \end{tikzcd}$$

in the functor category $\mathrm{Fun}(K,\mathscr{D})$.

The analogous uniqueness result holds for right Kan extensions as well, though we won't record it explicitly.

Proof. We have the map of $\mathrm{h}\,\mathscr{K}an$-enriched categories

$$i^* : \pi\,\mathrm{Fun}(\mathscr{C},\mathscr{D}) \to \pi\,\mathrm{Fun}(K,\mathscr{D})$$

and we lift the transformations $\zeta : \bar{F} \to Fi$ and $\zeta' : \bar{F} \to F'i$ which exhibit F and F' as left Kan extensions to unique morphisms $\alpha : F \to F'$ and $\alpha' : F \to F'$ which recover ζ and ζ' under the compositions $\bar{F} \to F'i \to Fi$ and $\bar{F} \to Fi \to F'i$. The composites $F \to F' \to F$ and $F' \to F \to F'$ then restrict to ζ and ζ' respectively, so that they are the respective identity morphisms in $\pi\,\mathrm{Fun}(\mathscr{C},\mathscr{D})$ by Proposition 15.21. $\square$

We observe in Section **??** below a kind of global uniqueness for Kan extensions. Namely, if $\mathscr{D}$ is cocomplete then the collection of all left Kan extensions to $\mathscr{D}$, along some map $i : K \to \mathscr{C}$ with K essentially small and $\mathscr{C}$ locally small, assemble into a left adjoint

$$\mathrm{Lan}_i : \mathrm{Fun}(K, \mathscr{D}) \to \mathrm{Fun}(\mathscr{C}, \mathscr{D})$$

to the restriction functor $\mathrm{Fun}(\mathscr{C}, \mathscr{D}) \to \mathrm{Fun}(K, \mathscr{D})$. Indeed, one can view the universal property for left Kan extensions as a type of local claim for adjunction. Similarly, we find below that completeness of $\mathscr{D}$ implies the existence of a right adjoint

$$\mathrm{Ran}_i : \mathrm{Fun}(K, \mathscr{D}) \to \mathrm{Fun}(\mathscr{C}, \mathscr{D})$$

to restriction which is realized, on objects, via right Kan extension.

15.5. **Kan extension under base change.** We only record the following results for left Kan extensions. The interested reader might translate immediately to their rightward facing analogs.

lem:4912 **Lemma 15.24.** *Let $i : K \to \mathscr{C}$ be an arbitrary map from a simplicial set to an ∞-category, and $\bar{F}, \bar{F}' : K \to \mathscr{D}$ be isomorphic functors to another ∞-category. Then for any isomorphism $\alpha : \bar{F}' \to \bar{F}$, and any transformation $\zeta : \bar{F} \to Fi$ which exhibits a functor $F : \mathscr{C} \to \mathscr{D}$ as a left Kan extension of $\bar{F}$, the composite $\zeta\alpha : \bar{F}' \to Fi$ exhibits F as a left Kan extension of $\bar{F}'$.*

Proof. Follows from stability of colimits under isomorphisms of diagrams, as expressed specifically in Proposition 13.17. □

lem:4920 **Lemma 15.25.** *Let $i' : K' \to \mathscr{C}$ be a map to an ∞-category, $t : K \to K'$ be a categorical equivalence, and $i = i't : K \to \mathscr{C}$ be the corresponding composite. Let $\bar{F}' : K' \to \mathscr{D}$ be an arbitrary functor and $\bar{F} : K \to \mathscr{D}$ be the corresponding composite. A transformation $\zeta' : \bar{F}' \to Fi$ exhibits a functor $F : \mathscr{C} \to \mathscr{D}$ as a left Kan extension of $\bar{F}'$ if and only if the composite $\zeta' t : \bar{F} \to Fi$ exhibits F as a left Kan extension of $\bar{F}$.*

Proof. By Lemma 15.16 the induced map $K_{/x} \to K'_{/x}$ is a categorical equivalence at each x in $\mathscr{C}$. So the result follows by Proposition 13.14. □

15.6. **Kan extensions along embeddings.**

Proposition 15.26 ([5, 02YV]). *Let $i : \mathscr{K} \to \mathscr{C}$ be a fully faithful functor between ∞-categories, and $\bar{F} : \mathscr{K} \to \mathscr{D}$ be arbitrary. If a transformation $\zeta : \bar{F} \to Fi$ exhibits a functor $F : \mathscr{C} \to \mathscr{D}$ as a Kan extension of $\bar{F}$ then ζ is a natural isomorphism.*

Proof. By Lemmas 15.24 and 15.25 it suffices to prove the result in the case where $\mathscr{K}$ is a full ∞-subcategory in $\mathscr{C}$. In this instance our claim is implied by [5, 02YV]. □

In the case of a fully faithful functor $i : \mathscr{K} \to \mathscr{C}$, and a Kan extension $F : \mathscr{C} \to \mathscr{D}$ of a given functor $\bar{F} : \mathscr{K} \to \mathscr{D}$, we now understand that composition function

$$\mathrm{Hom}_{\mathrm{Fun}(\mathscr{K},\mathscr{D})}(\bar{G}, Fi) \xrightarrow{\zeta_*} \mathrm{Hom}_{\mathrm{Fun}(\mathscr{K},\mathscr{D})}(\bar{G}, \bar{F})$$

is an isomorphism in $\mathrm{h}\mathscr{K}an$. It follows by the universal property from Proposition 15.21 that the restriction functor

$$\mathrm{Hom}_{\mathrm{Fun}(\mathscr{C},\mathscr{D})}(G, F) \to \mathrm{Hom}_{\mathrm{Fun}(\mathscr{K},\mathscr{D})}(Gi, Fi)$$

is an equivalence whenever a functor $F : \mathscr{C} \to \mathscr{D}$ is a left Kan extension for some functor $\bar{F} : \mathscr{K} \to \mathscr{D}$.

Corollary 15.27. *Suppose $\mathscr{K} \to \mathscr{C}$ is a fully faithful functor, and that $F : \mathscr{C} \to \mathscr{D}$ is a left Kan extension of some functor $\mathscr{K} \to \mathscr{D}$. Then at each functor G in* $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$ *the restriction map*

$$\mathrm{Hom}_{\mathrm{Fun}(\mathscr{C},\mathscr{D})}(G, F) \to \mathrm{Hom}_{\mathrm{Fun}(\mathscr{K},\mathscr{D})}(Gi, Fi)$$

is an equivalence.

16. Yoneda embedding

We recall that, for any ∞-category $\mathscr{C}$, a Hom functor $H : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{K}an$ is a classifying functor for the twisted arrow fibration $\mathscr{T}w(\mathscr{C}) \to \mathscr{C}^{\mathrm{op}} \times \mathscr{C}$.

thm:yoneda_embed

Theorem 16.1 (Yoneda embedding)**.** *Let $\mathscr{C}$ be any ∞-category. For any* Hom *functor $H : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{K}an$ the associated functor*

$$h : \mathscr{C} \to \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{K}an), \quad h(x) := H(-, x),$$

is fully faithful.

Via the identification of cocartesian fibrations $\mathscr{T}w(\mathscr{C}) \cong \mathscr{T}w(\mathscr{C}^{\mathrm{op}})$ we understand that Hom functors for $\mathscr{C}$ are Hom functors for $\mathscr{C}^{\mathrm{op}}$, and vide versa. So the Yoneda embedding is symmetric in the two factors. In particular, Theorem 16.1 has its opposite expression.

Theorem 16.2 (Yoneda embedding v.2)**.** *Let $\mathscr{C}$ be any ∞-category. For any* Hom *functor $H : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{K}an$ the associated functor*

$$h' : \mathscr{C}^{\mathrm{op}} \to \mathrm{Fun}(\mathscr{C}, \mathscr{K}an), \quad h'(x) := H(x, -),$$

is fully faithful.

16.1. **Colimits over constant diagrams.**

def:tensored

Definition 16.3. For objects x and y in an ∞-category, a morphism of simplicial sets $\beta : K \to \mathrm{Hom}_{\mathscr{C}}(x, y)$ is said to exhibit y as a tensor product of x by K if, for any z in $\mathscr{C}$, the composition

$$\mathrm{Hom}_{\mathscr{C}}(y, z) \times K \to \mathrm{Hom}_{\mathscr{C}}(y, z) \times \mathrm{Hom}_{\mathscr{C}}(x, y) \xrightarrow{\circ} \mathrm{Hom}_{\mathscr{C}}(x, z)$$

induces an isomorphism $\mathrm{Hom}_{\mathscr{C}}(y, z) \to \mathrm{Fun}(K, \mathrm{Hom}_{\mathscr{C}}(x, z))$ in $\mathrm{h}\,\mathscr{K}an$.

Now, the above definition unfortunately does not strictly make sense as stated, since the composition function for the mapping spaces is only explicitly defined in $\mathrm{h}\,\mathscr{K}an$, while K does not exist as an object in $\mathscr{K}an$. However, composition is defined up to natural isomorphism in Kan, so that the map

$$\mathrm{Hom}_{\mathscr{C}}(y, z) \times K \to \mathrm{Hom}_{\mathscr{C}}(x, z)$$

is defined up to a natural isomorphism. Subsequently, the induced map $\mathrm{Hom}_{\mathscr{C}}(y, z) \to \mathrm{Fun}(K, \mathrm{Hom}_{\mathscr{C}}(x, z))$ is defined uniquely as a morphism in $\mathrm{h}\,\mathscr{K}an$, and we see that evaluating this condition is independent of the choice of representative for the composition function.

For an example [5, 03F4], we can consider the case where K is a discrete set, so that a map $K \to \mathrm{Hom}_{\mathscr{C}}(x, y)$ is defined by a $K[0]$-collection of maps $i_k : x \to y$. These maps then define an object in the undercategory $\mathscr{C}_{\underline{x}/}$, where $\underline{x} : K \to$

$\mathscr{C}$ is the constant functor. We *expect* in this case that the original map $K \to \mathrm{Hom}_{\mathscr{C}}(x,y)$ exhibits y as a tensor product of x by K if and only if y is initial in this undercategory, i.e. if and only if y is a coproduct $\amalg_{k\in K[0]}x$. We show that this colimit interpretation of tensoring is valid in complete generalization.

Let us first, however, do some bookkeeping. A map $\beta : K \to \mathrm{Hom}_{\mathscr{C}}(x,y)$ defines, and is equivalent to, a map $\beta : \Delta^1 \to \mathrm{Fun}(K,\mathscr{C})$ which sends 0 to the constant diagram $\underline{x} : K \to \mathscr{C}$ and sends 1 to the constant diagram $\underline{y}$. In this way β can be understood equivalently as a transformation between constant diagram $\beta : \underline{x} \to \underline{y}$.

prop:tensor_colim **Proposition 16.4** ([5, 03F3]). *A map $\beta : K \to \mathrm{Hom}_{\mathscr{C}}(x,y)$ exhibits y as a tensor product of x by K if and only if the corresponding transformation $\beta : \underline{x} \to \underline{y}$ between constant functors on K exhibit y as a colimit of $\underline{x}$ in $\mathscr{C}$.*

Proof. Consider objects $a, b : * \to \mathscr{C}$. We the inclusion

$$\mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{a},\underline{b}) \subseteq \mathrm{Fun}(\Delta^1,\mathrm{Fun}(K,\mathscr{C})) \cong \mathrm{Fun}(K,\mathrm{Fun}(\Delta^1,\mathscr{C}))$$

which naturally identifies the morphisms $\mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{a},\underline{b})$ with the Kan complex $\mathrm{Fun}(K,\mathrm{Hom}_{\mathscr{C}}(x,y))$. This just follows from naturality of the adjunction $\mathrm{Fun}(-,\mathrm{Fun}(L,M)) \cong \mathrm{Fun}(L,\mathrm{Fun}(-,M))$.

Under this identification the composition

$$\mathrm{Hom}_{\mathscr{C}}(y,z) \to \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{y},\underline{z}) \xrightarrow{\beta^*} \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{x},\underline{z}) \qquad (37)$$ eq:5044

appearing in Definition 13.1 is identified with the composition

$$\mathrm{Hom}_{\mathscr{C}}(y,z) \to \mathrm{Fun}(K,\mathrm{Hom}_{\mathscr{C}}(y,z)) \to \mathrm{Fun}(K,\mathrm{Hom}_{\mathscr{C}}(x,z)), \qquad (38)$$ eq:5048

where the first map is dual to the terminal function $K \to *$ and the second map is induced by β. One checks directly that this map induced by β takes an n-simplex $s : \Delta^n \times K \to \mathrm{Hom}_{\mathscr{C}}(y,z)$ to the composite simplex

$$\Delta^n \times K \xrightarrow{[\sigma,\beta]^T} \mathrm{Hom}_{\mathscr{C}}(y,z) \times \mathrm{Hom}_{\mathscr{C}}(x,y) \xrightarrow{\circ} \mathrm{Hom}_{\mathscr{C}}(x,z),$$

where here we abuse notation to let β denote the composite of the projection $\Delta^1 \times K \to K$ with β. Hence the composite (38) sends an n-simplex $s : \Delta^n \to \mathrm{Hom}_{\mathscr{C}}(y,z)$ to the n-simplex

$$\Delta^n \times K \xrightarrow{s\times\beta} \mathrm{Hom}_{\mathscr{C}}(y,z) \times \mathrm{Hom}_{\mathscr{C}}(x,y) \xrightarrow{\circ} \mathrm{Hom}_{\mathscr{C}}(x,z).$$

So the map (38) is adjoint to the composite

$$\mathrm{Hom}_{\mathscr{C}}(y,z) \times K \xrightarrow{1\times\beta} \mathrm{Hom}_{\mathscr{C}}(y,z) \times \mathrm{Hom}_{\mathscr{C}}(x,y) \xrightarrow{\circ} \mathrm{Hom}_{\mathscr{C}}(x,z),$$

We now conclude that the composition (37) is an isomorphism in $\mathrm{h}\mathscr{K}\!an$ if and only if the map $\mathrm{Hom}_{\mathscr{C}}(y,z) \to \mathrm{Fun}(K,\mathrm{Hom}_{\mathscr{C}}(x,z))$ appearing in Definition 16.3 is an isomorphism in $\mathrm{h}\mathscr{K}\!an$. Rather, the transformation $\beta : \underline{x} \to \underline{y}$ exhibits y as a colimit of the constant functor $\underline{x} : K \to \mathscr{C}$ if and only if the corresponding map $\beta : K \to \mathrm{Hom}_{\mathscr{C}}(x,y)$ exhibits y as a tensor product of x by K. $\square$

We cover some quality of life results before closing out the subsection.

Lemma 16.5. (a) *Let $\beta, \beta' : K \to \mathrm{Hom}_{\mathscr{C}}(x,y)$ be isomorphic morphisms. Then β exhibits y as a tensor product of x by K if and only if β' exhibits y as a tensor product of x by K.*

(b) *Let $K \to K'$ be a categorical equivalence, $\beta' : K \to \mathrm{Hom}_{\mathscr{C}}(x,y)$ be an arbitrary* *map and $\beta : K \to \mathrm{Hom}_{\mathscr{C}}(x,y)$ be the corresponding composition. Then β' exhibits y as a tensor product of x by K' if and only if β exhibits y as a tensor product of x by K.*

Given a Kan-enriched category $\underline{A}$, the following is a consequence of the identification of $\mathrm{h}\,\mathscr{K}an$-enriched categories $\pi\underline{A} \cong \pi\,\mathrm{N}^{\mathrm{hc}}(\underline{A})$ from Proposition 7.6.

lem:simplicial_tensor **Lemma 16.6.** *Suppose $\underline{A}$ is a simplicial category which is enriched in* Kan *and take $\mathscr{A} = \mathrm{N}^{\mathrm{hc}}(\underline{A})$. Let $\theta : \underline{\mathrm{Hom}}_{\underline{A}}(x,y) \to \mathrm{Hom}_{\mathscr{A}}(x,y)$ be the equivalence of Theorems 5.27 and I-10.17. For a morphism $\beta : K \to \underline{\mathrm{Hom}}_{\underline{A}}(x,y)$, the composite map $\theta\beta : K \to \mathrm{Hom}_{\mathscr{A}}(x,y)$ exhibits y as a tensor product of x over K if and only if, at all z in $\underline{A}$, the composite*

$$\underline{\mathrm{Hom}}_{\underline{A}}(y,z) \times K \to \underline{\mathrm{Hom}}_{\underline{A}}(y,z) \times \underline{\mathrm{Hom}}_{\underline{A}}(x,y) \overset{\circ}{\to} \underline{\mathrm{Hom}}_{\underline{A}}(x,z)$$

induces a homotopy equivalence $\underline{\mathrm{Hom}}_{\underline{A}}(y,z) \to \mathrm{Fun}(K, \underline{\mathrm{Hom}}_{\underline{A}}(x,z))$.

We consider an example.

ex:space_tensors **Example 16.7** ([5, 03F8])**.** We consider the case of the category of spaces $\mathscr{C} = \mathscr{K}an$. Let $\mathscr{Y}$ be an arbitrary Kan complex and consider a morphism $\beta : K \to \mathrm{Hom}_{\mathscr{K}an}(*, \mathscr{Y})$. Up to equivalence we can assume that β factors through the equivalence $\mathscr{Y} \cong \mathrm{Fun}(*, \mathscr{Y}) \to \mathrm{Hom}_{\mathscr{K}an}(*, \mathscr{Y})$, and take $\bar{\beta} : K \to \mathscr{Y}$ the corresponding map to $\mathscr{Y}$. We claim that β exhibits $\mathscr{Y}$ as a tensor product of the point $*$ by K if and only if $\bar{\beta} : K \to \mathscr{Y}$ is a weak homotopy equivalence.

To see this it suffices to show, according to Lemma 16.6, that at each Kan complex $\mathscr{Z}$ the map

$$\bar{\beta}^* : \mathrm{Fun}(\mathscr{Y}, \mathscr{Z}) \to \mathrm{Fun}(K, \mathscr{Z})$$

is an equivalence if and only if $\bar{\beta}$ is a weak homotopy equivalence. However, this is simply the definition of a categorical equivalence [5, 03PJ, 03PJ]. We conclude now that an arbitrary map $\beta : K \to \mathrm{Hom}_{\mathscr{K}an}(*, \mathscr{Y})$ exhibits $\mathscr{Y}$ as a tensor product of $*$ by K if and only if β is a categorical equivalence.

16.2. **Yoneda's lemma and proof of Yoneda embedding.**

lem:5100 **Lemma 16.8.** *Let $\mathscr{C}$ be an ∞-category and $F : \mathscr{C} \to \mathscr{K}an$ be any functor. A map $1_x : * \to F(x)$ exhibits F as corepresented by an object x in $\mathscr{C}$ if and only if the map 1_x exhibits F as a left Kan extension of the constant functor $\Delta^0 : * \to \mathscr{K}an$ along the inclusion $x : * \to \mathscr{C}$.*

Proof. The functor F is corepresented by x, via the map 1_x, if and only if the composite

$$\mathrm{Hom}_{\mathscr{C}}(x,y) \xrightarrow{F} \mathrm{Hom}_{\mathscr{K}an}(F(x),F(y)) \xrightarrow{1_x^*} \mathrm{Hom}_{\mathscr{K}an}(*,F(y))$$

is an isomorphism in $\mathrm{h}\,\mathscr{K}an$. This follows by Proposition 11.4.

On the other hand, we have

$$*_{/y} = \mathrm{Hom}^{\mathrm{R}}_{\mathscr{C}}(x,y) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}}(x,y)$$

at each y in $\mathscr{C}$, and 1_x exhibits F as a left Kan extension of Δ^0 if and only if the transformation

$$\Delta^0|_{\mathrm{Hom}_{\mathscr{C}}(x,y)} \xrightarrow{1_x} F(x)|_{\mathrm{Hom}_{\mathscr{C}}(x,y)} \xrightarrow{F(\gamma)} F(y)|_{\mathrm{Hom}_{\mathscr{C}}(x,y)} \qquad (39)$$ eq:5117

realizes $F(x)$ as a colimit of the constant diagram $\Delta^0 : \mathrm{Hom}_{\mathscr{C}}(x,y) \to \mathscr{K}an$. Here

$$\gamma : \Delta^1 \times \mathrm{Hom}(x,y) \to \mathscr{C}$$

is adjoint to the inclusion $\mathrm{Hom}(x,y) \to \mathrm{Fun}(\Delta^1, \mathscr{C})$, i.e. the evaluation map, and $F(\gamma)$ is the composite of evaluation with F. So $F(\gamma)$ is adjoint to the map $F : \mathrm{Hom}_{\mathscr{C}}(x,y) \to \mathrm{Hom}_{\mathscr{K}an}(F(x), F(y))$, and the above composite (39) is adjoint to the composition of F with $1_x^* : \mathrm{Hom}_{\mathscr{K}an}(F(x), F(y)) \to \mathrm{Hom}_{\mathscr{K}an}(*, F(y))$.

By Proposition 16.4 the above transformation (39) realizes $F(y)$ as a colimit of the constant diagram at Δ^0 if and only if the map $1_x^* F : \mathrm{Hom}_{\mathscr{C}}(x,y) \to \mathrm{Hom}_{\mathscr{K}an}(*, F(y))$ exhibits $F(y)$ as a tensor product of $*$ by $\mathrm{Hom}_{\mathscr{C}}(x,y)$. As we saw in Example 16.7 this occurs if and only if the map $1_x^* F : \mathrm{Hom}_{\mathscr{C}}(x,y) \to \mathrm{Hom}_{\mathscr{K}an}(*, F(y))$ is an equivalence, i.e. if and only if 1_x exhibits F as corepresented by x. □

thm:yoneda_lem

Theorem 16.9 (Yoneda's lemma)**.** *Let $F : \mathscr{C} \to \mathscr{K}an$ be a corepresented by an object $x : * \to \mathscr{C}$, with corresponding initial vertex $1_x : * \to F(x)$. For any other functor $G : \mathscr{C} \to \mathscr{K}an$ the composition*

$$\mathrm{Hom}_{\mathrm{Fun}(\mathscr{C},\mathscr{K}an)}(F,G) \xrightarrow{-|_x} \mathrm{Hom}_{\mathscr{K}an}(F(x), G(x)) \xrightarrow{1_x^*} \mathrm{Hom}_{\mathscr{K}an}(*, G(x)) \xrightarrow{\theta^{-1}} G(x)$$

is an isomorphism in $\mathrm{h}\,\mathscr{K}an$*.*

Proof. By Lemma 16.8 the map $1_x : * \to F(x)$, which we interpret as a tranformation in $\mathrm{Fun}(*, \mathscr{K}an)$, exhibits $h : \mathscr{C} \to \mathscr{K}an$ as a left Kan extension of the constant functor of value $*$ in $\mathscr{K}an$. Hence the composition under consideration is an isomorphism in $\mathrm{h}\,\mathscr{K}an$ by the universal property of the left Kan extension, Proposition 15.21. □

Applying this result to $\mathscr{C}^{\mathrm{op}}$ yields the following.

thm:yoneda_lem_v2

Theorem 16.10 (Yoneda's lemma v.2)**.** *Let $F' : \mathscr{C}^{\mathrm{op}} \to \mathscr{K}an$ be a represented by an object $x : * \to \mathscr{C}$, with corresponding initial vertex $1_x : * \to F'(x)$. For any other functor $G : \mathscr{C}^{\mathrm{op}} \to \mathscr{K}an$ the composition*

$$\mathrm{Hom}_{\mathrm{Fun}(\mathscr{C}^{\mathrm{op}},\mathscr{K}an)}(F',G) \xrightarrow{-|_x} \mathrm{Hom}_{\mathscr{K}an}(F'(x), G(x)) \xrightarrow{1_x^*} \mathrm{Hom}_{\mathscr{K}an}(*, G(x)) \xrightarrow{\theta^{-1}} G(x)$$

is an isomorphism in $\mathrm{h}\,\mathscr{K}an$*.*

We now prove that the Yoneda embedding, Theorem 16.1.

Proof of Theorem 16.1. Let $H : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{K}an$ be a Hom functor for $\mathscr{C}$, i.e. a transport functor for the twisted arrow fibration. By Lemma 12.5 the identity morphism $id_x : * \to \mathscr{T}w(\mathscr{C})$ is initial in both of the fibers

$$\{x\} \times_{\mathscr{C}^{\mathrm{op}}} \mathscr{T}w(\mathscr{C}) \;\text{ and }\; \mathscr{T}w(\mathscr{C}) \times_{\mathscr{C}} \{x\}.$$

This implies that the corresponding vertex $1_x : * \to H(x,x)$ is initial for both of the functors $H(x,-) : \mathscr{C} \to \mathscr{K}an$ and $H(-,x) : \mathscr{C}^{\mathrm{op}} \to \mathscr{K}an$, i.e. simultaneously exhibits $H(x,-)$ as corepresented by x and $H(-,x)$ as represented by x.

Take $h : \mathscr{C} \to \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}\mathscr{K}an)$ proposed embedding, $h(x) = H(-,x)$. By Theorem 16.10 the map

$$h : \mathrm{Hom}_{\mathscr{C}}(x,y) \to \mathrm{Hom}_{\mathrm{Fun}(\mathscr{C}^{\mathrm{op}},\mathscr{K}an)}(h(x), h(y))$$

is an equivalence if and only if the composite

$$\mathrm{Hom}_{\mathscr{C}}(x,y) \xrightarrow{h} \mathrm{Hom}_{\mathrm{Fun}(\mathscr{C}^{\mathrm{op}},\mathscr{K}an)}(h(x), h(y)) \xrightarrow{\psi} h(y)(x) = H(x,y) \tag{40}$$

eq:5168

is an isomorphism in $\mathrm{h}\mathscr{K}an$, where the map ψ is the composite of Theorem 16.10.

By Proposition 7.6 the composite ψ is equal to the composite

$$\mathrm{Hom}_{\mathrm{Fun}(\mathscr{C}^{\mathrm{op}},\mathscr{K}an)}(h(x),h(y)) \overset{-|_x}{\to} \mathrm{Hom}_{\mathscr{K}an}(H(x,x),H(x,y))$$
$$\overset{\theta^{-1}}{\to} \mathrm{Fun}(H(x,x),H(x,y)) \overset{1_x^*}{\to} H(x,y),$$

as a morphism in $\mathrm{h}\mathscr{K}an$. Thus, for $H_x : \mathscr{C} \to \mathscr{K}an$ given by $H_x = H(x,-)$ the composite condences to the sequence

$$\mathrm{Hom}_{\mathscr{C}}(x,y) \overset{H_x}{\to} \mathrm{Hom}_{\mathscr{K}an}(H_x(x),H_x(y)) \cong \mathrm{Fun}(H_x(x),H_x(y)) \overset{1_x^*}{\to} H_x(y).$$

This sequence is in fact an isomorphism in $\mathrm{h}\mathscr{K}an$, since H_x is corepresented by x, and we see now that the map

$$h : \mathrm{Hom}_{\mathscr{C}}(x,y) \to \mathrm{Hom}_{\mathrm{Fun}(\mathscr{C}^{\mathrm{op}},\mathscr{K}an)}(h(x),h(y))$$

is an isomorphism at all pairs of objects $x, y : * \to \mathscr{C}$. It follows, by the definition, that the functor $h : \mathscr{C} \to \mathrm{Fun}(\mathscr{C}^{\mathrm{op}},\mathscr{K}an)$ is fully faithful. $\square$

16.3. **A useful characterization of limit diagrams.**

Lemma 16.11. *Any map of simplicial sets $A \to B$ admits a factorization $A \to A' \to B$ in which the first map $A \to A'$ is a monomorphism and the second map $A' \to B$ is a trivial Kan fibration.*

We have the following ∞-analog of Proposition I-4.8 which charactarizes those equivalences which are trivial Kan fibrations.

prop:isokan_equiv **Proposition 16.12.** *A functor between ∞-categories $F : \mathscr{C} \to \mathscr{D}$ is a trivial Kan fibration if and only if it is both an isofibration and an equivalence.*

Proof. If F is a trivial Kan fibration then it is conservative, and hence any lift of an isomorphism in $\mathscr{D}$ is an isomorphism in $\mathscr{C}$. In particular F is an isofibration. Consider now any simplicial set K and the equivalent lifting problems

$$\begin{tikzcd} A \arrow[r, "g_0"] \arrow[d, "i"'] & \mathrm{Fun}(K,\mathscr{C})^{\mathrm{Kan}} \arrow[d, "F_*"] \\ B \arrow[r, "g_1"'] \arrow[ur, dotted] & \mathrm{Fun}(K,\mathscr{D})^{\mathrm{Kan}} \end{tikzcd} \qquad \begin{tikzcd} A \times K \arrow[r] \arrow[d, "i\times K"'] & \mathscr{C} \arrow[d] \\ B \times K \arrow[r] \arrow[ur, dotted] & \mathscr{D} \end{tikzcd} \tag{41}$$ eq:5204

in which i is injective.

Since F is a trivial Kan fibration the second lifting problem admits a solution, and so there exists a map $g : B \to \mathrm{Fun}(K,\mathscr{C})$ which restricts fo g_0 on A and has $F_* g = g_1$. We need only show that g has image in the subcomplex $\mathrm{Fun}(K,\mathscr{C})^{\mathrm{Kan}}$. However, since F is conservative, the characterization of natural isomorphisms provided in Proposition I-7.9 tells us that the induced map $F_* : \mathrm{Fun}(K,\mathscr{C}) \to \mathrm{Fun}(K,\mathscr{D})$ is conservative. Hence the diagram

$$\begin{tikzcd} \mathrm{Fun}(K,\mathscr{C})^{\mathrm{Kan}} \arrow[r] \arrow[d] & \mathrm{Fun}(K,\mathscr{C}) \arrow[d] \\ \mathrm{Fun}(K,\mathscr{D})^{\mathrm{Kan}} \arrow[r] & \mathrm{Fun}(K,\mathscr{D}) \end{tikzcd}$$

is a pullback diagram. It follows that our splitting map $g : B \to \mathrm{Fun}(K, \mathscr{C})$ necessarily has image in $\mathrm{Fun}(K, \mathscr{C})$, and so we solve the first lifting problem in (41).

We have now established that the induced map $F_* : \mathrm{Fun}(K, \mathscr{C})^{\mathrm{Kan}} \to \mathrm{Fun}(K, \mathscr{D})^{\mathrm{Kan}}$ is a trivial Kan fibration at each simplicial set K, and hence an equivalence at each K by Proposition I-4.8. By Theorem I-5.43 we conclude that the original map $F : \mathscr{C} \to \mathscr{D}$ is an equivalence.

For the converse, suppose that F is both an equivalence and an isofibration. Factor F as a composition of an inclusion $\mathscr{C} \xrightarrow{i} \mathscr{C}'$ and a trivial Kan fibration $F' : \mathscr{C}' \to \mathscr{D}$. In this case $\mathscr{C}'$ is an ∞-category, F' is an equivalence by the above findings, and thus i is an equivalence as well.

The functor

$$\mathrm{Fun}(\mathscr{C}', \mathscr{C}) \to Fun(\mathscr{C}', \mathscr{D}) \times_{\mathrm{Fun}(\mathscr{C}, \mathscr{D})} \mathrm{Fun}(\mathscr{C}, \mathscr{C})$$

is now a isofibration by Proposition I-6.13 and an equivalence by Corollary I-6.24. In this case all objects in the fiber product lift to objects in $\mathrm{Fun}(\mathscr{C}', \mathscr{C})$, and we can lift the pairing $(F', id_{\mathscr{C}}$ in particular to a map $\pi : \mathscr{C}' \to \mathscr{C}$ which splits the diagram

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "id"] \arrow[d, "i"'] & \mathscr{C} \arrow[d, "F"] \\
\mathscr{C}' \arrow[r, "F'"'] \arrow[ur, "\pi"] & \mathscr{D}.
\end{tikzcd}$$

The above diagram, and the fact that F' is a trivial Kan fibration, implies that F is also a trivial Kan fibration. $\square$

cor:5245 **Corollary 16.13.** *Let $p : K \to \mathscr{C}$ be a diagram in an ∞-category and $\widetilde{p} : \{0\} \star K \to \mathscr{C}$ be an arbitrary extension of p. Then $\widetilde{p}$ is a limit diagram for p if and only if the forgetful functor $\mathscr{C}_{/\widetilde{p}} \to \mathscr{C}_{/p}$ is an equivalence.*

Proof. Note that $\widetilde{p}$ specifies an object in $\mathscr{C}_{/p}$, by the defintion of the overcategory. The claim now follows by Proposition 16.12 and the fact that the forgetful functor

$$(\mathscr{C}_{/p})_{/\widetilde{p}}$$

is a right fibration, and in particular an isofibration. $\square$

prop:lim_funcat **Proposition 16.14.** *Let $p : K \to \mathscr{C}$ be a diagram in an ∞-category and $\widetilde{p} : \{0\} \star K \to \mathscr{C}$ be an arbitrary extension of p. Then $\widetilde{p}$ is a limit diagram for p if and only if, at every object x in $\mathscr{C}$, the forgetful functor*

$$\mathrm{Hom}_{\mathrm{Fun}(K^{<}, \mathscr{C})}(\underline{x}, \widetilde{p}) \to \mathrm{Hom}_{\mathrm{Fun}(K, \mathscr{C})}(\underline{x}, p) \tag{42}$$ eq:5259

is an equivalence.

Proof. We have the diagram

$$\begin{tikzcd}
\mathscr{C}_{/\widetilde{p}} \arrow[r] \arrow[d] & \mathscr{C}_{/p} \arrow[d] \\
\mathscr{C} \times^{\mathrm{or}}_{\mathrm{Fun}(K^{<}, \mathscr{C})} \{\widetilde{p}\} \arrow[r] & \mathscr{C} \times^{\mathrm{or}}_{\mathrm{Fun}(K, \mathscr{C})} \{p\}
\end{tikzcd}$$

in which the vertical maps are equivalences. Here the vertical maps are, in particular, the slice diagonals from Section I-10.5. Hence the map $\mathscr{C}_{/\widetilde{p}} \to \mathscr{C}_{/p}$ is an equivalence if and only if the map $\mathscr{C} \times^{\mathrm{or}}_{\mathrm{Fun}(K^{<},\mathscr{C})} \{\widetilde{p}\} \to \mathscr{C} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \{p\}$ is an equivalence. Therefore, by Corollary 16.13, the latter map is an equivalence if and only if $\widetilde{p}$ is a limit diagram for p.

Since this latter map, let's call it f, fits into a diagram of right fibrations

$$\begin{tikzcd} \mathscr{C} \times^{\mathrm{or}}_{\mathrm{Fun}(K^{<},\mathscr{C})} \{\widetilde{p}\} \arrow[rr, "f"] \arrow[dr] & & \mathscr{C} \times^{\mathrm{or}}_{\mathrm{Fun}(K,\mathscr{C})} \{p\} \arrow[dl] \\ & \mathscr{C} & \end{tikzcd}$$

we see that f is an equivalence if and only if all of its fibers over $\mathscr{C}$ are equivalences (Theorem 3.8). However, the fiber of f at a given point $x : * \to \mathscr{C}$ simply recovers the original morphisms of interest (42). So we see that $\widetilde{p}$ is a limit diagram if and only if, at each x in $\mathscr{C}$, the maps (42) is an equivalence. □

16.4. **Yoneda embedding and limits.**

thm:yoneda_lim

Theorem 16.15. *Let $h : \mathscr{C} \to \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{K}an)$ be a Yoneda embedding for an ∞-category $\mathscr{C}$, and $p : K \to \mathscr{C}$ be a diagram in $\mathscr{C}$. For any extension $\widetilde{p} : \{0\} \star K \to \mathscr{C}$ of p the following are equivalent:*

(a) *$\widetilde{p}$ is a limit diagram in $\mathscr{C}$.*
(b) *$h\widetilde{p}$ is a limit diagram in $\mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{K}an)$.*

Proof. By Proposition 16.14 $\widetilde{p}$ is a limit diagram if and only if, at each x in $\mathscr{C}$, the map

$$\mathrm{Hom}_{\mathrm{Fun}(K^{<},\mathscr{C})}(\underline{x}, \widetilde{p}) \to \mathrm{Hom}_{\mathrm{Fun}(K,\mathscr{C})}(\underline{x}, p)$$

is an equivalence. For $\mathscr{E} = \{x\} \times^{\mathrm{or}}_{\mathscr{C}} \mathscr{C}$ the above map is identified with the restriction functor $\mathrm{Fun}_{\mathscr{C}}(K^{<}, \mathscr{E}) \to \mathrm{Fun}_{\mathscr{C}}(K, \mathscr{E})$.

For $E' = \mathscr{E} \times_{\mathscr{C}} K^{<}$ and $E = \mathscr{E} \times_{\mathscr{C}} K$ we have the pullback diagram of left fibrations

$$\begin{tikzcd} E \arrow[r] \arrow[d] & E' \arrow[d] \\ K \arrow[r] & K^{<} \end{tikzcd}$$

and identifications

$$\mathrm{Fun}_{\mathscr{C}}(K^{<}, \mathscr{E}) \cong \mathrm{Fun}_{K^{<}}(K^{<}, E'), \quad \mathrm{Fun}_{\mathscr{C}}(K, \mathscr{E}) \cong \mathrm{Fun}_{K}(K, E)$$

under which the above restriction functor becomes the restriction functor

$$\mathrm{Fun}_{K^{<}}(K^{<}, E') \to \mathrm{Fun}_{K}(K, E).$$

This final map is an equivalence if and only if the composite

$$\begin{tikzcd} & & \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{K}an) \arrow[dr, dashed, "x^*"] & \\ K^{<} \arrow[r] & \mathscr{C} \arrow[ur, dashed, "h"] \arrow[rr] & & \mathscr{K}an \end{tikzcd}$$

is a limit diagram for the restricted map $K \to \mathscr{K}an$, by the diffraction criterion of Theorem 14.8 and Theorem 14.25.

We now understand that the extension $\widetilde{p}$ is a limit diagram in $\mathscr{C}$ if and only if, at each x in $\mathscr{C}$, the composite

$$K^{<} \to \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{K}an) \xrightarrow{x^*} \mathscr{K}an$$

is a limit diagram in $\mathscr{K}an$. However this occurs if and only if the map $K^{<} \to \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{K}an)$ is a limit diagram, by Proposition 13.28. □

As we saw in the above proof, the characterization of limits in $\mathrm{Fun}(\mathscr{C}, \mathscr{K}an)$ provided in Proposition 16.14 implies the following alternate expression of Theorem 16.15.

`cor:lim_coreps`

Corollary 16.16. *Let $p : K \to \mathscr{C}$ be a diagram in an ∞-category. An extension $\widetilde{p} : \{0\} \star K \to \mathscr{C}$ of p is a limit diagram in $\mathscr{C}$ if and only if, for each object x in $\mathscr{C}$ and functor $h^x : \mathscr{C} \to \mathscr{K}an$ which is corepresented by x, the composite*

$$\{0\} \star K \xrightarrow{\widetilde{p}} \mathscr{C} \xrightarrow{h^x} \mathscr{K}an$$

is a limit diagram in $\mathscr{K}an$.

We apply this result in the simplicial setting, in conjunction with Proposition 11.6, to obtain the following

`cor:simp_lim_coreps`

Corollary 16.17. *Let $\underline{A}$ be a Kan-enriched simplicial category, $\mathscr{A} = \mathrm{N}^{\mathrm{hc}}(\underline{A})$, and $p : K \to \mathscr{A}$ be an arbitrary diagram. An extension $\widetilde{p} : \{0\} \star K \to \mathscr{A}$ of p is a limit diagram in $\mathscr{A}$ if and only if, for each object x in $\mathscr{A}$, the composite*

$$\underline{\mathrm{Hom}}_{\mathscr{A}}(x, -) \circ p : \{0\} \star K \to \mathscr{K}an$$

is a limit diagram in $\mathscr{K}an$.

We note that, since the Yoneda embedding is fully faithful and hence conservative, Theorem 16.15 is equivalent to the statement that the Yoneda embedding commutes with all limits in $\mathscr{C}$. Applying the above theorem to $\mathscr{C}^{\mathrm{op}}$ provides a corresponding characterization of colimit diagrams in $\mathscr{C}$.

`thm:yoneda_colim`

Theorem 16.18. *Let $h' : \mathscr{C}^{\mathrm{op}} \to \mathrm{Fun}(\mathscr{C}, \mathscr{K}an)$ be a contravariant Yoneda embedding for an ∞-category $\mathscr{C}$, and $q : K \to \mathscr{C}$ be a diagram in $\mathscr{C}$. For any extension $\widetilde{q} : K \star \{1\} \to \mathscr{C}$ of q the following are equivalent:*

(a) *$\widetilde{q}$ is a colimit diagram in $\mathscr{C}$.*
(b) *$h'(\widetilde{q})^{\mathrm{op}}$ is a limit diagram in $\mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{K}an)$.*

16.5. **Aside: limits and colimits of essentially small spaces.**

`lem:smallkan_lim`

Lemma 16.19. *The full subcategory $\mathscr{K}an^{\mathrm{sm}} \subseteq \mathscr{K}an$ of essentially small Kan complexes is closed under the formation of all limits and colimits in $\mathscr{K}an$.*

Proof. Given a small diagram $\bar{p} : K \to \mathscr{K}an^{\mathrm{sm}} \subseteq \mathscr{K}an$ we have an inner anodyne map $K \to \mathscr{K}$ to a small ∞-category by Proposition 15.9. We can lift the given diagram to a diagram $p : \mathscr{K} \to \mathscr{K}an^{\mathrm{sm}}\mathscr{K}an$ and find that the limits over K and $\mathscr{K}$ agree, by Proposition 13.14. For $\mathscr{E} = \int_{\mathscr{K}} p$ we have the explicit expression

$$\lim(p) = \mathrm{Fun}_{\mathscr{K}}(\mathscr{K}, \mathscr{E})$$

in $\mathscr{K}an$ by Corollary 14.27.

Now, since p has image in $\mathscr{K}an^{\mathrm{sm}}$ the fibration $\mathscr{E} \to \mathscr{K}$ is essentially small and $\mathscr{E}$ itself is therefore essentially small by Proposition 15.13. So after replacing $\mathscr{E}$

with an equivalent ∞-category we may assume $\mathscr{E}$ is small. In this case the functor category $\mathrm{Fun}(\mathscr{K}, \mathscr{E})$ is small as well, so that the limit of p is essentially small.

As for the colimit of p, we have $\mathrm{colim}(p) = \mathscr{E}[W^{-1}]$ where the localization can be produced by fibrant replacement in the model category of marked simplicial sets. This fibrant replacement occurs specifically in the model category sSet^+ of marked simplicial sets in our medium sized universe. However, since $\mathscr{E}$ is small we can perform such a fibrant replacement $\mathscr{E} \to \mathscr{E}[W^{-1}]^{\mathrm{sm}}$ in the subcategory of small marked simplicial sets. By the small object argument [3, Proposition A.1.2.5] the localization map $\mathscr{E} \to \mathscr{E}[W^{-1}]^{\mathrm{sm}}$ can furthermore be chosen to be marked anodyne [3, Proposition A.1.2.5] in the category of small marked simplicial sets. Since the inclusion of small marked siplicial sets into medium sized marked simplicial sets preserves marked anodyne morphisms we see that $\mathscr{E} \to \mathscr{E}[W^{-1}]^{\mathrm{sm}}$ remains marked anodyne in this larger category, and hence remains a Cartesian equivalence in the larger category [3, Remark 3.1.3.4]. So we can take $\mathscr{E}[W^{-1}] = \mathscr{E}[W^{-1}]^{\mathrm{sm}}$ and we observe that $\mathrm{colim}(p)$ is essentially small. □

16.6. **Yoneda embedding in the locally small setting.** We consider the case of a locally small ∞-category $\mathscr{C}$. Then for any Hom functor $H : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{K}an$ the values $H(x,y)$ are essentially small, since they are equivalent to the corresponding mapping spaces $\mathrm{Hom}_{\mathscr{C}}(x,y)$ by Corollary 12.7. Hence the Yoneda embedding for $\mathscr{C}$ factors as

$$h : \mathscr{C} \to \mathrm{Fun}(\mathscr{C}, \mathscr{K}an^{\mathrm{sm}}) \to \mathrm{Fun}(\mathscr{C}, \mathscr{K}an).$$

lem:full_funcat

Lemma 16.20. *Let $\mathscr{D}'$ be a full ∞-subcategory in an ∞-category $\mathscr{D}$. Then for any simplicial set K the inclusion $\mathrm{Fun}(K, \mathscr{D}') \to \mathrm{Fun}(K, \mathscr{D})$ identifies $\mathrm{Fun}(K, \mathscr{D}')$ with the full ∞-subcategory of functors in $\mathrm{Fun}(K, \mathscr{D})$ which take values in $\mathscr{D}'$.*

Proof. Consider a map $F : \Delta^n \times K \to \mathscr{D}$ whose restrictions $F|_{\{i\}\times K} : \{i\} \times K \to \mathscr{D}$ have image in $\mathscr{D}'$. It suffices to show that F itself has image in $\mathscr{D}'$. Since $\mathscr{D}'$ is full in $\mathscr{D}$, F has image in $\mathscr{D}'$ if and only if $F(z)$ has image in $\mathscr{D}'$ for each object $z : * \to \Delta^n \times K$. However, the objects in $\Delta^n \times K$ are specifically pairs of objects (i, x) with i in $\Delta^n[0] = [n]$ and x in $K[0]$, and hence each object lies in some subcomplex $\{i\} \times K$. By our assumption on F it follows that F sends each object in $\Delta^n \times K$ to an object in $\mathscr{D}$, and therefore has image in $\mathscr{D}'$ as desired. □

Lemma 16.20, in conjunction with Theorem 16.1 and the preceding discussion, provide a refinement for the Yoneda embedding in the locally small setting.

thm:yoneda_locsmall

Theorem 16.21. *Suppose $\mathscr{C}$ is a locally small ∞-category. Then any* Hom *functor $H : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{K}an$ has image in $\mathscr{K}an^{\mathrm{sm}}$ and the associated functor*

$$h : \mathscr{C} \to \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{K}an^{\mathrm{sm}}), \quad h(x) := H(-, x),$$

is fully faithful.

We can apply this finding specifically in the essentially small setting.

cor:yoneda_locsmall

Corollary 16.22. *If $\mathscr{C}$ is essentially small, then any* Hom *functor $H : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{K}an$ has image in $\mathscr{K}an^{\mathrm{sm}}$ and the associated functor*

$$h : \mathscr{C} \to \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{K}an^{\mathrm{sm}}), \quad h(x) := H(-, x),$$

is fully faithful.

We note that, by Lemma 16.19, the full ∞-subcategory $\mathrm{Fun}(\mathscr{C}, \mathscr{K}an^{\mathrm{sm}})$ is both complete and cocomplete, and the inclusion $\mathrm{Fun}(\mathscr{C}, \mathscr{K}an^{\mathrm{sm}}) \to \mathrm{Fun}(\mathscr{C}, \mathscr{K}an)$ is both continuous and cocontinuous. So Theorems 16.15 and 16.18, in conjunction with Theorem 16.21, provide the following.

Corollary 16.23. *Suppose $\mathscr{C}$ is a locally small ∞-category. Then the Yoneda embedding $h : \mathscr{C} \to \mathrm{Fun}(\mathscr{C}, \mathscr{K}an^{\mathrm{sm}})$ both preserves and detects limits in $\mathscr{C}$.*

Though the relevance of this refinement is not immediately clear, within the context of this text at lease, one of the primary implications is the following.

Corollary 16.24. *Any essentially small ∞-category $\mathscr{C}$ admits a fully faithful functor $F : \mathscr{C} \to \mathscr{C}'$ into an ∞-category $\mathscr{C}'$ with the following properties:*

(a) *$\mathscr{C}'$ is locally small, complete, and cocomplete.*
(b) *F commutes with all limits which exist in $\mathscr{C}$.*

The embedding of Corollary 16.22 forms the foundations of various cocompletion constructions which appear in [3, §5.3.5, 5.5.8]. Local smallness of the functor category $\mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{K}an^{\mathrm{sm}})$ is essential in assuring the existences of Kan extensions from various subcategories $\mathscr{C}'' \subseteq \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{K}an^{\mathrm{sm}})$ along the Yoneda embedding.

Appendix A. Proof of Proposition 8.2

We now provide the details for our primary foundational result. We recall the situation: We have a cocartesian fibration $q : X \to S$ and an injective map of simplicial sets $i : K \to S$. For $q_K : X_K \to K$ the fiber of q along i, we have the pullback functor on the spaces of transport functors with witnessing data

$$i^* : \mathscr{T}\mathscr{W}it(q) \to \mathscr{T}\mathscr{W}it(q_K).$$

We've claimed in Proposition 8.2 that i^* is a trivial Kan fibration.

We decompose the proof of Proposition 8.2 into two subclaims. First, we claim that i^* is an inner fibration. We then argue that it is an isofibration, and hence a Kan fibration since both spaces in question are Kan complexes. Triviality follows since both spaces are contractible. Let us demonstrate this first claim.

A.1. Restriction is an inner fibration.

lem:tdata_res

Lemma A.1. *Under the hypotheses of Proposition 8.2, the restriction functor $i^* : \mathscr{T}\mathscr{W}it(q) \to \mathscr{T}\mathscr{W}it(q_K)$ is an inner fibration.*

Proof. Suppose we have a transport functor $F_K : \Delta^n \times K \to \mathscr{C}at_\infty$ for $\Delta^n \times q_K$ with witnessing data

$$\begin{tikzcd} \Delta^n \times X_K \arrow[rr, "g_K"] \arrow[dr, "\Delta^n \times q_K"'] & & \mathscr{E}_K \arrow[dl, "p_K"] \\ & \Delta^n \times K & \end{tikzcd},$$

where $\mathscr{E}_K = \int_{\Delta^n \times K} F_K$ with its associated pullback fibration p_K to $\Delta^n \times K$. Recall that, by definition, g_K is an equivalence of cocartesian fibrations. This data defines an n-simplex

$$(F_K, g_K) : \Delta^n \to \mathscr{T}\mathscr{W}it(q_K).$$

Suppose now that we have an inner horn $(F_\Lambda, g_\Lambda) : \Lambda^n_l \to \mathscr{TWit}(q)$ which restricts to $(F_K, g_K)|_{\Lambda^n_l}$ along i. Equivalently, we have a choice of functor $F_\Lambda : \Lambda^n_l \times S \to \mathscr{C}at_\infty$ and a witnessing data

$$\begin{tikzcd} \Lambda^n_l \times X \arrow[rr, "g_\Lambda"] \arrow[dr, "\Lambda^n_l \times q"'] & & \mathscr{E}_\Lambda \arrow[dl, "p_\Lambda"] \\ & \Lambda^n_l \times S & \end{tikzcd}$$

where $\mathscr{E}' = \int_{\Lambda^n \times S} F_\Lambda$.

Consider the union

$$M = (\Delta^n \times K) \coprod_{(\Lambda^n_l \times K)} (\Lambda^n_l \times S).$$

By Lemma I-5.4 the inclusion $M \to \Delta^n \times S$ is inner anodyne, so that the restriction functor

$$\mathrm{Fun}(\Delta^n \times S, \mathscr{C}at_\infty) \to \mathrm{Fun}(M, \mathscr{C}at_\infty)$$

is a trivial Kan fibration by Corollary I-5.8. Hence the pair of functors

$$(F_K, F_\Lambda) : * \to \mathrm{Fun}(M, \mathscr{C}at_\infty)$$

extend uniquely to a functor $F : \Delta^n \times S \to \mathscr{C}at_\infty$. Let us take now

$$\mathscr{E} = \int_{\Delta^n \times S} F,$$

and note the identifications $\Lambda^n_l \times_{\Delta^n} \mathscr{E} = \mathscr{E}_\Lambda$ and $K \times_S \mathscr{E} = \mathscr{E}_K$.

Now, the inclusion $\Lambda^n \times X \to \Delta^n \times X$ is inner anodyne and the map $\Delta^n \times X_K \to \Delta^n \times X$ is injective so that the map from the union

$$Y = (\Lambda^n \times X) \coprod_{(\Lambda^n \times X_K)} (\Delta^n \times X_k) \to \Delta^n \times X$$

is inner anodyne as well, again by Lemma I-5.4. Hence the induced map

$$\mathrm{Fun}(\Delta^n \times X, \mathscr{E}) \to \mathrm{Fun}(Y, \mathscr{E}) \times_{\mathrm{Fun}(Y, \Delta^n \times S)} \mathrm{Fun}(\Delta^n \times X, \Delta^n \times S)$$

is a trivial Kan fibration, by Proposition I-5.7. It follows that the compatible collection $(g_K, g_\Lambda, \Delta^n \times S) : * \to \mathrm{Fun}(Y, \mathscr{E})$ extend uniquely to a map $G : \Delta^n \times X \to \mathscr{E}$ of isofibrations over $\Delta^n \times S$.

We claim that G preserves cocartesian edges, and hence is a map of cocartesian fibrations over $\Delta^n \times S$. To see this, we note that the sequence

$$\Lambda^n_l \times X \xrightarrow{g_\Lambda} \mathscr{E}_\Lambda \xrightarrow{\mathrm{incl}} \mathscr{E}$$

preserves cocartesian fibrations, since both of the constituent maps preserve cocartesian edges. Since, for any simplicial set T, $T \times q$-cocartesian edges in $T \times X$ are precisely edges those edges of the form (α, β) where α is arbitrary in T and β is q-cocartesian in X. From this we see that all cocartesian edges in $\Delta^n \times X$ are realized as composites of cocartesian edges which exist in the subcomplex $\Lambda^n_l \times X$. Since cocartesian edges are preserved under composition it follows that the functor $G : \Delta^n \times X \to \mathscr{E}$ does in fact preserve cocartesian edges.

Finally, we claim that the functor G is an equivalence. This property can be checked on the fibers over $\Delta^n \times S$, by Theorem 3.8, and so can be checked after pulling back to the subcomplex $\Delta^n \times K$. (Recall that all vertices in S are in the image of the inclusion $K \to S$ by hypothesis.) However, pulling back G along the inclusion $\Delta^n \times K \to \Delta^n \times S$ recovers the functor $g_K : \Delta^n \times K \to \mathscr{E}_K$, which was already chosen to be an equivalence. So we see that G is an equivalence, and the pair (F, G) therefore defines an n-simplex

$$(F, G) : \Delta^n \to \mathscr{T}\mathscr{W}it(q)$$

which splits the given diagram

$$\begin{tikzcd} \Lambda^n \arrow[r, "(F_\Lambda, g_\Lambda)"] \arrow[d] & \mathscr{T}\mathscr{W}it(q) \arrow[d, "i^*"] \\ \Delta^n \arrow[r, "(F_K, g_K)"'] \arrow[ur, "(F,G)"] & \mathscr{T}\mathscr{W}it(q_K). \end{tikzcd}$$

We conclude that i^* is an inner fibration. □

A.2. **Proof of Proposition 8.2.** We now return to the proof of Proposition 8.2.

Proof of Proposition 8.2. By Lemma A.1 the restriction functor $i^* : \mathscr{T}\mathscr{W}it(q) \to \mathscr{T}\mathscr{W}it(q_K)$ is an inner fibration. Since the spaces $\mathscr{T}\mathscr{W}it(q)$ and $\mathscr{T}\mathscr{W}it(q_K)$ are Kan complexes the functor i^* is then a Kan fibration if and only if it is an isofibration. This follows by Corollary I-5.34, for example. Triviality is then a consequence of contractibility of both spaces (see Proposition I-4.9).

Consider a pairing of a functor $F_K : \Delta^1 \times K \to \mathscr{C}at_\infty$ which defines a natural isomorphism in $\mathrm{Fun}(K, \mathscr{C}at_\infty)$ and an equivalence $g_K : \Delta^1 \times X_K \to \mathscr{E}_K$ of cocartesian fibrations over $\Delta^1 \times K$, where $\mathscr{E}_K = \int_{\Delta^1 \times K} F_K$. This data defines a map $\Delta^1 \to \mathscr{T}\mathscr{W}it(q_K)$ whose image in $\mathrm{Fun}(K, \mathscr{C}at_\infty)$ along the forgetful functor $\mathscr{T}\mathscr{W}it(q_K) \to \mathrm{Fun}(K, \mathscr{C}at_\infty)$ is an isomorphism. (All maps in $\mathscr{T}\mathscr{W}it(q_K)$ have this property, as they are all isomorphisms, but whatever.)

Fix a pairing of a functor $F_0 : S \to \mathscr{C}at_\infty$ and an equivalence $g_0 : X \to \mathscr{E}_0$, where $\mathscr{E}_0 = \int_S F$, and suppose that restricting along the inclusion $i : K \to S$ recovers the pair $(F_K|_{\{0\}\times K}, g_K|_{\{0\}\times X_K})$. Equivalently, we consider a lifting problem

$$\begin{tikzcd} \{0\} \arrow[r, "(F_0, g_0)"] \arrow[d] & \mathscr{T}\mathscr{W}it(q) \arrow[d, "i^*"] \\ \Delta^1 \arrow[r, "(F_K, g_K)"'] \arrow[ur, dotted] & \mathscr{T}\mathscr{W}it(q_K) \end{tikzcd}$$

By Corollary I-6.14 the restriction map $\mathrm{Fun}(S, \mathscr{C}at_\infty) \to \mathrm{Fun}(K, \mathscr{C}at_\infty)$ is an isofibration, so that F_K lifts to a natural isomorphism $F : \Delta^1 \to \mathrm{Fun}(S, \mathscr{C}at_\infty)$, which we might also view as a functor $F : \Delta^1 \times S \to \mathscr{C}at_\infty$. We consider the associated cocartesian fibration

$$p : \mathscr{E} = \int_{\Delta^1 \times S} F \to \Delta^1 \times S$$

and note that we have the two inclusions

$$j_0 : \mathscr{E}_0 \to \mathscr{E} \quad \text{and} \quad j_K : \mathscr{E}_K \to \mathscr{E}$$

which recover the implicit fibrations $p_0 : \mathscr{E}_0 \to S$ and $p_K : \mathscr{E}_K \to \Delta^1 \times K$ upon restriction.

By Theorem 2.7 there is a unique transformation $G' : \Delta^1 \times X \to \mathscr{E}$ which fits into a diagram

$$\begin{tikzcd}
\{0\} \times X \arrow[r] \arrow[d] & \mathscr{E} \arrow[d] \\
\Delta^1 \times X \arrow[r, "\Delta^1 \times q"'] \arrow[ur, "G'"] & \Delta^1 \times S
\end{tikzcd}$$

and which sends each edge $(\alpha, x) : \Delta^1 \to \Delta^1 \times X$, at any given vertex $x : * \to \mathscr{E}_0$, to a cocartesian edge in $\mathscr{E}$. We note that G' restricts to such a cocartesian transformation on $\Delta^1 \times X_K$ as well. Since $g_K : \Delta^1 \times X_K \to \mathscr{E}$ provides another such transformation, the uniqueness claim of Theorem 2.7 implies the existence of an isomorphism $g_K \cong G'|_{\Delta^1 \times K}$.

Take $L = (\{0\} \times X) \cup (\Delta^1 \times K)$. Since the restriction functor

$$\mathrm{Fun}(\Delta^1 \times X, \mathscr{E}) \to \mathrm{Fun}(L, \mathscr{E}) \times_{\mathrm{Fun}(L, \Delta^1 \times S)} \mathrm{Fun}(\Delta^1 \times X, \Delta^1 \times S)$$

is an isofibration (Proposition I-6.13) we can therefore replace G' with an isomorphic transformation $G : \Delta^1 \times X \to \mathscr{E}$ which also fits into a diagram of the form (**??**) and has $G|_{\Delta^1 \times K} = g_K$.

We claim that G is a map of cocartesian fibrations. Since cocartesian edges are preserved under isomorphism, it suffices to prove that $G' : \Delta^1 \times X \to \mathscr{E}$ preserves cocartesian edges. However, this is clear since G' sends each edge of the form $(\alpha, x) : \Delta^1 = \Delta^1 \times \{x\} \to \Delta^1 \times X$ to a cocartesian edge in $\mathscr{E}$, and since g_0 preserves cocartesian edges we also see that G' sends each edge of the form $(0, \beta) : \Delta^1 \to \Delta^1 \times X$ with β cocartesian in X to a cocartesian edge in $\mathscr{E}$. Since every cocartesian edge in $\Delta^1 \times X$ is a composite of such basic coartesian edges, and cocartesian edges are stable under composition, we see that G' preserves cocartesian edges. Hence G preserves cocartesian edges as well.

Finally, we claim that $G : \Delta^1 \times X \to \mathscr{E}$ is an equivalence. Since this property can be checked after taking fibers over the base $\Delta^1 \times S$, by Theorem **??**, it suffices to show that the pullback along the inclusion $\Delta^1 \times K \to \Delta^1 \times K$ is an equivalence. (This follows from the fact that every vertex in $\Delta^1 \times S$ lies in the subcomplex $\Delta^1 \times K$, by hypothesis.) However, this pullback is just g_K, by construction, and g_K was chosen to be an equivalence. So G is in fact an equivalence of cocartesian fibrations. We have now found the required solution

$$\begin{tikzcd}
\{0\} \arrow[r, "(F_0, g_0)"] \arrow[d] & \mathscr{T}\mathscr{W}it(q) \arrow[d, "i^*"] \\
\Delta^1 \arrow[r, "F_K, g_K"'] \arrow[ur, "(F,G)"] & \mathscr{T}\mathscr{W}it(q_K)
\end{tikzcd}$$

to our lifting problem. □

Appendix B. Proof of Theorem 8.11

A foundational result from the foundational text [3] is the straightening and unstraightening theorem, which proposes (after some translation [1]) the existence of mutually inverse equivalences

$$\mathrm{St}^{\mathrm{Lur}} : \mathscr{C}ocart(S) \to \mathrm{Fun}(S, \mathscr{C}at_\infty) \quad \text{and} \quad \mathrm{Un}^{\mathrm{Lur}} : \mathrm{Fun}(S, \mathscr{C}at_\infty) \to \mathscr{C}ocart(S).$$

These equivalences are produced via some sequence relatively opaque manipulations with the model categories of marked simplicial sets over S and functors into simplicial sets. Or rather, manipulations which are relatively opaque *to me*.

However, the ultimate conclusion that the ∞-category of cocartesian fibrations over a given base S is equivalent to the ∞-category of $\mathscr{C}at_\infty$-valued functors from S seems unavoidable from the discussions we've had above. Let us explore these higher issues from the perspective of this text.

B.1. **Lifting to** $\mathrm{h}\,\mathscr{K}an$**.** The functor $\mathrm{h}\,\mathrm{St}$ from Corollary 8.10 admits a canonical $\mathrm{h}\,\mathscr{K}an$-enrichment $\pi\,\mathrm{St} : \pi\mathscr{C}ocart(S) \to \pi\,\mathrm{Fun}(S, \mathscr{C}at_\infty)$ which is obtained by applying a transport construction to the mapping spaces. One similarly obtains an enrichment $\pi\,\mathrm{Un} : \pi\,\mathrm{Fun}(S, \mathscr{C}at_\infty) \to \pi\mathscr{C}ocart(S)$ via an application of Corollary 2.8, and one can show that these enriched functors are again mutually inverse.

Having approached the topic in this manner, it is relatively clear that we should have mutually inverse equivalences at the level of ∞-categories

$$\mathrm{St} : \mathscr{C}ocart(S) \to \mathrm{Fun}(S, \mathscr{C}at_\infty) \quad \text{and} \quad \mathrm{Un} : \mathrm{Fun}(S, \mathscr{C}at_\infty) \to \mathscr{C}ocart(S)$$

which lift the equivalences appearing in Corollary 8.10 and the above discussions. Since both of the $\mathrm{h}\,\mathscr{K}an$-enriched functors $\pi\,\mathrm{St}$ and $\pi\,\mathrm{Un}$ are equivalences, it suffices to demonstrate the existence of a lift for one of either $\pi\,\mathrm{St}$ or $\pi\,\mathrm{Un}$.

B.2. **Lifting to the ∞-setting.** If we consider the supposed functor St, the category $\mathscr{C}ocart(S)$ is obtained as the localization of the discrete category $\mathrm{Cocart}(S)$ relative to the class of equivalences

$$\mathscr{C}ocart(S) = \mathrm{Cocart}(S)[\mathrm{Equiv}^{-1}]$$

[3, Proposition 3.1.3.7, Corollary 3.1.4.4] [4, Theorem 1.3.4.20]. So producing a functor $\mathscr{C}ocart(S) \to \mathrm{Fun}(S, \mathscr{C}at_\infty)$ is equivalent to producing a functor $\mathrm{Cocart}(S) \to \mathrm{Fun}(S, \mathscr{C}at_\infty)$ which preserves equivalences, this in turn is equivalent to producing a functor $S \times \mathrm{Cocart}(S) \to \mathscr{C}at_\infty$ which sends equivalences in the second factor to equivalences in $\mathscr{C}at_\infty$, and this final claim is equivalent to the construction of a cocartesian fibration

$$\bar{q}_{\mathrm{St}} : E_{\mathrm{St}} \to S \times \mathrm{Cocart}(S).$$

On obvious choice for the fibration E is given by the weighted nerve

$$E_{\mathrm{St}} = \mathrm{N}^{forget}(\mathrm{Cocart}(S))$$

of the forgetful functor $\mathrm{Cocart}(S) \to \mathrm{Cat}_\infty$. The structure map

$$E_{\mathrm{St}} = \mathrm{N}^{forget}(\mathrm{Cocart}(S)) \to \mathrm{N}^{\underline{S}}(\mathrm{Cocart}(S)) = S \times \mathrm{Cocart}(S)$$

is provided by the natural transformation to the constant functor $forget \to \underline{S}$ induced by the structure maps on objects in $\mathrm{Cocart}(S)$.

Note that the fibers of E_{St} over each point $id_S \times q : S \to S \times \mathrm{Cocart}(S)$ returns the chosen fibration q, i.e. we observe a pullback diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "q"'] & E_{\mathrm{St}} \arrow[d, "\bar{q}_{\mathrm{St}}"] \\
S \arrow[r, "1\times q"'] & S \times \mathrm{Cocart}(S),
\end{tikzcd}$$

and similarly see that pulling back along any map of cocartesian fibrations $F : \Delta^1 \to \mathrm{Cocart}(S)$ returns the fibration

$$E_{\mathrm{St}} \times_{\mathrm{Cocart}(S)} \Delta^1 = \mathrm{N}^F(\Delta^1) \to \Delta^1 \times S.$$

We take $\bar{\mathrm{St}} : S \times \mathrm{Cocart}(S) \to \mathscr{C}at_\infty$ the corresponding transport functor, observe from Proposition 8.8 that $\bar{\mathrm{St}}$ preserves equivalences in the second factor, and take

$$\mathrm{St} : S \times \mathscr{C}ocart(S) \to \mathscr{C}at_\infty$$

the unique extension of $\bar{\mathrm{St}}$ to the localization $S \times \mathscr{C}ocart(S) = S \times \mathrm{Cocart}(S)[\mathrm{Equiv}^{-1}]$. Take $q_{\mathrm{St}} : \mathscr{E}_{\mathrm{St}} \to S \times \mathscr{C}ocart(S)$ the cocartesian fibration corresponding to $|opnSt$.

It is clear from construction that the induced map on homotopy categories $\mathrm{h}\,\mathrm{St}$, which is calculable from the restricted functor $\bar{\mathrm{St}}$, recovers our equivalence from Corollary 8.10. The action of the functor St on mapping spaces can be calculated by applying transport to the pullback fibration

$$\begin{tikzcd}
\mathscr{E}_{X,Y} \arrow[r] \arrow[d] & \mathscr{E}_{\mathrm{St}} \arrow[d] \\
S \times \Delta^1 \times \mathrm{Hom}_{\mathscr{C}ocart}(X,Y) \arrow[r, "1\times\mathrm{ev}"'] & S \times \mathscr{C}ocart(S).
\end{tikzcd}$$

This fibration has $\mathscr{E}_{X,Y}|_0 = \mathrm{Hom}_{\mathscr{C}ocart}(X,Y) \times X$ and $\mathscr{E}_{X,Y}|_1 = \mathrm{Hom}_{\mathscr{C}ocart}(X,Y) \times Y$, and the unique cocartesian lifting

$$\begin{tikzcd}
\{0\} \times \mathrm{Hom}(X,Y) \times X \arrow[r] \arrow[d] & \mathscr{E}_{X,Y} \arrow[d] \\
\Delta^1 \times \mathrm{Hom}(X,Y) \times X \arrow[r] & \Delta^1 \times \mathrm{Hom}(X,Y) \times S
\end{tikzcd}$$

provides maps

$$\mathrm{Hom}(X,Y) \times X = \{1\} \times \mathrm{Hom}(X,Y) \times X \to \mathrm{Hom}(X,Y) \times Y$$

which appear as $[p_1 \;\; ev']$, where ev' is some map $ev' : \mathrm{Hom}(X,Y) \times X \to Y$. These maps ev' provide, via adjunctions, a unique morphism

$$\mathrm{Hom}_{\mathscr{C}ocart}(X,Y) \to \mathrm{Fun}_S(X,Y)$$

which fits into a

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