

GAUGE INVARIANTS FROM THE POWERS OF ANTIPODES

CRIS NEGRON AND SIU-HUNG NG

ABSTRACT. Let S be the antipode of a finite-dimensional Hopf algebra H with the Chevalley property. We prove that the trace of S^n is gauge invariant for any integer n . As a consequence, the orders of S and S^2 are invariant under Drinfeld twists. The invariance of the order of S^2 is closely related to a question of Shimizu on the pivotal cover of finite tensor category $H\text{-mod}$, which is affirmatively answered for Hopf algebras with the Chevalley property.

1. INTRODUCTION

The present paper is dedicated to a study of the traces of the powers of the antipode of a Hopf algebra, and an approach to the Frobenius-Schur indicators of non-semisimple Hopf algebras (following [14, 33]).

The antipode of a Hopf algebra has emerged as an object of importance in the study of Hopf algebras. It has been proved by Radford that the order of the antipode S of any finite-dimensional Hopf algebra H is finite [28]. Moreover, the trace of S^2 is nonzero if, and only if, H is semisimple and cosemisimple [18]. If the base field \mathbb{k} is of characteristic zero, $\text{Tr}(S^2) = \dim H$ or 0, which characterizes respectively whether H semisimple or nonsemisimple [17]. This means semisimplicity of H is characterized by the value of $\text{Tr}(S^2)$. In particular, $\text{Tr}(S^2)$ is an invariant of the finite tensor category $H\text{-mod}$.

Using a recent theory of indicators introduced in [14], one can also show that the trace of the antipode $\text{Tr}(S)$ is an invariant as well, and one can use a related result of Shimizu [33] to get invariance of the trace of S^2 in arbitrary characteristic. Indeed, the second and zeroth generalized indicators $\nu_2^{\text{KMN}}(H)$ and $\nu_0^{\text{KMN}}(H)$ are $\text{Tr}(S)$ and $\text{Tr}(S^2)$ respectively, and the indicators have been shown to be invariants of the category $H\text{-mod}$.

The invariance of $\text{Tr}(S)$ and $\text{Tr}(S^2)$ alludes to the following question to be investigated in this paper:

Question 1.1. For any finite-dimensional Hopf algebra H with the antipode S , is the sequence $\{\text{Tr}(S^n)\}_{n \in \mathbb{N}}$ an invariant of the finite tensor category $H\text{-mod}$?

For the purposes of this paper, we will always assume \mathbb{k} to be an algebraic closed field of characteristic zero, and all Hopf algebras are finite-dimensional over \mathbb{k} .

Recall that a finite-dimensional Hopf algebra H has the Chevalley property if its Jacobson radical is a Hopf ideal. Equivalently, H has the Chevalley property if the full subcategory of sums of irreducible modules in $H\text{-mod}$ forms a tensor subcategory. We provide a positive answer of Question 1.1 for Hopf algebras with the Chevalley property.

Theorem A (Thm 4.3). *Let H and K be finite-dimensional Hopf algebras over \mathbb{k} with antipodes S_H and S_K respectively. Suppose H has the Chevalley property and that $H\text{-mod}$ and $K\text{-mod}$ are equivalent as tensor categories. Then we have*

$$\mathrm{Tr}(S_H^n) = \mathrm{Tr}(S_K^n)$$

for all integers n .

In a categorical language, the theorem tells us that for any finite tensor category \mathcal{C} with the Chevalley property which admits a fiber functor to the category of vector spaces, the “traces of the powers of the antipode” are well defined invariants which are independent of the choice of fiber functor. Rather, these scalars are intrinsic to the category \mathcal{C} itself.

Etingof has asked the question whether, for any finite-dimensional H , $\mathrm{Tr}(S^{2m}) = 0$ provided $\mathrm{ord}(S^2) \nmid m$ [30, p186]. This question is affirmatively answered for pointed and dual pointed Hopf algebras in [30]. However, the odd powers of the antipode may have nonzero traces in general. It may be worth noting that the above result covers both the even and odd powers of the antipode. We also note that the trace of the first power of the antipode is an invariant for any finite-dimensional Hopf algebras over any field [14].

Theorem A also implies that orders of the first two powers of the antipode of a Hopf algebra with the the Chevalley property is also an invariant.

Corollary A (Cor. 4.4). *Let H and K be finite-dimensional Hopf algebras over \mathbb{k} with antipodes S_H and S_K respectively. Suppose H has the Chevalley property and that $H\text{-mod}$ and $K\text{-mod}$ are equivalent tensor categories. Then $\mathrm{ord}(S_H) = \mathrm{ord}(S_K)$ and hence $\mathrm{ord}(S_H^2) = \mathrm{ord}(S_K^2)$.*

The questions under consideration here are closely related to recent investigations of Shimizu into the Frobenius-Schur indicators of non-semisimple Hopf algebras [33]. Namely, both projects essentially reduce to an analysis of certain units associated to *Drinfeld twists* of a Hopf algebra H (see 2.1).

The 2nd Frobenius-Schur indicator $\nu_2(V)$ of an irreducible complex representation of a finite group was introduced in [10]; the notion was then extended to semisimple Hopf algebras, quasi-Hopf algebras, certain C^* -fusion categories and conformal field theory (cf. [19, 21, 11, 2]). Higher Frobenius-Schur indicators $\nu_n(V)$ for semisimple Hopf algebra have been extensively studied in [15]. In the most general context, Frobenius-Schur (FS-) indicators can be defined for each object V in a *pivotal* tensor category \mathcal{C} , and they are invariants of these tensor categories [24].

The n th Frobenius-Schur indicators $\nu_n(H)$ of the regular representation of a semisimple Hopf algebra H in particular is an invariant of the fusion category $H\text{-mod}$ (cf. [24] and [25, Thm. 2.2]). For this special representation it is obtained in [15] that

$$\nu_n(H) = \text{Tr}(S \circ P_{n-1}) \quad (1.1)$$

where P_k denotes the k th convolution power of identity map id_H in $\text{End}_{\mathbb{k}}(H)$. On elements, the map $S \circ P_{n-1}$ is given by $h \mapsto S(h_1 \dots h_{n-1})$.

The importance of the FS-indicators is illustrated in their applications to semisimple Hopf algebras and *spherical* fusion categories (see for examples [15], [23], [26], [4], [3] and [27]). The arithmetic properties of the values of the FS-indicators have played an integral role in all these applications, and remains the main interest of FS-indicators (see for example [12], [13], [32] and [31]).

It would be tempting to extend the notion of FS-indicators for the study of finite tensor categories or non-semisimple Hopf algebras. One would expect that such a *generalized* indicator for a general Hopf algebra H should coincide with the existing one when H is semisimple.

The introduction of (what we refer to as) the KMN-indicators $\nu_n^{\text{KMN}}(H)$ in [14] is an attempt at this endeavor. Note that the right hand side of (1.1), $\text{Tr}(S \circ P_{n-1})$ is well-defined for any finite-dimensional Hopf algebra over any based field, and we denote it $\nu_n^{\text{KMN}}(H)$. It has been shown in [14] that the numbers $\nu_n^{\text{KMN}}(H)$ are an invariant of the finite tensor category $H\text{-mod}$ for all positive integers n . In particular, $\nu_2^{\text{KMN}}(H) = \text{Tr}(S)$ is an invariant of $H\text{-mod}$. However, this definition for the indicators for the regular representation in $H\text{-mod}$ cannot be extended to other object in $H\text{-mod}$.

In [33] Shimizu lays out an alternate, categorial approach to generalized indicators for a non-semisimple Hopf algebra H . He first constructs a *universal pivotalization* $(H\text{-mod})^{\text{piv}}$ of $H\text{-mod}$, i.e. a pivotal tensor category with a fixed monoidal functor $\Pi : (H\text{-mod})^{\text{piv}} \rightarrow H\text{-mod}$ which is universal among all such categories. Then, the scalar $\nu_n^{\text{KMN}}(H)$ can be recovered from a new version of the n th indicator $\nu_n^{\text{Sh}}(\mathbf{R}_H^*)$ of the dual of the *regular object* \mathbf{R}_H in $(H\text{-mod})^{\text{piv}}$ (see 5.3). The universal pivotalization is natural in the sense that for any monoidal functor $\mathcal{F} : H\text{-mod} \rightarrow K\text{-mod}$, where K is a Hopf algebra, there exists a unique pivotal functor $\mathcal{F}^{\text{piv}} : (H\text{-mod})^{\text{piv}} \rightarrow (K\text{-mod})^{\text{piv}}$ compatible with both Π and \mathcal{F} .

However, the invariance of $\nu_n^{\text{KMN}}(H)$ does not follow immediately from this categorial framework. Instead, it would be a consequence of an isomorphism $\mathcal{F}^{\text{piv}}(\mathbf{R}_H) \cong \mathbf{R}_K$ associated to any monoidal equivalence $\mathcal{F} : H\text{-mod} \rightarrow K\text{-mod}$. While the latter condition remains open in general, we show below that the regular objects are preserved under monoidal equivalence for Hopf algebras with the Chevalley property.

Theorem B. *Let H and K be Hopf algebras with the Chevalley property and $\mathcal{F} : H\text{-mod} \rightarrow K\text{-mod}$ an equivalence of tensor categories. Then*

the induced pivotal equivalence $\mathcal{F}^{\text{piv}} : (H\text{-mod})^{\text{piv}} \rightarrow (K\text{-mod})^{\text{piv}}$ on the universal pivotalizations satisfies $\mathcal{F}^{\text{piv}}(\mathbf{R}_H) \cong \mathbf{R}_K$.

This gives a positive solution to [33, Quest. 5.12]. From Theorem B we recover the gauge invariance result of [14], in the specific case of Hopf algebras with the Chevalley property.

Corollary B (cf. [14, Thm. 2.2]). *Suppose H and K are Hopf algebras with the Chevalley property and have equivalent tensor categories of representations. Then $\nu_n^{\text{KMN}}(H) = \nu_n^{\text{KMN}}(K)$.*

The paper is organized as follows: Section 2 recalls some basic notions and results on Hopf algebras and pivotal tensor categories. In Section 3, we prove that a specific element γ_F for any Drinfeld twist F of a semisimple Hopf algebra H is fixed by the antipode of H , using the pseudounitary structure of $H\text{-mod}$. We proceed to prove Theorem A and Corollary A in Section 4. In Section 5, we recall the universal pivotalization $H\text{-mod}^{\text{piv}}$ of Shimizu for $H\text{-mod}$, his definition of n th indicators for an object in $H\text{-mod}^{\text{piv}}$ and their relations to $\nu_n^{\text{KMN}}(H)$. In Section 6, we introduce finite pivotalizations of $H\text{-mod}$ and, in particular, the exponential pivotalization which contains all the possible pivotal categories defined on $H\text{-mod}$. In Section 7, we answer the question of Shimizu on the preservation of regular objects for Hopf algebra with the Chevalley property.

2. PRELIMINARIES

Throughout this paper, we assume some basic definitions on Hopf algebras and monoidal categories. We denote the antipode of a Hopf algebra H by S_H or, when no confusion will arise, simply by S . Tensor categories in this paper are \mathbb{k} -linear abelian monoidal category with simple unit object $\mathbf{1}$. A monoidal functor between two tensor categories is a pair (\mathcal{F}, ξ) in which \mathcal{F} is a \mathbb{k} -linear functor satisfying $\mathcal{F}(\mathbf{1}) = \mathbf{1}$, and $\xi_{V,W} : \mathcal{F}(V) \otimes \mathcal{F}(W) \rightarrow \mathcal{F}(V \otimes W)$ is the coherence isomorphism. If the context is clear, we may simply write \mathcal{F} for the pair (\mathcal{F}, ξ) . The readers are referred to [16, 22] for the details.

2.1. Gauge equivalence, twists, and the antipode. Let H be a finite-dimensional Hopf algebras over \mathbb{k} with the antipode S , comultiplication Δ and counit ϵ . The category $H\text{-mod}$ of finite-dimensional representations of H is a finite tensor categories in the sense of [7]. For $V \in H\text{-mod}$, the dual vector space V^* of V admits the natural right H -action \leftarrow given by

$$(v^* \leftarrow h)(v) = v^*(hv)$$

for $h \in H$, $v^* \in V^*$ and $v \in V$. The left dual V^* of V is the vector space V^* endowed with the left H -action defined by

$$hv^* = v^* \leftarrow S(h)$$

for $h \in H$, $v^* \in V'$, the usual evaluation $\text{ev} : V^* \otimes V \rightarrow \mathbb{k}$ and the dual basis map as the coevaluation $\text{coev} : \mathbb{k} \rightarrow V \otimes V^*$. The right dual of V is defined similarly with S replaced by S^{-1} .

Suppose K is another finite-dimensional Hopf algebra over \mathbb{k} such that $K\text{-mod}$ and $H\text{-mod}$ are equivalent monoidal categories. It follows from [25, Thm. 2.2] that there is a gauge transformation $F = \sum_i f_i \otimes g_i \in H \otimes H$ (cf. [16]), which is an invertible element satisfying

$$(\epsilon \otimes \text{id})(F) = 1 = (\text{id} \otimes \epsilon)(F),$$

such that the map $\Delta^F : H \rightarrow H \otimes H, h \mapsto F\Delta(h)F^{-1}$ together with the counit ϵ and the algebra structure of H form a bialgebra H^F and that $K \cong_{\sigma} H^F$ as bialgebras. In particular, H^F is a Hopf algebra with the antipode give by

$$S_F(h) = \beta_F S(h) \beta_F^{-1} \quad (2.1)$$

where $\beta_F = \sum_i f_i S(g_i)$. Following the terminology of [16] (cf. [14]), we say that K and H are *gauge equivalent* if the categories of their finite-dimensional representations are equivalent monoidal categories. A quantity $f(H)$ obtained from a finite-dimensional Hopf algebra H is called a *gauge invariant* if $f(H) = f(K)$ for any Hopf algebra K gauge equivalent H . For instance, $\text{Tr}(S)$ and $\text{Tr}(S^2)$ are gauge invariants of H .

If $F^{-1} = \sum_i d_i \otimes e_i$, then $\beta_F^{-1} = \sum_i S(d_i) e_i$. For the purpose of this paper, we set $\gamma_F = \beta_F S(\beta_F^{-1})$ and so, by (2.1), we have

$$S_F^2(h) = \gamma_F S^2(h) \gamma_F^{-1} \quad (2.2)$$

for $h \in H$.

Since the associativities of K and H are given by $1 \otimes 1 \otimes 1$, the gauge transformation F satisfies the condition

$$(1 \otimes F)(\text{id} \otimes \Delta)(F) = (F \otimes 1)(\Delta \otimes \text{id})(F). \quad (2.3)$$

This is a necessary and sufficient condition for Δ^F to be coassociative. A gauge transformation $F \in H \otimes H$ satisfying equation (2.3) is often called a *Drinfeld twist* or simply a *twist*.

Suppose $F \in H \otimes H$ is a twist and $K \cong_{\sigma} H^F$ as Hopf algebras. Following [16], one can define a monoidal equivalence $(\mathcal{F}_{\sigma}, \xi^F) : H\text{-mod} \rightarrow K\text{-mod}$. For $V \in H\text{-mod}$, $\mathcal{F}_{\sigma}(V)$ is the left K -module with the action given by $k \cdot v := \sigma(k)v$ for $k \in K$ and $v \in V$. The assignment $V \mapsto \mathcal{F}_{\sigma}(V)$ defines \mathbb{k} -linear equivalence from $H\text{-mod}$ to $K\text{-mod}$ with identity action on the morphisms. Together with the natural isomorphism $\xi^F : \mathcal{F}_{\sigma}(V) \otimes \mathcal{F}_{\sigma}(W) \rightarrow \mathcal{F}_{\sigma}(V \otimes W)$ defined by the action of F^{-1} on $V \otimes W$, the pair $(\mathcal{F}_{\sigma}, \xi^F) : H\text{-mod} \rightarrow K\text{-mod}$ is an equivalence of monoidal categories. If $K = H^F$ for some twist $F \in H \otimes H$, then $(\text{Id}, \xi^F) : H\text{-mod} \rightarrow H^F\text{-mod}$ is an equivalence of monoidal categories since \mathcal{F}_{id} is the identity functor Id .

2.2. Pivotal categories. For any finite tensor category \mathcal{C} with the unit object $\mathbf{1}$, the left duality can define a functor $(-)^* : \mathcal{C} \rightarrow \mathcal{C}$ and the double dual functor $(-)^{**} : \mathcal{C} \rightarrow \mathcal{C}$ is an equivalence of monoidal categories. A pivotal structure of \mathcal{C} is an isomorphism $j : \text{Id} \rightarrow (-)^{**}$ of monoidal functors. Associated with a pivotal structure j is the notions of *trace* and *dimension*: For any $V \in \mathcal{C}$ and $f : V \rightarrow V$, one can define $\underline{\text{ptr}}(f)$ as the scalar of the composition

$$\underline{\text{ptr}}(f) := (\mathbf{1} \xrightarrow{\text{coev}} V \otimes V^* \xrightarrow{f \otimes V^*} V \otimes V^* \xrightarrow{j \otimes V^*} V^{**} \otimes V^* \xrightarrow{\text{ev}} \mathbf{1})$$

and $d(V) = \underline{\text{ptr}}(\text{id}_V)$. A finite tensor category with a specified pivotal structure is called a *pivotal category*.

Suppose \mathcal{C} and \mathcal{D} pivotal categories with the pivotal structures j and j' respectively, and $(\mathcal{F}, \xi) : \mathcal{C} \rightarrow \mathcal{D}$ is a monoidal functor. Then there exists a unique natural isomorphism $\tilde{\xi} : \mathcal{F}(V^*) \rightarrow \mathcal{F}(V)^*$ which is determined by either of the following commutative diagrams (cf. [24, p67]):

$$\begin{array}{ccc} \mathcal{F}(V^*) \otimes \mathcal{F}(V) & \xrightarrow{\tilde{\xi} \otimes \mathcal{F}(V)} & \mathcal{F}(V^*)^* \otimes \mathcal{F}(V) \\ \downarrow \xi & & \downarrow \xi \\ \mathcal{F}(V^* \otimes V) & \xrightarrow{\mathcal{F}(\text{ev})} & \mathbf{1} \end{array} \quad \text{or} \quad \begin{array}{ccc} \mathcal{F}(V^*) \otimes \mathcal{F}(V) & \xrightarrow{\tilde{\xi} \otimes \mathcal{F}(V)} & \mathcal{F}(V^*)^* \otimes \mathcal{F}(V) \\ \downarrow \xi & & \uparrow \text{coev} \\ \mathcal{F}(V^* \otimes V) & \xleftarrow{\mathcal{F}(\text{coev})} & \mathbf{1}. \end{array} \quad (2.4)$$

The monoidal functor (\mathcal{F}, ξ) is said to be *pivotal* if it preserves the pivotal structures, which means the commutative diagram

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{\mathcal{F}(j_V)} & \mathcal{F}(V^{**}) \\ j'_{\mathcal{F}(V)} \downarrow & & \downarrow \tilde{\xi} \\ \mathcal{F}(V)^{**} & \xrightarrow{\tilde{\xi}^*} & \mathcal{F}(V^*)^* \end{array} \quad (2.5)$$

is satisfied for $V \in \mathcal{C}$. It follows immediately from definitions that pivotal monoidal equivalence preserves dimensions. More precisely, if $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ is an equivalence of pivotal tensor categories, then $d(V) = d(\mathcal{F}(V))$ for $V \in \mathcal{C}$.

3. SEMISIMPLE HOPF ALGEBRAS AND PSEUDO-UNITARY FUSION CATEGORIES

In general, a finite tensor category may not have a pivotal structure. However, all the known semisimple finite tensor categories, also called *fusion categories*, over \mathbb{k} , admits a pivotal structure. It remains to be an open question whether every fusion category admits a pivotal structure (cf. [6]). We present an equivalent definition of *pseudo-unitary* fusion categories obtained in [6] as in the following proposition.

Proposition 3.1. [6] *A fusion category \mathcal{C} over \mathbb{k} is pseudo-unitary if \mathcal{C} admits a pivotal structure $j^{\mathcal{C}}$ whose associated dimensions are all real and nonnegative. In this case, this pivotal structure is unique and the associated pivotal dimension is identical to the Frobenius-Perron dimension.*

The pivotal structure of a pseudo-unitary \mathcal{C} with real and nonnegative dimensions will be called the *canonical* pivotal structure of \mathcal{C} . In particular, the fusion category $H\text{-mod}$ of a finite-dimensional semisimple quasi-Hopf algebra H is pseudo-unitary and the pivotal dimension of an H -module V associated with the canonical pivotal structure of $H\text{-mod}$ is simply the ordinary dimension of V (cf. [6]).

The canonical pivotal structure j^{Vec} on the *trivial* fusion category Vec of finite-dimensional \mathbb{k} -linear space is just the usual vector space isomorphism $V \rightarrow V^{**}$, which sends an element $v \in V$ to the evaluation function $\hat{v} : V^* \rightarrow \mathbb{k}$, $f \mapsto f(v)$.

Let H be a finite-dimensional semisimple Hopf algebra over \mathbb{k} . Then the antipode S of H satisfies $S^2 = \text{id}$ (cf. [17]). Thus, for $V \in H\text{-mod}$, the natural isomorphism $j^{\text{Vec}} : V \rightarrow V^{**}$ of vector space is an H -module map. In fact, j^{Vec} provides a pivotal structure of $H\text{-mod}$ and the associated pivotal dimension $d(V)$ of V , given by the composition map

$$\mathbb{k} \xrightarrow{\text{coev}} V \otimes V^* \xrightarrow{j^{\otimes V^*}} V^{**} \otimes V^* \xrightarrow{\text{ev}} \mathbb{k},$$

is equal to its ordinary dimension $\dim V$, which is real and nonnegative. Therefore, j^{Vec} is the canonical pivotal structure of $H\text{-mod}$.

By [24, Cor. 6.2], the canonical pivotal structure of a pseudo-unitary fusion category is preserved by any monoidal equivalence of fusion categories. For the purpose of this article, we restate this statement in the context of semisimple Hopf algebras.

Corollary 3.2. [24, Cor. 6.2] *Let H and K be finite-dimensional semisimple Hopf algebras over \mathbb{k} . If $(\mathcal{F}, \xi) : H\text{-mod} \rightarrow K\text{-mod}$ defines a monoidal equivalence, then (\mathcal{F}, ξ) preserves their canonical pivotal structures, i.e. they satisfy the commutative diagram (2.5). In particular, if $K \stackrel{\sigma}{\cong} H^F$ as Hopf algebras for some twist $F \in H \otimes H$, then the monoidal equivalence $(\mathcal{F}_\sigma, \xi^F) : H\text{-mod} \rightarrow K\text{-mod}$ preserves their canonical pivotal structures.*

Now, we can prove the following condition on a twist of a semisimple Hopf algebra.

Theorem 3.3. *Let H be a semisimple Hopf algebra over \mathbb{k} with the antipode S , $F = \sum_i f_i \otimes g_i \in H \otimes H$ a twist and $\beta_F = \sum_i f_i S(g_i)$. Then*

$$S(\beta_F) = \beta_F.$$

Proof. Let $F^{-1} = \sum_i d_i \otimes e_i$. then $\beta^{-1} = \sum_i S(d_i)e_i$ (cf. Section 2), where β_F is simply abbreviated as β . For $V \in H\text{-mod}$, we denote by V^* and V^\vee respectively the left duals of V in $H\text{-mod}$ and $H^F\text{-mod}$. It follows from (2.4) that the duality transformation $\tilde{\xi}^F : V^* \rightarrow V^\vee$, for $V \in H\text{-mod}$, of the monoidal equivalence $(\text{Id}, \xi^F) : H\text{-mod} \rightarrow H^F\text{-mod}$, is given by

$$\tilde{\xi}^F(v^*) = v^* \leftarrow \beta^{-1} \tag{3.1}$$

for all $v^* \in V^*$. Since both H and H^F are semisimple, their canonical pivotal structures are the same as the usual natural isomorphism j^{Vec} of finite-dimensional vector spaces over \mathbb{k} . Since (Id, ξ^F) preserves the canonical pivotal structures, by (2.5), we have

$$\begin{aligned} \tilde{\xi}^F(j^{\text{Vec}}(v))(v^*) &= (\tilde{\xi}^F)^*(j^{\text{Vec}}(v))(v^*) \\ &= j^{\text{Vec}}(v)(\tilde{\xi}^F(v^*)) = (v^* \leftarrow \beta^{-1})(v) = v^*(\beta^{-1}v) \end{aligned}$$

for all $v \in V$ and $v^* \in V^*$. Rewriting the first term of this equation, we find

$$v^*(S(\beta^{-1}v)) = v^*(\beta^{-1}v).$$

This implies $\beta^{-1} = S(\beta^{-1})$ by taking $V = H$ and $v = 1$. \square

4. HOPF ALGEBRAS WITH THE CHEVALLEY PROPERTY

A finite-dimensional Hopf algebra H over \mathbb{k} is said to have the *Chevalley property* if the Jacobson radical $J(H)$ of H is a Hopf ideal. In this case, $\overline{H} = H/J(H)$ is a semisimple Hopf algebra and the natural surjection $\pi : H \rightarrow \overline{H}$ is a Hopf algebra map. Let $F \in H \otimes H$ be a twist of H . Then

$$\overline{F} := (\pi \otimes \pi)(F) \in \overline{H} \otimes \overline{H}$$

is a twist and so

$$\pi(\beta_F) = \beta_{\overline{F}} = \overline{S}(\beta_{\overline{F}}) = \pi(S(\beta_F))$$

by Theorem 3.3, where \overline{S} denotes the antipode of \overline{H} . Therefore, $S(\beta_F) \in \beta_F + J(H)$ and this proves

Lemma 4.1. *Let H be a finite-dimensional Hopf algebra over \mathbb{k} with the Chevalley property. For any twist $F \in H \otimes H$,*

$$S(\beta_F) \in \beta_F + J(H).$$

We will need the following lemma.

Lemma 4.2. *Let A be a finite-dimensional algebra over \mathbb{k} and T an algebra endomorphism or anti-endomorphism of A .*

(i) *For any $x \in J(A)$ and $a \in A$,*

$$l(x)r(a)T, \quad l(a)r(x)T$$

are nilpotent operators, where $l(x)$ and $r(x)$ respectively denote the left and the right multiplication by x .

(ii) *For any $a, a', b, b' \in A$ such that $a' \in a + J(A)$ and $b' \in b + J(A)$, we have*

$$\text{Tr}(l(a)r(b)T) = \text{Tr}(l(a')r(b')T).$$

Proof. (i) Let n be a positive integer such that $J(A)^n = 0$. We first consider the case when T is an algebra endomorphism of A . Then

$$\begin{aligned} (l(a)r(x)T)^n &= l(a)l(T(a)) \cdots l(T^{n-1}(a))r(T^{n-1}(x)) \cdots r(T(x))r(x)T^n \\ &= l(aT(a) \cdots T^{n-1}(a))r(T^{n-1}(x) \cdots T(x)x)T^n. \end{aligned}$$

Since $J(A)^n = 0$ and $x, T(x), \dots, T^{n-1}(x) \in J(A)$,

$$T^{n-1}(x) \cdots T(x)x = 0.$$

Therefore, $(l(a)r(x)T)^n = 0$. By the same argument, one can show that $(l(x)r(a)T)^n = 0$. In particular, they are nilpotent operators.

If T is an algebra anti-automorphism on A , then

$$(l(a)r(x)T)^2 = l(aT(x))r(T(a)x)T^2.$$

Since T^2 is an algebra endomorphism of A and $aT(x) \in J(A)$, we have $(l(a)r(x)T)^{2n} = 0$. Similarly, $(l(x)r(a)T)^{2n} = 0$.

(ii) Let $a' = a + x$ and $b' = b + y$ for some $x, y \in J(A)$.

$$l(a')r(b')T = l(a)r(b)T + l(x)r(b')T + l(a)r(y)T.$$

By (i), $l(x)r(b')T$ and $l(a)r(y)T$ are nilpotent operators, and so the result follows. \square

We can now prove that the traces of the powers of the antipode of a Hopf algebra with the Chevalley property are gauge invariants.

Theorem 4.3. *Let H be a Hopf algebra over \mathbb{k} with the antipode S . Suppose H has the Chevalley property. Then for any twist $F \in H \otimes H$, we have*

$$\mathrm{Tr}(S_F^n) = \mathrm{Tr}(S^n)$$

for all integers n , where S_F is the antipode of H^F . Moreover, if K is another Hopf algebra over \mathbb{k} with antipode S' which is gauge equivalent to H , then

$$\mathrm{Tr}(S^n) = \mathrm{Tr}(S'^n)$$

for all integers n .

Proof. By (2.1), the antipode S_F of H^F is given by

$$S_F(h) = \beta_F S(h) \beta_F^{-1}$$

for $h \in H$. Recall from (2.2) that

$$S_F(h) = \gamma_F S(h) \gamma_F^{-1}$$

where $\gamma_F = \beta_F S(\beta_F^{-1})$. Then, for any nonnegative integer n , we can write $S_F^n = l(u_n)r(u_n^{-1})S^n$ where $u_0 = 1$ and

$$u_n = \begin{cases} \gamma_F S^2(\gamma_F) \cdots S^{n-2}(\gamma_F) & \text{if } n \text{ is positive and even,} \\ \beta_F S(u_{n-1}^{-1}) & \text{if } n \text{ is odd.} \end{cases}$$

Thus, if n is an even positive integer, $u_n \in 1 + J(H)$ by Lemma 4.1. It follows from Lemma 4.2 that

$$\mathrm{Tr}(S_F^n) = \mathrm{Tr}(l(u_n)r(u_n^{-1})S^n) = \mathrm{Tr}(l(1)r(1)S^n) = \mathrm{Tr}(S^n).$$

From now, we assume n is odd. Then $u_n \in \beta_F + J(H)$ and so we have

$$\begin{aligned} \mathrm{Tr}(S_F^n) &= \mathrm{Tr}(l(u_n)r(u_n^{-1})S^n) = \mathrm{Tr}(l(\beta_F)r(\beta_F^{-1})S^n) \\ &= \mathrm{Tr}(l(\beta_F)r(S^n(\beta_F^{-1}))S^n). \end{aligned} \tag{4.1}$$

The last equality of the above equation follows from Lemmas 4.1 and 4.2(ii).

Let Λ be a left integral of H and λ a right integral of H^* such that $\lambda(\Lambda) = 1$. By [29, Thm. 2],

$$\mathrm{Tr}(T) = \lambda(S(\Lambda_2)T(\Lambda_1))$$

for any \mathbb{k} -linear endomorphism T on H , where $\Delta(\Lambda) = \Lambda_1 \otimes \Lambda_2$ is the Sweedler notation with the summation suppressed. Thus, by (4.1), we have

$$\mathrm{Tr}(S_F^n) = \lambda(S(\Lambda_2)\beta_F S^n(\Lambda_1)S^n(\beta_F^{-1})) = \lambda(S(\Lambda_2)\beta_F S^n(\beta_F^{-1}\Lambda_1)). \quad (4.2)$$

Recall from [29, p591] that

$$\Lambda_1 \otimes a\Lambda_2 = S(a)\Lambda_1 \otimes \Lambda_2$$

for all $a \in H$. Using this equality and (4.2), we find

$$\begin{aligned} \mathrm{Tr}(S_F^n) &= \lambda(S(\Lambda_2)\beta_F S^n(\beta_F^{-1}\Lambda_1)) = \lambda(S(S^{-1}(\beta_F^{-1})\Lambda_2)\beta_F S^n(\Lambda_1)) \\ &= \lambda(S(\Lambda_2)\beta_F^{-1}\beta_F S^n(\Lambda_1)) = \lambda(S(\Lambda_2)S^n(\Lambda_1)) = \mathrm{Tr}(S^n). \quad \square \end{aligned}$$

The second statement of the theorem then follows immediately from Corollary 3.2.

Corollary 4.4. *If H is a finite-dimensional Hopf algebra over \mathbb{k} with the Chevalley property, then $\mathrm{ord}(S)$ is a gauge invariant. In particular, $\mathrm{ord}(S^2)$ is a gauge invariant.*

Proof. Since \mathbb{k} is of characteristic zero, $\mathrm{Tr}(S^n) = \dim H$ if and only if $S^n = \mathrm{id}$. In particular, $\mathrm{ord}(S)$ is the smallest positive integer n such that $\mathrm{Tr}(S^n) = \dim H$. If K is a Hopf algebra (over \mathbb{k}) with the antipode S' and is gauge equivalent to H , then $\dim K = \dim H$ by Corollary 3.2. Hence, by Theorem 4.3, $\mathrm{ord}(S) = \mathrm{ord}(S')$. Note that S has odd order if, and only if, S is the identity. Therefore, the last statement follows. \square

5. PIVOTALIZATION AND INDICATORS

5.1. KMN-indicators. For the regular representation H of a semisimple Hopf algebra H over \mathbb{k} with the antipode S , the formula of the n th Frobenius-Schur indicator $\nu_n(H)$ was obtained in [15] and is given by (1.1). Since a monoidal equivalence between the module categories of two finite-dimensional Hopf algebras preserves their regular representation [25, Thm. 2.2] and Frobenius-Schur indicators are invariant under monoidal equivalence (cf. [24, Cor. 4.4] or [25, Prop. 3.2]), $\nu_n(H)$ is an invariant of $\mathrm{Rep}(H)$ if H is semisimple.

The formula (1.1) is well-defined even for a non-semisimple Hopf algebra H without any pivotal structure in $H\text{-mod}$. In fact, the gauge invariance of these scalars has been recently proved in [14] which is stated as the following theorem.

Theorem 5.1 ([14, Thm. 2.2]). *For any finite-dimensional Hopf algebra H over any field \mathbb{k} , we define $\nu_n^{\text{KMN}}(H)$ as in (1.1). If H and K are gauge equivalent finite-dimensional Hopf algebras over \mathbb{k} , then we have*

$$\nu_n^{\text{KMN}}(H) = \nu_n^{\text{KMN}}(K).$$

In general, these indicators $\nu_n^{\text{KMN}}(H)$ can *only* be defined for the regular representation of H . The proof Theorem 5.1 relies heavily on Corollary 3.2 and theory of Hopf algebras. We would like to have a categorical framework for the definition of $\nu_n^{\text{KMN}}(H)$ in order to extend the definitions of the indicators to other objects in $H\text{-mod}$ and give a categorical proof of gauge invariance of these indicators.

5.2. The universal pivotalization. Shimizu proposes in [33] the notion of universal pivotalization \mathcal{C}^{piv} of a finite tensor category \mathcal{C} in order to produce indicators for pairs consisting of an object V in \mathcal{C} along with a chosen isomorphism to its double dual. Under this categorical framework, $\nu_n^{\text{KMN}}(H)$ is the n th indicator of a special (or regular) object in $(H\text{-mod})^{\text{piv}}$. We recall some constructions and results from [33] here.

Let \mathcal{C} be a finite tensor category. In [33] Shimizu constructs a universal pivotalization $\Pi_{\mathcal{C}} : \mathcal{C}^{\text{piv}} \rightarrow \mathcal{C}$ of \mathcal{C} , which is referred to as the *pivotal cover* of \mathcal{C} therein.¹ The category \mathcal{C}^{piv} is the abelian, rigid, monoidal category of pairs (V, ϕ_V) of an object V and an isomorphism $\phi_V : V \rightarrow V^{**}$ in \mathcal{C} . Morphisms $(V, \phi_V) \rightarrow (W, \phi_W)$ in \mathcal{C}^{piv} are maps $f : V \rightarrow W$ in \mathcal{C} which satisfy $\phi_W f = f^{**} \phi_V$. Note that the forgetful functor $\Pi_{\mathcal{C}} : \mathcal{C}^{\text{piv}} \rightarrow \mathcal{C}$ is faithful.

The category \mathcal{C}^{piv} will be monoidal under the obvious tensor product $(V, \phi_V) \otimes (W, \phi_W) := (V \otimes W, \phi_V \otimes \phi_W)$ (where we suppress the natural isomorphism $(V \otimes W)^{**} \cong V^{**} \otimes W^{**}$), and (left) rigid under the dual $(V, \phi_V)^* = (V^*, (\phi_V^{-1})^*)$. There is a natural pivotal structure $j : \text{Id}_{\mathcal{C}^{\text{piv}}} \rightarrow (-)^{**}$ on \mathcal{C}^{piv} which, on each object (V, ϕ_V) , is simply given by $j_{(V, \phi_V)} := \phi_V$.

The construction \mathcal{C}^{piv} is universal in the sense that any monoidal functor $\mathcal{F} : \mathcal{D} \rightarrow \mathcal{C}$ from a pivotal tensor category \mathcal{D} factors uniquely through \mathcal{C}^{piv} . By faithfulness of the forgetful functor $\Pi_{\mathcal{C}} : \mathcal{C}^{\text{piv}} \rightarrow \mathcal{C}$, the factorization $\tilde{\mathcal{F}} : \mathcal{D} \rightarrow \mathcal{C}^{\text{piv}}$, which is a monoidal functor preserving the pivotal structures, is determined uniquely by where it sends objects. This factorization is described as follows.

Theorem 5.2 ([33, Thm. 4.3]). *Let j denote the pivotal structure on \mathcal{D} and $(\mathcal{F}, \xi) : \mathcal{D} \rightarrow \mathcal{C}$ a monoidal functor. Then the factorization $\tilde{\mathcal{F}} : \mathcal{D} \rightarrow \mathcal{C}^{\text{piv}}$ sends each object V in \mathcal{D} to the pair $(\mathcal{F}(V), (\xi^*)^{-1} \tilde{\xi} \mathcal{F}(j_V))$, where $\tilde{\xi}$ is the duality transformation as in Section 2.*

From the universal property for \mathcal{C}^{piv} one can conclude that the construction $(-)^{\text{piv}}$ is functorial, which means a monoidal functor $\mathcal{F} : \mathcal{D} \rightarrow \mathcal{C}$ induces

¹We accept the term pivotal cover, but adopt the term pivotalization as it is consistent with the constructions of [9] and admits adjectives more readily.

a unique pivotal functor $\mathcal{F}^{\text{piv}} : \mathcal{D}^{\text{piv}} \rightarrow \mathcal{C}^{\text{piv}}$ which satisfies the commutative diagram

$$\begin{array}{ccc} \mathcal{D}^{\text{piv}} & \xrightarrow{\mathcal{F}^{\text{piv}}} & \mathcal{C}^{\text{piv}} \\ \Pi_{\mathcal{D}} \downarrow & & \downarrow \Pi_{\mathcal{C}} \\ \mathcal{D} & \xrightarrow{\mathcal{F}} & \mathcal{C} \end{array}$$

of monoidal functors.

5.3. Shimizu's version of indicators via \mathcal{C}^{piv} . Following [24], for any $V, W \in \mathcal{C}$, we denote by $A_{V,W}$ and $D_{V,W}$ for the natural isomorphisms $\text{Hom}_{\mathcal{C}}(\mathbf{1}, V \otimes W) \rightarrow \text{Hom}_{\mathcal{C}}(V^*, W)$ and $\text{Hom}_{\mathcal{C}}(V, W) \rightarrow \text{Hom}_{\mathcal{C}}(W^*, V^*)$ respectively. Thus,

$$T_{V,W} := A_{W,V^{**}}^{-1} \circ D_{V^*,W} \circ A_{V,W}$$

is a natural isomorphism from $\text{Hom}_{\mathcal{C}}(\mathbf{1}, V \otimes W) \rightarrow \text{Hom}_{\mathcal{C}}(\mathbf{1}, W \otimes V^{**})$. We also define $V^{\otimes 0} = \mathbf{1}$ and $V^{\otimes n} = V \otimes V^{\otimes(n-1)}$ for any positive integer n inductively.

Similar to the definition provided in [24, p71], for any $\mathbf{V} = (V, \phi_V) \in \mathcal{C}^{\text{piv}}$ and positive integer n , one can define the map $E_{\mathbf{V}}^{(n)} : \text{Hom}_{\mathcal{C}}(\mathbf{1}, V^{\otimes n}) \rightarrow \text{Hom}_{\mathcal{C}}(\mathbf{1}, V^{\otimes n})$ by

$$E_{\mathbf{V}}^{(n)}(f) := \Phi^{(n)} \circ (\text{id} \otimes \phi_V^{-1}) \circ T_{V,W}(f)$$

where $W = V^{\otimes(n-1)}$ and $\Phi^{(n)} : W \otimes V \rightarrow V \otimes W$ is the unique map obtained by the associativity isomorphisms. Shimizu's version of the n th FS-indicator of \mathbf{V} is defined as

$$\nu_n^{\text{Sh}}(\mathbf{V}) = \text{Tr}(E_{\mathbf{V}}^{(n)}).$$

This indicator is preserved by monoidal equivalence in the following sense:

Theorem 5.3 ([33, Thm. 5.3]). *If $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ is an equivalence of monoidal categories, for any $\mathbf{V} \in \mathcal{C}^{\text{piv}}$ and positive integer n , we have $\nu_n^{\text{Sh}}(\mathbf{V}) = \nu_n^{\text{Sh}}(\mathcal{F}^{\text{piv}}(\mathbf{V}))$.*

Remark 5.4. Shimizu's version of the n th FS-indicator $\nu_n^{\text{Sh}}(\mathbf{V})$ of \mathbf{V} is different from the definition $\nu_n(\mathbf{V})$ introduced in [24], in which $E_{\mathbf{V}}^{(n)}$ is defined on the space $\text{Hom}_{\mathcal{C}^{\text{piv}}}(\mathbf{1}, \mathbf{V}^{\otimes n})$ instead. It is natural to ask the question whether or how these two notions of indicators are related.

In the case of a finite-dimensional Hopf algebra $\mathcal{C} = H\text{-mod}$, we take $\mathbf{R}_H = (H, \varphi_H)$ to be the object in \mathcal{C}^{piv} , in which H is the left regular H -module and $\varphi_H : H \rightarrow H^{**}$ is the composition $j^{\text{Vec}} \circ S^2 : H \rightarrow \mathcal{F}_{S^2}(H) \cong H^{**}$. We call \mathbf{R}_H the *regular object* in \mathcal{C}^{piv} . Shimizu proves

Theorem 5.5 ([33, Thm. 5.7]). *Suppose $\mathcal{C} = H\text{-mod}$. Then for each integer n we have $\nu_n^{\text{Sh}}(\mathbf{R}_H^*) = \nu_n^{\text{KMN}}(H)$.*

The theorem provides a convincing argument to pursue this categorical framework of FS-indicator. However, this framework does not yield another proof for the gauge invariance of $\nu_n^{\text{KMN}}(H)$ (cf. Theorem 5.1). The gauge invariance of $\nu_n^{\text{KMN}}(H)$ will follow if the following question raised by Shimizu can be positively answered.

Question 5.6 ([33]). Let H and K be two gauge equivalent Hopf algebras, and let $\mathcal{F} : H\text{-mod} \rightarrow K\text{-mod}$ be a monoidal equivalence. Do we have $\mathcal{F}^{\text{piv}}(\mathbf{R}_H) \cong \mathbf{R}_K$ in $(K\text{-mod})^{\text{piv}}$?

If the question is affirmatively answered for gauge equivalent Hopf algebras H and K , then we have $\mathcal{F}^{\text{piv}}(\mathbf{R}_H) \cong \mathbf{R}_K$ in $(K\text{-mod})^{\text{piv}}$ for any monoidal equivalence $\mathcal{F} : H\text{-mod} \rightarrow K\text{-mod}$. Thus, $\mathcal{F}^{\text{piv}}(\mathbf{R}_H^*) \cong (\mathcal{F}^{\text{piv}}(\mathbf{R}_H))^* \cong \mathbf{R}_K^*$. It follows from [33, Thm. 5.3] that

$$\nu_n^{\text{KMN}}(H) = \nu_n^{\text{Sh}}(\mathbf{R}_H^*) = \nu_n^{\text{Sh}}(\mathcal{F}^{\text{piv}}(\mathbf{R}_H^*)) = \nu_n^{\text{Sh}}(\mathbf{R}_K^*) = \nu_n^{\text{KMN}}(K).$$

An affirmative answer to the question for semisimple H has been provided in [33, Prop 5.10], and we will give in Theorem 7.4 a positive answer for H having the Chevalley property. As discussed above, an affirmative answer to the above question yields a categorical proof of Theorem 5.1.

6. FINITE PIVOTALIZATIONS FOR HOPF ALGEBRAS

In this section we just remark that the universal pivotalization \mathcal{C}^{piv} , which is not finite-dimensional in general, has a finite alternative for module categories of Hopf algebras.

For any \mathbb{k} -linear map $\tau : V \rightarrow V^{**}$ we let $\underline{\tau} \in \text{Aut}_{\mathbb{k}}(V)$ denote the automorphism $\underline{\tau} := (j^{\text{Vec}})^{-1} \circ \tau$.

Definition 6.1. For a Hopf algebra H we let H^{piv} denote the smash product $H \rtimes \mathbb{Z}$, where the generator x of \mathbb{Z} acts on H by S^2 . Similarly, for any positive integer N with $\text{ord}(S^2) \mid N$, we take $H^{N\text{piv}} = H \rtimes (\mathbb{Z}/N\mathbb{Z})$, where again the generator x of $\mathbb{Z}/N\mathbb{Z}$ acts as S^2 .

The smash product H^{piv} and $H^{N\text{piv}}$ admits a unique Hopf structures so that the inclusions $H \rightarrow H^{\text{piv}}$ and $H \rightarrow H^{N\text{piv}}$ are Hopf algebra maps and x is grouplike.

It has been pointed out in [33, Rem. 4.5] that $H^{\text{piv}}\text{-mod}$ is isomorphic to $(H\text{-mod})^{\text{piv}}$ as pivotal tensor categories. To realize the identification $\Theta : H^{\text{piv}}\text{-mod} \xrightarrow{\cong} \mathcal{C}^{\text{piv}}$ one takes a H^{piv} -module V to the H -module V along with the isomorphism $\phi_V := j^{\text{Vec}} \circ l(x) : V \rightarrow \mathcal{F}_{S^2}(V) \cong V^{**}$. On elements, $\phi_V(v) = j^{\text{Vec}}(x \cdot v)$. So we see that the inverse functor $\Theta^{-1} : \mathcal{C}^{\text{piv}} \rightarrow H^{\text{piv}}\text{-mod}$ takes the pair (V, ϕ_V) to the H -module V along with the action of the grouplike $x \in H^{\text{piv}}$ by $x \cdot v = \phi_V(v)$.

From the above description of \mathcal{C}^{piv} for Hopf algebras we see that \mathcal{C}^{piv} will not usually be a finite tensor category.

Note that, for any integer N as above, we have the Hopf projection $H^{\text{piv}} \rightarrow H^{N\text{piv}}$ which is the identity on H and sends x (in H^{piv}) to x (in $H^{N\text{piv}}$).

Dually, we get a fully faithful embedding of tensor categories $H^{N\text{piv}}\text{-mod} \rightarrow H^{\text{piv}}\text{-mod}$.

Definition 6.2. For any positive integer N which is divisible by the order of S^2 , we let $\mathcal{C}^{N\text{piv}}$ denote the full subcategory of \mathcal{C}^{piv} which is the image of $H^{N\text{piv}}\text{-mod} \subset H^{\text{piv}}\text{-mod}$ along the isomorphism $\Theta : H^{\text{piv}}\text{-mod} \rightarrow \mathcal{C}^{\text{piv}}$.

From this point on if we write $H^{N\text{piv}}$ or $\mathcal{C}^{N\text{piv}}$ we are assuming that N is a positive integer with $\text{ord}(S^2)|N$. We see, from the descriptions of the isomorphisms Θ and Θ^{-1} given above, that $\mathcal{C}^{N\text{piv}}$ is the full subcategory consisting of all pairs (V, ϕ_V) so that the associated automorphism $\underline{\phi}_V \in \text{Aut}_{\mathbb{k}}(V)$ has order dividing N .

Lemma 6.3. *The category $\mathcal{C}^{N\text{piv}}$ is a pivotal finite tensor subcategory in the pivotal (non-finite) tensor category \mathcal{C}^{piv} which contains \mathbf{R}_H .*

Proof. Since the map $\Theta : H^{\text{piv}}\text{-mod} \rightarrow \mathcal{C}^{\text{piv}}$ is a tensor equivalence, it follows that $\mathcal{C}^{N\text{piv}}$, which is defined as the image of $H^{N\text{piv}}\text{-mod}$ in \mathcal{C}^{piv} , is a full tensor subcategory in \mathcal{C}^{piv} . The category $\mathcal{C}^{N\text{piv}}$ is pivotal with its pivotal structure inherited from \mathcal{C}^{piv} . The fact that $\mathbf{R}_H = (H, j^{\text{Vec}} \circ S^2)$ is in $\mathcal{C}^{N\text{piv}}$ just follows from the fact the order of $S^2 = \underline{\phi}_{\mathbf{R}_H}$ is assumed to divide N . \square

Remark 6.4. There is another interesting object \mathbf{A}_H introduced in [33, Sect. 6.1 & Thm. 7.1]. This object is the adjoint representation H_{ad} of H along with the isomorphism $\phi_{\mathbf{A}_H} = j^{\text{Vec}} \circ S^2$. We will have that \mathbf{A}_H is also in $\mathcal{C}^{N\text{piv}}$ for any N .

Some choices for N which are of particular interest are $N = \text{ord}(S^2)$ or $N = \text{qexp}(H)$, where $\text{qexp}(H)$ is the quasi-exponent of H . Recall that the quasi-exponent $\text{qexp}(H)$ of H is defined as the unipotency index of the Drinfeld element u in the Drinfeld double $D(H)$ of H (see [5]). This number is always finite and divisible by the order of S^2 [5, Prop 2.5]. More importantly, $\text{qexp}(H)$ is a gauge invariant of H .

When we would like to pivotalize with respect to the the quasi-exponent we take $H^{E\text{piv}} = H^{\text{qexp}(H)\text{piv}}$ and $\mathcal{C}^{E\text{piv}} = \mathcal{C}^{\text{qexp}(H)\text{piv}}$. We call $\mathcal{C}^{E\text{piv}}$ the *exponential pivotalization* of $\mathcal{C} = H\text{-mod}$.

If \mathcal{C} admits any pivotal structures, one can show that the exponential pivotalization contains a copy of (\mathcal{C}, j) for any choice of pivotal structure j on \mathcal{C} as a full pivotal subcategory. More specifically, for any choice of pivotal structure j on \mathcal{C} the induced map $(\mathcal{C}, j) \rightarrow \mathcal{C}^{\text{piv}}$ will necessarily have image in $\mathcal{C}^{E\text{piv}}$. In this way, the indicators for \mathcal{C} calculated with respect to any choice of pivotal structure can be recovered from the (Shimizu-)indicators on $\mathcal{C}^{E\text{piv}}$.

For some Hopf algebras H , the integer $\text{qexp}(H)$ is minimal so that $\mathcal{C}^{N\text{piv}}$ has this property. For example, when we take the generalized Taft algebra

$$H_{n,d}(\zeta) = k\langle g, x \rangle / (g^{nd} - 1, x^d, gx - \zeta xg),$$

where ζ is a *primitive* d th root of unity (cf. [34],[8, Def 3.1]). We have $\text{ord}(S^2) = d$ and $nd = \text{qexp}(H_{n,d}(\zeta))$ by [5, Thm. 4.6]. The grouplike element g provides a pivotal structure j on $H_{n,d}(\zeta)\text{-mod}$, and the resulting map into $(H_{n,d}(\zeta)\text{-mod})^{\text{piv}}$ has image in $(H_{n,d}(\zeta)\text{-mod})^{N\text{piv}}$ if and only if $\text{qexp}(H_{n,d}(\zeta))|N$. This relationship can be seen as a consequence of the general fact that $\text{qexp}(H) = \text{qexp}(G(H))$ for any pointed Hopf algebra H [5, Thm. 4.6].

Our functoriality result for the finite pivotalizations is the following.

Proposition 6.5. *For any monoidal equivalence $\mathcal{F} : H\text{-mod} \rightarrow K\text{-mod}$, where H and K are Hopf algebras, the functor \mathcal{F}^{piv} restricts to an equivalence $\mathcal{F}^{E\text{piv}} : (H\text{-mod})^{E\text{piv}} \rightarrow (K\text{-mod})^{E\text{piv}}$. Furthermore, when H has the Chevalley property \mathcal{F}^{piv} restricts to an equivalence $\mathcal{F}^{N\text{piv}} : (H\text{-mod})^{N\text{piv}} \rightarrow (K\text{-mod})^{N\text{piv}}$ for each N (in particular $N = \text{ord}(S_H^2) = \text{ord}(S_K^2)$).*

The proof of the proposition is given in the appendix.

7. PRESERVATION OF THE REGULAR OBJECT

In this section we show that for a gauge equivalence $\mathcal{F} : H\text{-mod} \rightarrow K\text{-mod}$ of Hopf algebras with the Chevalley property we will have $\mathcal{F}^{\text{piv}}(\mathbf{R}_H) \cong \mathbf{R}_K$. From this we recover Theorem 5.1 for Hopf algebras with the Chevalley property.

Let H be a finite-dimensional Hopf algebra with antipode S , and $F \in H \otimes H$ a twist of H . We let $\mathcal{C} = H\text{-mod}$, $\mathcal{C}_F = H^F\text{-mod}$, and let $F = (\mathcal{F}_{\text{id}}, \xi^F)$ denote the associated equivalence from \mathcal{C} to \mathcal{C}_F , by abuse of notation.

For this section we will be making copious use of the isomorphism $j^{\text{Vec}} : V \rightarrow V^{**}$, and adopt the shorthand $\hat{v} = j^{\text{Vec}}(v) \in V^{**}$ for $v \in V$. Recall that \hat{v} is just the evaluation map $V^* \rightarrow \mathbb{k}$, $\eta \mapsto \eta(v)$.

7.1. Preservation of regular objects. Recall that the antipode S_F of H^F is given by $S_F(h) = \beta_F S(h) \beta_F^{-1}$ and that $\gamma_F = \beta_F S(\beta_F)^{-1}$. For any positive integer k , define

$$\gamma_F^{(k)} = \gamma_F S^2(\gamma_F) \cdots S^{2k-2}(\gamma_F).$$

Then we have $S_F^{2k}(h) = \gamma_F^{(k)} S^{2k}(h) (\gamma_F^{(k)})^{-1}$ for all positive integers k and $h \in H$. The following lemma is well-known and it follows immediately from [1, Eq.(6)].

Lemma 7.1. *The element $\gamma_F^{(\text{ord}(S^2))}$ is grouplike element in H^F .*

Proof. Take $N = \text{ord}(S^2)$. We have from [1, Eq.(6)] that

$$\Delta(\gamma_F) = F^{-1}(\gamma_F \otimes \gamma_F)(S^2 \otimes S^2)(F)$$

(see also [20]). Hence

$$\Delta(\gamma_F^{(n)}) = F^{-1}(\gamma_F^{(n)} \otimes \gamma_F^{(n)})(S^{2n} \otimes S^{2n})(F)$$

for each n and therefore

$$\Delta_F(\gamma_F^{(N)}) = F^{-1}\Delta(\gamma_F^{(N)})F = \gamma_F^{(N)} \otimes \gamma_F^{(N)}. \quad \square$$

We have the following concrete description of the (universal) pivotalization of an equivalence $F : \mathcal{C} \rightarrow \mathcal{C}_F$ induced by a twist F on H .

Lemma 7.2. *The functor $F^{\text{piv}} : \mathcal{C}^{\text{piv}} \rightarrow \mathcal{C}_F^{\text{piv}}$ sends an object (V, ϕ_V) in \mathcal{C}^{piv} to the pair consisting of the object V along with the isomorphism*

$$V \rightarrow V^{**}, \quad v \mapsto j^{\text{Vec}}(\gamma_F \phi_V(v)).$$

In particular, $F^{\text{piv}}(\mathbf{R}_H) = (H^F, j^{\text{Vec}} \circ l(\gamma_F) \circ S^2)$.

Proof. Take $\beta = \beta_F$, $\gamma = \gamma_F$ and $\xi = \xi^F$. Recall that $F(V^*) = F(V)^* = V^*$ as vector spaces for each V in \mathcal{C} . It follows from (3.1) that, for any object V in \mathcal{C} , $\tilde{\xi} : F(V^*) \rightarrow F(V)^*$ is given by

$$\tilde{\xi}(f) = f \leftarrow \beta^{-1} \quad \text{for } f \in V^*.$$

This implies

$$\tilde{\xi}(\hat{v})(f) = (\hat{v} \leftarrow \beta^{-1})(f) = \hat{v}(\beta^{-1} \cdot f) = f(S(\beta^{-1})v) = j^{\text{Vec}}(S(\beta^{-1})v)(f)$$

for $\hat{v} \in F(V^{**})$ and $f \in F(V^*)$. Thus,

$$\begin{aligned} (\tilde{\xi}^*)^{-1}\tilde{\xi}(\hat{v})(f) &= (\tilde{\xi}^*)^{-1}j^{\text{Vec}}(S(\beta^{-1})v)(f) = j^{\text{Vec}}(S(\beta^{-1})v)(\tilde{\xi}^{-1}(f)) \\ &= j^{\text{Vec}}(S(\beta^{-1})v)(f \leftarrow \beta) = f(\beta S(\beta^{-1})v) = f(\gamma v) = j^{\text{Vec}}(\gamma v)(f) \end{aligned}$$

for $\hat{v} \in F(V^{**})$ and $f \in F(V)^*$. By Theorem 5.2, $F^{\text{piv}}(V, \phi_V) = (V, (\tilde{\xi}^*)^{-1}\tilde{\xi}\phi_V)$ and

$$(\tilde{\xi}^*)^{-1}\tilde{\xi}\phi_V(v) = (\tilde{\xi}^*)^{-1}\tilde{\xi}j^{\text{Vec}}\phi_V(v) = j^{\text{Vec}}(\gamma\phi_V(v))$$

for $v \in V$. The last statement follows immediately from the definition of $\mathbf{R}_H = (H, j^{\text{Vec}} \circ S^2)$. This completes the proof. \square

In the following proposition we let \bar{S}^2 denote the automorphism of $H/J(H)$ induced by S^2 .

Proposition 7.3. *Let $F \in H \otimes H$ be a twist. The following statements are equivalent.*

- (i) $F^{\text{piv}}(\mathbf{R}_H) \cong \mathbf{R}_{HF}$ in $\mathcal{C}_F^{\text{piv}}$;
- (ii) there is a unit t in H which satisfies the equation

$$S^2(t)\gamma_F^{-1} - t = 0; \quad (7.1)$$

- (iii) there is a unit \bar{t} in $H/J(H)$ which satisfies the equation

$$\bar{S}^2(\bar{t})\bar{\gamma}_F^{-1} - \bar{t} = 0. \quad (7.2)$$

Proof. We take $N = \text{ord}(S^2)$. By Lemma 7.2, $F^{\text{piv}}(\mathbf{R}_H) = (H^F, j^{\text{Vec}} \circ l(\gamma_F) \circ S^2)$. An isomorphism $F^{\text{piv}}(\mathbf{R}_H) \cong \mathbf{R}_{HF}$ is determined by a H^F -module

automorphism of H^F , which is necessarily given by right multiplication by a unit $t \in H^F$, producing a diagram

$$\begin{array}{ccccc}
H^F & \xrightarrow{l(\gamma_F)S^2} & H^F & \xrightarrow{j^{\text{Vec}}} & (H^F)^{**} \\
r(t) \downarrow & & \downarrow r(t) & & \downarrow r(t)^{**} \\
H^F & \xrightarrow{S_F^2} & H^F & \xrightarrow{j^{\text{Vec}}} & (H^F)^{**}.
\end{array}$$

Equivalently, we are looking for a unit t such that

$$\gamma_F S^2(h)t = S_F^2(ht) = \gamma_F S^2(h)S^2(t)\gamma_F^{-1}$$

for all $h \in H$. This equation is equivalent to

$$S^2(t)\gamma_F^{-1} = t. \quad (7.3)$$

Let σ denote the \mathbb{k} -linear automorphism $r(\gamma_F^{-1}) \circ S^2 = r(\gamma_F)^{-1} \circ S^2$ of H^F , and Σ be the subgroup generated by σ in $\text{Aut}_{\mathbb{k}}(H^F)$. Then we have $\sigma^N = r(\gamma_F^{(N)})^{-1} \circ S^2 = r(\gamma_F^{(N)})^{-1}$. Since $\gamma_F^{(N)}$ is grouplike in H^F , it has a finite order. Therefore σ^N has finite order, as does σ , and Σ is a finite cyclic group.

Since $J(H)$ is a σ -invariant, the exact sequence

$$0 \rightarrow J(H) \rightarrow H \rightarrow H/J(H) \rightarrow 0$$

is in $\text{Rep}(\Sigma)$. Applying the exact functor $(-)^{\Sigma}$, we get another exact sequence

$$0 \rightarrow J(H)^{\Sigma} \rightarrow H^{\Sigma} \rightarrow (H/J(H))^{\Sigma} \rightarrow 0. \quad (7.4)$$

Recall that an element in H is a unit if, and only if, its image in $H/J(H)$ is a unit. So from the exact sequence (7.4), we conclude that there is a unit in $(H/J(H))^{\Sigma}$ if and only if there is a unit in H^{Σ} . Rather, there exists a unit \bar{t} solving the equation $\sigma \cdot X - X = 0$ in $H/J(H)$ if, and only if, there exists a unit t solving the equation in H . Since $\sigma \cdot \bar{t} = \bar{S}^2(\bar{t})\bar{\gamma}_F^{-1}$ and $\sigma \cdot t = S^2(t)\gamma_F^{-1}$, the equation $\bar{S}^2(X)\bar{\gamma}_F^{-1} - X = 0$ has a unit solution in \bar{H} if, and only if, a the equation $S^2(X)\gamma_F^{-1} - X = 0$ has a unit solution in H . \square

As an immediate corollary of this proposition, we can prove preservation of regular object for Hopf algebras with the Chevalley property.

Theorem 7.4. *Suppose H and K are gauge equivalent finite-dimensional Hopf algebras with the Chevalley property, and $\mathcal{F} : H\text{-mod} \rightarrow K\text{-mod}$ is a monoidal equivalence. Then we have $\mathcal{F}^{\text{piv}}(\mathbf{R}_H) \cong \mathbf{R}_K$ in $(K\text{-mod})^{\text{piv}}$.*

Proof. In view of [25, Thm. 2.2], it suffices to assume $K = H^F$ for some twist $F \in H \otimes H$, and that \mathcal{F} is the associated equivalence $F : H\text{-mod} \rightarrow H^F\text{-mod}$. Let S be the antipode of H . It follows from Lemma 4.1 that $\bar{\gamma}_F = \bar{1}$ and $\bar{S}^2 = \text{id}$. Therefore, every unit $t \in H/J(H)$ satisfies $\bar{S}^2(t)\bar{\gamma}_F^{-1} - t = 0$. The proof is then completed by Proposition 7.3. \square

As a corollary we recover Theorem 5.1 for Hopf algebras with the Chevalley property. Before giving the result let us say clearly that the invariance theorem from [14] holds for *all* (finite-dimensional) Hopf algebras.

Corollary 7.5 (cf. [14, Thm. 2.2]). *If $\mathcal{F} : H\text{-mod} \rightarrow K\text{-mod}$ is a gauge equivalence and H has the Chevalley property then we have*

$$\nu_n^{\text{KMN}}(H) = \nu_n^{\text{KMN}}(K)$$

for all $n \geq 0$.

Proof. We have $\mathcal{F}^{\text{piv}}(\mathbf{R}_H) \cong \mathbf{R}_K$ by Theorem 7.4. Since a gauge equivalence preserves duals this implies $\mathcal{F}^{\text{piv}}(\mathbf{R}_H^*) \cong \mathbf{R}_K^*$ as well. Hence, using [33, Thm. 5.3 & 5.7], we have

$$\nu_n^{\text{KMN}}(H) = \nu_n^{\text{Sh}}(\mathbf{R}_H^*) = \nu_n^{\text{Sh}}(\mathbf{R}_K^*) = \nu_n^{\text{KMN}}(K).$$

□

APPENDIX A. FUNCTORIALITY OF FINITE PIVOTALIZATIONS

We adopt the notation introduced at the beginning of Section 6. Recall that the subcategory $\mathcal{C}^{N\text{piv}} \subset \mathcal{C}^{\text{piv}}$ is the full subcategory consisting of all pairs (V, ϕ_V) such that the associated automorphism $\underline{\phi}_V \in \text{Aut}_{\mathbb{k}}(V)$ satisfies $\text{ord}(\underline{\phi}_V) | N$.

Lemma A.1. *Let $F \in H \otimes H$ be a twists and consider the functor $F : \mathcal{C} \rightarrow \mathcal{C}_F$. Then, for any N divisible by $\text{ord}(S^2)$, the following statements are equivalent:*

- (i) F^{piv} restricts to an equivalence $F^{N\text{piv}} : \mathcal{C}^{N\text{piv}} \rightarrow \mathcal{C}_F^{N\text{piv}}$.
- (ii) $\gamma_F^{(N)} = 1$.

Furthermore, the existence of an isomorphism $F^{\text{piv}}(\mathbf{R}_H) \cong \mathbf{R}_{HF}$ implies (i) and (ii).

Proof. Consider any (V, ϕ_V) in $\mathcal{C}^{N\text{piv}}$. We have $F^{\text{piv}}(V, \phi_V) = (V, j^{\text{Vec}} \circ l(\gamma_F) \circ \phi_V)$, by Lemma 7.2. So $\underline{\phi}_{F^{\text{piv}}(V, \phi_V)} = l(\gamma_F) \circ \underline{\phi}_V$. Since $\underline{\phi}_V$, considered as an H -module map, is a map from V to $\mathcal{F}_{S^2}(V)$, we find by induction that

$$(l(\gamma_F) \circ \underline{\phi}_V)^n = l(\gamma_F^{(n)}) \circ \underline{\phi}_V^n$$

for each n . In particular,

$$(l(\gamma_F) \circ \underline{\phi}_V)^N = l(\gamma_F^{(N)}) \tag{A.1}$$

since $\underline{\phi}_V^N = 1$, by the assumption that (V, ϕ_V) is in $\mathcal{C}^{N\text{piv}}$.

From equation (A.1) we see that $F^{\text{piv}}(V, \phi_V)$ lay in $\mathcal{C}_F^{N\text{piv}}$ if and only if $l(\gamma_F^{(N)}) = \text{id}_V$. Whence we have the implication (ii) \Rightarrow (i). Applying (A.1) to the case $(V, \phi_V) = \mathbf{R}_H$ gives the converse implication (i) \Rightarrow (ii) as well as the implication $F^{\text{piv}}(\mathbf{R}_H) \cong \mathbf{R}_{HF} \Rightarrow$ (ii), since \mathbf{R}_{HF} is in each $\mathcal{C}_F^{N\text{piv}}$. □

We can now give the

Proof of Proposition 6.5. In view of [25, Thm. 2.2], it suffices to assume $K = H^F$ for some twist $F \in H \otimes H$ and the monoidal equivalence $F : H\text{-mod} \rightarrow H^F\text{-mod}$.

For Hopf algebras with the Chevalley property: Recall $\text{ord}(S^2) = \text{ord}(S_F^2)$ by Corollary 4.4. So we can pivotalize both H and H^F with respect to any N divisible by $\text{ord}(S^2)$. We have already seen that $\mathcal{F}^{\text{piv}}(\mathbf{R}_H) \cong \mathbf{R}_{H^F}$. We see, as in the proof of Lemma A.1, that this implies $\gamma_F^{(N)}$ for arbitrary N with $\text{ord}(S^2) | N$. Now by Lemma A.1 F^{piv} restricts to an equivalence $F^{N\text{piv}} : \mathcal{C}^{N\text{piv}} \rightarrow \mathcal{C}_F^{N\text{piv}}$.

The general case: From [5, Prop. 3.2] and the proof of [5, Prop. 3.3] we have $\gamma_F^{(\text{qexp}(H))} = 1$. By Lemma A.1 it follows that F^{piv} restricts to an equivalence $F^{E\text{piv}} : \mathcal{C}^{E\text{piv}} \rightarrow \mathcal{C}_F^{E\text{piv}}$. \square

REFERENCES

- [1] E. Aljadeff, P. Etingof, S. Gelaki, and D. Nikshych. On twisting of finite-dimensional Hopf algebras. *J. Algebra*, 256(2):484–501, 2002.
- [2] P. Bantay. The Frobenius-Schur indicator in conformal field theory. *Phys. Lett. B*, 394(1-2):87–88, 1997.
- [3] P. Bruillard, S.-H. Ng, E. C. Rowell, and Z. Wang. Rank-finiteness for modular categories. *J. Amer. Math. Soc.*, 29(3):857–881, 2016.
- [4] C. Dong, X. Lin, and S.-H. Ng. Congruence property in conformal field theory. *Algebra Number Theory*, 9(9):2121–2166, 2015.
- [5] P. Etingof and S. Gelaki. On the quasi-exponent of finite-dimensional Hopf algebras. *Math. Res. Lett.*, 9(2-3):277–287, 2002.
- [6] P. Etingof, D. Nikshych, and V. Ostrik. On fusion categories. *Ann. of Math. (2)*, 162(2):581–642, 2005.
- [7] P. Etingof and V. Ostrik. Finite tensor categories. *Mosc. Math. J.*, 4(3):627–654, 782–783, 2004.
- [8] P. Etingof and C. Walton. Pointed Hopf actions on fields, II. *J. Algebra*, 460:253–283, 2016.
- [9] P. I. Etingof, S. Gelaki, D. Nikshych, V. Ostrik, S. Gelaki, and D. Nikshych. *Tensor categories*, volume 205. American Mathematical Society, 2015.
- [10] G. Frobenius and I. Schur. Über die reellen Darstellungen der endlichen Gruppen. *Berl. Ber.*, 1906:186–208, 1906.
- [11] J. Fuchs, A. C. Ganchev, K. Szlachányi, and P. Vecsernyés. S_4 symmetry of $6j$ symbols and Frobenius-Schur indicators in rigid monoidal C^* categories. *J. Math. Phys.*, 40(1):408–426, 1999.
- [12] R. Guralnick and S. Montgomery. *preprint 2005*.
- [13] M. Iovanov, G. Mason, and S. Montgomery. FSZ -groups and Frobenius-Schur indicators of quantum doubles. *Math. Res. Lett.*, 21(4):757–779, 2014.
- [14] Y. Kashina, S. Montgomery, and S.-H. Ng. On the trace of the antipode and higher indicators. *Israel J. Math.*, 188:57–89, 2012.
- [15] Y. Kashina, Y. Sommerhäuser, and Y. Zhu. On higher Frobenius-Schur indicators. *Mem. Amer. Math. Soc.*, 181(855):viii+65, 2006.
- [16] C. Kassel. *Quantum groups*. Springer-Verlag, New York, 1995.
- [17] R. G. Larson and D. E. Radford. Finite-dimensional cosemisimple Hopf algebras in characteristic 0 are semisimple. *J. Algebra*, 117(2):267–289, 1988.
- [18] R. G. Larson and D. E. Radford. Semisimple cosemisimple Hopf algebras. *Amer. J. Math.*, 110(1):187–195, 1988.

- [19] V. Linchenko and S. Montgomery. A Frobenius-Schur theorem for Hopf algebras. *Algebr. Represent. Theory*, 3(4):347–355, 2000. Special issue dedicated to Klaus Roggenkamp on the occasion of his 60th birthday.
- [20] S. Majid. *Foundations of quantum group theory*. Cambridge university press, 2000.
- [21] G. Mason and S.-H. Ng. Central invariants and Frobenius-Schur indicators for semisimple quasi-Hopf algebras. *Adv. Math.*, 190(1):161–195, 2005.
- [22] S. Montgomery. *Hopf algebras and their actions on rings*, volume 82 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1993.
- [23] S.-H. Ng and P. Schauenburg. Frobenius-Schur indicators and exponents of spherical categories. *Adv. Math.*, 211(1):34–71, 2007.
- [24] S.-H. Ng and P. Schauenburg. Higher Frobenius-Schur Indicators for Pivotal Categories. In *Hopf Algebras and Generalizations*, volume 441 of *Contemp. Math.*, pages 63–90. Amer. Math. Soc., Providence, RI, 2007.
- [25] S.-H. Ng and P. Schauenburg. Central invariants and higher indicators for semisimple quasi-Hopf algebras. *Trans. Amer. Math. Soc.*, 360(4):1839–1860, 2008.
- [26] S.-H. Ng and P. Schauenburg. Congruence subgroups and generalized Frobenius-Schur indicators. *Comm. Math. Phys.*, 300(1):1–46, 2010.
- [27] V. Ostrik. Pivotal fusion categories of rank 3. *Mosc. Math. J.*, 15(2):373–396, 405, 2015.
- [28] D. E. Radford. The order of the antipode of a finite dimensional Hopf algebra is finite. *Amer. J. Math.*, 98(2):333–355, 1976.
- [29] D. E. Radford. The trace function and Hopf algebras. *J. Algebra*, 163(3):583–622, 1994.
- [30] D. E. Radford and H.-J. Schneider. On the even powers of the antipode of a finite-dimensional Hopf algebra. *J. Algebra*, 251(1):185–212, 2002.
- [31] P. Schauenburg. Frobenius-Schur indicators for some fusion categories associated to symmetric and alternating groups. *Algebr. Represent. Theory*, 19(3):645–656, 2016.
- [32] K. Shimizu. On indicators of Hopf algebras. *Israel J. Math.*, 207(1):155–201, 2015.
- [33] K. Shimizu. The pivotal cover and Frobenius-Schur indicators. *J. Algebra*, 428:357–402, 2015.
- [34] E. J. Taft. The order of the antipode of finite-dimensional Hopf algebra. *Proc. Nat. Acad. Sci. U.S.A.*, 68:2631–2633, 1971.

E-mail address: negronc@lsu.edu

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MA 02139, USA

E-mail address: rng@math.lsu.edu

DEPARTMENT OF MATHEMATICS, LOUISIANA STATE UNIVERSITY, BATON ROUGE, LA U.S.A.