SPECTRAL SEQUENCES FOR THE COHOMOLOGY RINGS OF A SMASH PRODUCT

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Abstract. Stefan and Guichardet have provided Lyndon-Hochschild-Serre type spectral sequences which converge to the Hochschild cohomology and Ext groups of a smash product. We show that these spectral sequences carry natural multiplicative structures, and that these multiplicative structures can be used to calculate the cup product on Hochschild cohomology and the Yoneda product on an Ext algebra.

1. Introduction

Let $\mathbf{k}$ be a field of arbitrary characteristic, and take $\otimes = \otimes_{\mathbf{k}}$. Fix a ($\mathbf{k}$-)algebra A and a Hopf algebra Γ acting on A. We assume that the antipode on Γ is bijective, and that the action of Γ on A gives it the structure of a Γ-module algebra [11, Definition 4.1.1]. We can then form the smash product algebra $A\#\Gamma$, which is the vector space $A \otimes \Gamma$ with multiplication

$$(a \otimes \gamma)(a' \otimes \gamma') := \sum_i a(\gamma_{i_1} \cdot a') \otimes \gamma_{i_2}\gamma',$$

where $\Delta(\gamma) = \sum_i \gamma_{i_1} \otimes \gamma_{i_2}$. In the case that Γ is the group algebra of a group G we use the notation $A\#G$ as a shorthand for the smash product $A\#\mathbf{k}G$.

Smash products have appeared in a number of different contexts in the literature. Under certain conditions, the smash product can serve as a replacement for the invariant algebra A^Γ [11, Section 4.5] [1] [5, Proposition 5.2]. Smash products have also appeared as a means of untwisting, or unbraiding, certain twisted structures. For example, one can untwists a twisted Calabi-Yau algebra with a smash product [12, 6], or unbraid a braided Hopf algebra [3, Section 1.5]. In a more classical context, smash products have played an integral role in a classification program proposed by Andruskiewitsch and Schneider, which began at [2].

In this paper we equip some known spectral sequences, which converge to the Hochschild cohomology and Ext groups of a smash product, with multiplicative structures. These spectral sequences, with their new multiplicative structures, can then be used to compute the products on these cohomologies. Specifically, we provide spectral sequences which converge to the Hochschild cohomology $\mathrm{HH}(A\#\Gamma, B)$, along with the cup product, and the extension algebra $\mathrm{Ext}_{A\#\Gamma\text{-mod}}(M, M)$, along with the standard Yoneda product. Here we allow M to be any (left) $A\#\Gamma$-module and B to be any algebra extension of the smash product, i.e. any algebra equipped with an algebra map $A\#\Gamma \to B$.

Recall that the Hochschild cohomology $\mathrm{HH}(R, M) = \oplus_i \mathrm{HH}^i(R, M)$ of an algebra R, with "coefficients" in a R-bimodule M, is defined to be the graded Ext group $\mathrm{HH}(R, M) = \mathrm{Ext}_{R\text{-bimod}}(R, M)$. In the case that M is an algebra extension of R, the Hochschild cohomology $\mathrm{HH}(R, M)$ carries a natural product called the cup product. The definition of the cup product is reviewed in Section 5.

The Hochschild cohomology ring $\mathrm{HH}(R, R)$ is known to be an invariant of the derived category $D^b(R)$ [13]. In addition to providing a relatively refined derived invariant, the cup product can also help us to analyze the module category of a given algebra. Snashall and Solberg have put forward a theory of support varieties for Artin algebras by way of the Hochschild cohomology ring $\mathrm{HH}(R, R)$. They assign to an Artin algebra R, and any pair of R-modules, a subvariety of the maximal ideal spectrum of (a subalgebra of) the Hochschild cohomology. The cup product can also help us get a handle on some of the additional structures on Hochschild cohomology, such as the Gerstenhaber bracket. The applications of Hochschild cohomology

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rings are, however, limited by a scarcity of examples and by the fact that the cup product can be difficult to compute in general.

In the theorem below, by a multiplicative spectral sequence we mean a spectral sequence $E = (E_r)$ equipped with bigraded products $E_r \otimes E_r \to E_r$ which are compatible with the differentials and structural isomorphism $E_{r+1} \cong \mathrm{H}(E_r)$. (One can refer to Section 6 for a more precise definition.) We say that a multiplicative spectral sequence converges to a graded algebra H if H carries an additional filtration and there is an isomorphism of bigraded algebras $E_\infty \cong \mathrm{grH}$. One of our main result is the following.

Theorem 1.1 (Corollary 6.8)**.** *For any algebra extension B of the smash product $A\#\Gamma$, there are two multiplicative spectral sequences*

$$E_2 = \mathrm{Ext}_{\Gamma\text{-mod}}(\mathbf{k}, \mathrm{HH}(A, B)) \Rightarrow \mathrm{HH}(A\#\Gamma, B)$$

and

$$'E_1 = \mathrm{Ext}_{\Gamma\text{-mod}}(\mathbf{k}, \mathrm{RHom}_{A\text{-bimod}}(A, B)) \Rightarrow \mathrm{HH}(A\#\Gamma, B)$$

which converge to the Hochschild cohomology as an algebra.

To be clear, we mean that there is some Γ-module algebra structure on $\mathrm{HH}(A, B)$ and that the second term E_2 is the bigraded algebra $\mathrm{Ext}_{\Gamma\text{-mod}}(\mathbf{k}, \mathrm{HH}(A, B))$. Similarly, for the term $'E_1$, we mean there is some particular model for $\mathrm{RHom}_{A\text{-bimod}}(A, B))$ which is a Γ-module (dg) algebra and that $'E_1$ is the given Ext algebra. We also provide a version of the above theorem for the cohomology rings $\mathrm{Ext}_{A\#\Gamma\text{-mod}}(M, M)$.

Theorem 1.2 (Corollary 7.7)**.** *For any $A\#\Gamma$-module M, there are two multiplicative spectral sequences*

$$\bar{E}_2 = \mathrm{Ext}_{\Gamma\text{-mod}}(\mathbf{k}, \mathrm{Ext}_{A\text{-mod}}(M, M)) \Rightarrow \mathrm{Ext}_{A\#\Gamma\text{-mod}}(M, M)$$

and

$$'\bar{E}_1 = \mathrm{Ext}_{\Gamma\text{-mod}}(\mathbf{k}, \mathrm{RHom}_{A\text{-mod}}(M, M)) \Rightarrow \mathrm{Ext}_{A\#\Gamma\text{-mod}}(M, M)$$

which converge to $\mathrm{Ext}_{A\#\Gamma\text{-mod}}(M, M)$ as an algebra.

In the text it is shown that all four of the above spectral sequences exist as explicit isomorphism at the level of cochains. Let us explain what is meant by this statement in the case of Hochschild cohomology.

Let B be an algebra extension of $A\#\Gamma$, as in Theorem 1.1. For a free A-bimodule resolution $K \to A$, equipped with a Γ-action satisfying certain natural conditions, and any resolution $L \to \mathbf{k}$ of the trivial Γ-module, we produce a dg algebra structure on the double complex

$$\mathrm{Hom}_{\Gamma\text{-mod}}(L, \mathrm{Hom}_{A\text{-bimod}}(K, B)). \tag{1.1}$$

From this data we also produce a $A\#\Gamma$-bimodule resolution $\mathcal{K}$ of $A\#\Gamma$, and dg algebra structure on the associated complex $\mathrm{Hom}_{A\#\Gamma\text{-bimod}}(\mathcal{K}, B)$. The dg algebra structure is chosen so that the homology of $\mathrm{Hom}_{A\#\Gamma\text{-bimod}}(\mathcal{K}, B)$ is the Hochschild cohomology $\mathrm{HH}(A\#\Gamma, B)$ with the cup product.

In Theorem 6.5, which can be seen as a lifting of Theorem 1.1 to the level of cochains, we show that there is an explicit isomorphism of dg algebras

$$\mathrm{Hom}_{A\#\Gamma\text{-bimod}}(\mathcal{K}, B) \overset{\cong}{\to} \mathrm{Hom}_{\Gamma\text{-mod}}(L, \mathrm{Hom}_{A\text{-bimod}}(K, B)).$$

It follows then that the Hochschild cohomology ring of the smash product can be computed as the homology of the double complex $\mathrm{Hom}_{\Gamma\text{-mod}}(L, \mathrm{Hom}_{A\text{-bimod}}(K, B))$. We get Theorem 1.1 as an easy corollary of this fact.

The full power of Theorem 6.5 is employed to compute some examples in a follow up paper. Let us discuss just one example here. Suppose $\mathbf{k}$ is of characteristic 0 and let $q \in \mathbf{k}$ be a nonzero scalar which is not a root of unity. Let $\mathbf{k}_q[x, y]$ denote the skew polynomial ring in 2-variables,

$$\mathbf{k}_q[x, y] = \frac{\mathbf{k}\langle x, y\rangle}{(yx - qxy)}.$$

This algebra is twisted Calabi-Yau. If we let $\mathbb{Z} = \langle \phi \rangle$ act on $\mathbf{k}_q[x, y]$ by the automorphism $\phi : x \mapsto q^{-1}x$, $y \mapsto qy$, then, according to [12, Proposition 7.3] and [6], the smash product $\mathbf{k}_q[x, y]\#\mathbb{Z}$ will be Calabi-Yau. By way of Theorem 6.5, we can provide the following computation.

Theorem 1.3. *Let λ, ε, ξ_i, ζ, and η_i be a variables of respective degrees* 0, 1, 1, 2, *and* 2. *Then there is an isomorphism of graded algebras*

$$\mathrm{HH}(\mathbf{k}_q[x,y]\#\mathbb{Z}) \cong \frac{\mathbf{k}[\varepsilon,\lambda,\xi_1,\xi_2,\zeta]}{(\lambda\zeta-\xi_1\xi_2,\zeta^2)} \times_{\mathbf{k}[\varepsilon]} \frac{\mathbf{k}[\varepsilon,\eta_i : i\in\mathbb{Z}-\{-1\}]}{(\eta_i^2,\eta_i\eta_j)}.$$

Furthermore, there is a natural embedding of graded algebras $\mathrm{HH}(\mathbf{k}_q[x,y]) \to \mathrm{HH}(\mathbf{k}_q[x,y]\#\mathbb{Z})$ *identifying* $\mathrm{HH}(\mathbf{k}_q[x,y])$ *with the subalgebra generated by* ξ_1, ξ_2, *and* η_0.

In the statement of the above theorem $\mathbf{k}[X_1,\dots,X_n]$ denotes the free graded commutative algebra on graded generators X_i, $\mathbf{k}[X_1,\dots,X_n]=\mathbf{k}\langle X_1,\dots,X_n\rangle/(X_iX_j-(-1)^{|X_i||X_j|}X_jX_i)$.

1.1. **Relation to the work of Stefan, Guichardet, and others.** As suggested in the abstract, Theorems 1.1 and 1.2 can be seen as a refinement of results of Stefan and Guichardet given in [18] and [8] respectively. However, both Stefan and Guichardet work with classes of algebras that are slightly different than general smash products. Guichardet provides spectral sequences

$$\mathrm{Ext}_{\mathbf{k}G\text{-mod}}(\mathbf{k},\mathrm{HH}(A,M)) \Rightarrow \mathrm{HH}(A\#_\alpha G,M)$$

for crossed product algebras $A\#_\alpha G$, where G is a group, while Stefan provides spectral sequences

$$\mathrm{Ext}_{\Gamma\text{-mod}}(\mathbf{k},\mathrm{HH}(A,M)) \Rightarrow \mathrm{HH}(E,M)$$

for Hopf Galois extensions $A\to E$. Guccione and Guccione extend the results of Guichardet to allow for crossed products with arbitrary Hopf algebras in [7]. None of these spectral sequences carry any multiplicative structures. For definitions of these different classes of algebras one can see [11]. Let us only mention that there are strict containments

$$\{\text{Smash Products}\} \subsetneq \{\text{Crossed Products}\} \subsetneq \{\text{Hopf Galois Extensions}\}.$$

So these more limited results (taken together) do apply to larger classes of algebras.

Let us mention here that Guccione and Guccione also provide spectral sequences for Ext groups in [7], by way of a standard relation [19, Lemma 9.1.9]. Also, some results involving multiplicative structures are given by Sanada, in a rather constrained setting, in [14]. Further analysis of the situation can be found in [4].

The main point of comparison here is that our spectral sequences can be used to compute the cup product, while those of Stefan, Guichardet, and Guccione-Guccione can not (at least after restricting to the case of smash products). However, there are also differences in the methods used in the three sources. As a consequence, the usefulness of the results vary in practice. For example, Stefan produces his spectral sequence as a Grothendieck spectral sequence, whereas those of Guccione and Guccione are derived from filtrations on a certain (rather large) complex. Guichardet shows that the Hochschild cohomology of a crossed product can be computed by the double complex $C(G,C(A,M))$, where C denotes the standard Hochschild cochain complex. Indeed, Guichardet provides quasi-isomorphisms of chain complexes $C(G,C(A,M)) \leftrightarrows C(A\#_\alpha G,M)$. It does not appear that either of the given maps are dg algebra maps, and so the cup product remains obscured.

The spectral sequences produced in this paper are those associated to the first quadrant double complex (1.1), which may in some cases be chosen to be relatively small. Our methods are most closely related those of Guichardet. In fact, by standard techniques, one may move from Guichardet's double Hochschild cochain complex to our double complex(es). To summarize the situation, we have the following chart

	Class of algebras for which the spectral sequences apply	Type of spectral sequences	Accounts for the cup/Yoneda product
Stefan	Hopf Galois extensions	Grothendieck	No
Guichardet	Crossed products with groups	Double complex	No
Guccione-Guccione	Crossed products	Filtration	No
Present paper	Smash products	Double complex	Yes

Of the works discussed, Theorems 4.3, 6.5, and 7.5 below provide the most computationally accessible approach to the cohomology of a smash product, irrespective of the cup product. Grothendieck spectral sequences, for example, require the use of injective resolutions, which are very difficult to come by in general. The methods used here are also more natural than those given in [7] in the sense that many of the constructions we employ are functorial.

1.2. **Contents.** Throughout we consider a Hopf algebra Γ acting on an algebra A. In Section 2 we produce a resolution of the Hopf algebra Γ which carries enough structure to admit a smash product construction. In particular, we construct a complex of projective Γ-bimodules with an additional (compatible) coaction, and quasi-isomorphism to Γ which preserves the given structure. We call such a resolution a Hopf bimodule resolution.

In Section 3 we propose a smash product construction for complexes of Hopf bimodules and complexes of, so called, equivariant bimodules over A (Definition 3.1). This smash product construction for complexes is used to produce, from the Hopf bimodule resolution of Section 2 and an equivariant resolution of A, a bimodule resolution of $A\#\Gamma$. In Section 4 we use the aforementioned bimodule resolution of $A\#\Gamma$ to construct an explicit isomorphism

$$\Xi : \mathrm{RHom}_{A\#\Gamma\text{-bimod}}(A\#\Gamma, M) \overset{\cong}{\to} \mathrm{RHom}_{\Gamma\text{-mod}}(\mathbf{k}, \mathrm{RHom}_{A\text{-bimod}}(A, M))$$

for any complex M of $A\#\Gamma$-bimodules.

In Section 5 we review the products on both the domain and codomain of the above isomorphism Ξ (when evaluated at an algebra extension of $A\#\Gamma$), and in Section 6 we show that the map Ξ is an isomorphism of dg algebras when appropriate. Theorem 1.1 is also proved in this section. Finally, in Section 7 we give versions of our main theorems for the Ext algebras $\mathrm{Ext}_{A\#\Gamma\text{-mod}}(M, M)$, for arbitrary M.

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Conventions

For any coalgebra Γ, the coproduct of an element $\gamma \in \Gamma$ will always be expressed using Sweedler's notation

$$\gamma_1 \otimes \gamma_2 = \Delta(\gamma).$$

(So "$\gamma_1 \otimes \gamma_2$" is a symbol representing a sum of elements $\sum_i \gamma_{i_1} \otimes \gamma_{i_2}$ in $\Gamma \otimes \Gamma$.) Given Γ-modules M and N the tensor product $M \otimes N$ is taken to be a Γ-module under the standard action

$$\gamma(m \otimes n) := (\gamma_1 m) \otimes (\gamma_2 n).$$

As mentioned previously, *a Hopf algebra will always mean a Hopf algebra with bijective antipode.* Let Γ be a Hopf algebra and A be a Γ-module algebra. Following [15], we denote the action of Γ on A by a superscript ${}^{\gamma}a := \gamma \cdot a$. Elements in the smash product $A\#\Gamma$ will be denoted by juxtaposition $a\gamma := a \otimes \gamma \in A\#\Gamma = A \otimes \Gamma$. Hence, the multiplication on $A\#\Gamma$ can be written $(a\gamma)(b\gamma') = a({}^{\gamma_1}b)\gamma_2\gamma'$. For an algebra A we let A^e denote the enveloping algebra $A^e = A^{op} \otimes A$. *All modules are left modules unless stated otherwise.* We do not distinguish between the category of A-bimodules and the category of (right or left) A^e-modules.

In computations, all elements in graded vectors spaces are chosen to be homogenous. For homogeneous x, in a graded space X, we let $|x|$ denote its degree. For any algebra A and A-complexes X and Y we let $\mathrm{Hom}_A(X, Y)$ denote the standard hom complex. Recall that the nth homogenous piece of the hom complex consists of all degree n maps $f : X \to Y$, and for any $f \in \mathrm{Hom}_A(X, Y)$ the differential is given by $f \mapsto d_Y f - (-1)^{|f|} f d_X$.

2. A Hopf bimodule resolution of Γ

Let Γ be a Hopf algebra. We have the canonical algebra embedding

$$\begin{aligned} \Delta^{tw} : \Gamma &\to \Gamma^e = \Gamma^{op} \otimes \Gamma \\ \gamma &\mapsto S(\gamma_1) \otimes \gamma_2. \end{aligned}$$

This map will be referred to as the *twisted diagonal map.* The twisted diagonal map gives Γ^e a left Γ-module structure. On elements, this left action is given by $\delta \cdot (\gamma \otimes \gamma') = \gamma S(\delta_1) \otimes \delta_2 \gamma'$, for $\delta \in \Gamma$, $\gamma \otimes \gamma' \in \Gamma^e$.

Note that, since the antipode of Γ is bijective, there is an isomorphism of left Γ-modules $S \otimes id : \Gamma^e \to \Gamma \otimes \Gamma$. The module $\Gamma \otimes \Gamma$ is known to be free over Γ. (One can use the fundamental theorem of Hopf modules [11, Theorem 1.9.4] to show this, for example.) So we get the following

Lemma 2.1. *The enveloping algebra Γ^e is a free, and hence flat, left Γ-module.*

Note that the left action of Γ on Γ^e is compatible with the standard (outer) bimodule structure on $\Gamma^e = {}_\Gamma\Gamma \otimes \Gamma_\Gamma$. Indeed, the module structure induced by the twisted diagonal map utilizes the inner bimodule structure $\Gamma_\Gamma \otimes {}_\Gamma\Gamma$ exclusively. So we see that Γ^e is a (Γ-Γ^e)-bimodule, and that the induced module $M \otimes_\Gamma \Gamma^e$ of a right Γ-module M is a Γ-bimodule. To be clear, the left and right actions of Γ on $M \otimes_\Gamma \Gamma^e$ are given by

$$\delta \cdot (m \otimes_\Gamma (\gamma \otimes \gamma')) := m \otimes_\Gamma (\delta\gamma \otimes \gamma')$$

and

$$(m \otimes_\Gamma (\gamma \otimes \gamma')) \cdot \delta := m \otimes_\Gamma (\gamma \otimes \gamma'\delta)$$

respectively, where $\delta \in \Gamma$ and $m \otimes_\Gamma (\gamma \otimes \gamma') \in M \otimes_\Gamma \Gamma^e$. The same analysis holds when we replace M with a complex of right Γ-modules.

Notation 2.2. Given any right Γ-module (resp. complex) M, we let we let $M^\uparrow$ denote the induced module (resp. complex) $M \otimes_\Gamma \Gamma^e$.

The following result is proven, in less detail, in [17, Section 3]. However, as we will be needing all the details, a full proof is given here.

Lemma 2.3. *Let $\xi : L \to \mathbf{k}$ be a resolution of the trivial right Γ-module $\mathbf{k} = \Gamma/\ker\epsilon$.*

(1) The induced complex $L^\uparrow$ is a complex of projective Γ-bimodules.

(2) The map $\xi^\uparrow : L^\uparrow \to \Gamma$, $\ell \otimes_\Gamma (\gamma \otimes \gamma') \mapsto \xi(\ell)\gamma\gamma'$, is a quasi-isomorphism of complexes of Γ-bimodules.

Statements (1) and (2) together say that $L^\uparrow$ is a projective bimodule resolution of Γ.

Proof. In each degree i we have the adjunction

$$\mathrm{Hom}_{\Gamma^e}(L^i \otimes_\Gamma \Gamma^e, -) = \mathrm{Hom}_\Gamma(L^i, \mathrm{Hom}_{\Gamma^e}(\Gamma^e, -)).$$

Whence the functor on the left is seen to be exact. So $L^\uparrow$ is a complex of projective bimodules.

As an intermediate step in proving (2), let us consider the Γ-bimodule map

$$\begin{gathered}\varphi : \mathbf{k} \otimes_\Gamma \Gamma^e \to \Gamma\\ 1 \otimes_\Gamma (\gamma \otimes \gamma') \mapsto \gamma\gamma'.\end{gathered}$$

For any $\gamma \in \Gamma$ we have $\varphi(1 \otimes_\Gamma (1 \otimes \gamma)) = \gamma$. So φ is surjective. Also, the computation

$$\begin{aligned}1 \otimes_\Gamma (\gamma \otimes \gamma') &= 1 \otimes (\gamma_1\epsilon(\gamma_2) \otimes \gamma')\\ &= \epsilon(\gamma_2) \otimes_\Gamma (\gamma_1 \otimes \gamma')\\ &= 1 \cdot \gamma_2 \otimes_\Gamma (\gamma_1 \otimes \gamma')\\ &= 1 \otimes_\Gamma (\gamma_1 S(\gamma_2) \otimes \gamma_3\gamma')\\ &= 1 \otimes_\Gamma (\epsilon(\gamma_1) \otimes \gamma_2\gamma')\\ &= 1 \otimes_\Gamma (1 \otimes \gamma\gamma')\end{aligned} \tag{2.1}$$

makes it clear that any element in $\mathbf{k} \otimes_\Gamma \Gamma^e$ is of the form $1 \otimes_\Gamma (1 \otimes \gamma)$ for some γ. Whence φ is seen to be injective as well, and therefore an isomorphism of bimodules. Now, for (2), simply note that $\xi \otimes_\Gamma \Gamma^e : L^\uparrow \to \mathbf{k} \otimes_\Gamma (\Gamma^e)$ is a quasi-isomorphism, since ξ is a quasi-isomorphism and Γ^e is flat over Γ, and that $\xi^\uparrow$ can be given as the composition of the isomorphism φ with the quasi-isomorphism $\xi \otimes_\Gamma \Gamma^e$. □

Definition 2.4. Given any right Γ-module M, we define the left Γ-comodule structure ρ_M on $M^\uparrow$ by

$$\begin{gathered}\rho_M : M^\uparrow \to \Gamma \otimes M^\uparrow\\ m \otimes_\Gamma (\gamma \otimes \gamma') \mapsto (\gamma_1\gamma'_1) \otimes (m \otimes_\Gamma (\gamma_2 \otimes \gamma'_2)).\end{gathered} \tag{2.2}$$

Following the standard notation, for any $\mathrm{m} \in M^\uparrow$, we denote the element $\rho_M(\mathrm{m})$ by $\mathrm{m}_{-1} \otimes \mathrm{m}_0$.

There is something of a question of whether or not this coaction is well defined. Certainly we can define a coaction on the $\mathbf{k}$-tensor product

$$\tilde{\rho}_M : M \otimes \Gamma^e \to \Gamma \otimes (M \otimes_\Gamma \Gamma^e)$$

by the same formula as (2.2). A direct computation shows that $\tilde{\rho}_M\big(m\delta \otimes (\gamma \otimes \gamma') - m \otimes \delta(\gamma \otimes \gamma')\big) = 0$, i.e. that $\tilde{\rho}_M$ vanishes on the relations for the tensor product $M \otimes_\Gamma \Gamma^e = M^\uparrow$. Whence the coaction ρ_M can be given as the map induced on the quotient $M^\uparrow = M \otimes_\Gamma \Gamma^e$.

Definition 2.5. By a *Hopf bimodule* we will mean a Γ-bimodule N equipped with a left Γ-coaction $N \to \Gamma \otimes M$ which is a map of Γ-bimodules (where Γ acts diagonally on the tensor product $\Gamma \otimes N$). Maps of Hopf bimodules are maps that are simultaneously Γ-bimodule maps and Γ-comodule maps.

The algebra Γ itself becomes a Hopf bimodule under the regular bimodule structure and coaction given by the comultiplication. In the notation of [11, Section 1.9], a Hopf bimodule is an object in the category ${}^{\Gamma}_{\Gamma}\mathcal{M}_{\Gamma}$.

Proposition 2.6. *Let M and N be right Γ-modules and $f : M \to N$ be a morphism of Γ-modules . For any $\mathrm{m} \in M^{\uparrow}$ and $\gamma \in \Gamma$ the following equations hold:*

(1) $\rho_N(f^{\uparrow}(\mathrm{m})) = \mathrm{m}_{-1} \otimes f^{\uparrow}(\mathrm{m}_0)$
(2) $\rho_M(\mathrm{m} \cdot \gamma) = \mathrm{m}_{-1}\gamma_1 \otimes \mathrm{m}_0\gamma_2$
(3) $\rho_M(\gamma \cdot \mathrm{m}) = \gamma_1\mathrm{m}_{-1} \otimes \gamma_2\mathrm{m}_0$.

Said another way, $(-)^{\uparrow}$ is a functor from mod-Γ *to the category of Hopf bimodules.*

The reader should be aware that we will be using the $^{\uparrow}$ notation on maps in a slightly more flexible manner throughout the paper. We will be generally be looking at the induced map composed with some convenient isomorphism.

Proof. These can all be checked directly from the definitions. For example, for (1), we have

$$\begin{aligned} \rho_M(f^{\uparrow}(m \otimes_{\Gamma} (\gamma \otimes \gamma'))) &= \rho_M(f(m) \otimes_{\Gamma} (\gamma \otimes \gamma')) \\ &= \gamma_1\gamma_1' \otimes f(m) \otimes_{\Gamma} (\gamma_2 \otimes \gamma_2') = \gamma_1\gamma_1' \otimes f^{\uparrow}(m \otimes_{\Gamma} (\gamma_2 \otimes \gamma_2')). \end{aligned}$$

□

Corollary 2.7. *For any complex X of right Γ-modules the induced complex $X^{\uparrow}$ is a complex of Hopf bimodules.*

Proof. This follows from part (1) of the pervious proposition and the fact that the differentials on X are Γ-linear. □

Proposition 2.8. *The quasi-isomorphism $\xi^{\uparrow} : L^{\uparrow} \to \Gamma$ of Lemma 2.3 is a quasi-isomorphism of complexes of Hopf bimodules.*

Proof. This can be checked directly from the definition of $\xi^{\uparrow}$ and the definitions of the coactions on $L^{\uparrow}$ and Γ. □

3. Bimodule resolutions of $A\#\Gamma$ via a smash product construction

Let Γ be a Hopf algebra and A be a Γ-module algebra. We recall here that a $\mathbf{k}$-linear map $M \to N$ of (right or left) $A\#\Gamma$-modules is $A\#\Gamma$-linear if and only if it is A-linear and Γ-linear independently. The following definition was given by Kaygun in [9].

Definition 3.1. A vector space M is called a Γ-equivariant A-bimodule if it is both a Γ-module and A-bimodule, and the structure maps $A \otimes M \to M$ and $M \otimes A \to M$ are maps of Γ-modules. Morphisms of Γ-equivariant A-bimodules are maps which are A^e-linear and Γ-linear independently. The category of such modules will be denoted $\mathrm{EQ}_{\Gamma}A^e$-mod. We define Γ-equivariant A^e-complexes similarly.

To ease notation we may at times write "equivariant bimodule" instead of the full Γ-equivariant A-bimodule. One example of an equivariant bimodule is A itself. One can think of an equivariant bimodule as an A-bimodule internal to the monoidal category (Γ-mod,$\otimes$).

Kaygun has shown that the category $\mathrm{EQ}_{\Gamma}A^e$-mod is actually the module category of a certain smash product $A^e\#\Gamma$ [9, Lemma 3.3]. Whence $\mathrm{EQ}_{\Gamma}A^e$-mod is seen to be abelian with enough projectives. Additionally, $\mathrm{EQ}_{\Gamma}A^e$-mod comes equipped with restriction functors (forgetful functors) to A^e-modules and Γ-modules. Since $A^e\#\Gamma$ is free over both A^e and Γ, one can verify that these restriction functors preserve projectives.

In this section we produce a projective bimodule resolution of $A\#\Gamma$ via the smash product construction outlined below.

Definition 3.2. Let X be any Γ-equivariant A^e-complex and let Y be any complex of Hopf bimodules. The smash product complex $X\#Y$ is defined to be the tensor complex $X\otimes Y$ with the left and right $A\#\Gamma$-actions

$$a\cdot(x\otimes y):=(ax)\otimes y,\ \gamma\cdot(x\otimes y):=(\gamma_1x)\otimes(\gamma_2y),\ (x\otimes y)\cdot a:=x({}^{y_{-1}}a)\otimes y_0$$

and

$$(x\otimes y)\cdot\gamma:=x\otimes(y\gamma),$$

for $x\in X$, $y\in Y$, $a\in A$ and $\gamma\in\Gamma$.

Obviously, we can define the smash product of an equivariant bimodule with a Hopf bimodule by considering them to be complexes concentrated in degree 0. The smash product construction is (bi)functorial in the sense of the following

Lemma 3.3. *If $f:X\to X'$ and $g:Y\to Y'$ are maps of complexes of Γ-equivariant A-bimodules and complexes of Hopf bimodules respectively, then the product map $f\otimes g:X\#Y\to X'\#Y'$ is a map of complexes of $A\#\Gamma$-bimodules.*

Proof. Left A-linearity of $f\otimes g$ follows from left A-linearity of f and right Γ-linearity follows from right Γ-linearity of g. Left Γ-linearity of $f\otimes g$ follows from the fact that both f and g are left Γ-linear. Finally, right A-linearity of $f\otimes g$ follows from right A-linearity of f and Γ-colinearity of g. $\square$

Now, let K be a projective resolution of A as an A-bimodule, with quasi-isomorphism $\tau:K\to A$. We will assume that K has the following additional properties:

(I) there is a Γ-action on K giving it the structure of a complex of Γ-equivariant A-bimodules, and the quasi-isomorphism $\tau:K\to A$ is Γ-equivariant.

(II) K is free over A^e on a graded base space $\bar{K}\subset K$ which is also a Γ-submodule.

An example of a resolution of A satisfying the above conditions is the bar resolution

$$BA=\cdots\to A\otimes A^{\otimes 2}\otimes A\to A\otimes A\otimes A\to A\otimes A\to 0,$$

with its standard differential

$$b\otimes a_1\otimes\ldots\otimes a_n\otimes b'\mapsto\begin{array}{l}ba_1\otimes\ldots\otimes b'+(-1)^nb\otimes\ldots\otimes a_nb'\\ \quad+\sum_{i=1}^{n-1}(-1)^ib\otimes\ldots\otimes a_ia_{i+1}\otimes\ldots\otimes b'.\end{array}\tag{3.1}$$

We give BA the natural diagonal Γ-action

$$\gamma\cdot(b\otimes a_1\otimes\ldots\otimes a_n\otimes b')=\gamma_1b\otimes\gamma_2a_1\otimes\ldots\otimes\gamma_{n+1}a_n\otimes\gamma_{n+2}b'.$$

In this case, $\overline{BA}$ will be the graded subspace $\overline{BA}=\bigoplus_n\mathbf{k}\otimes A^{\otimes n}\otimes\mathbf{k}$. One can also use the reduced bar complex or, if A is a Koszul algebra and Γ acts by graded endomorphisms, we can take K to be the Koszul resolution.

For any K satisfying (I) and (II), and any resolution L of the trivial right Γ-module $\mathbf{k}$, we can form the smash product complex $K\#L^{\uparrow}$ using the coaction on $L^{\uparrow}$ defined in the previous section. We will see that the smash product complex $K\#L^{\uparrow}$ provides a projective resolution of $A\#\Gamma$.

Lemma 3.4. *Suppose $\tau:K\to A$ is a bimodule resolution of A satisfying (I) and (II), and let $\xi:L\to\mathbf{k}$ be any projective resolution of the trivial right Γ-module. Let $\xi^{\uparrow}:L^{\uparrow}\to\Gamma$ be the quasi-isomorphism of Lemma 2.3. Then the product map $\tau\otimes\xi^{\uparrow}:K\#L^{\uparrow}\to A\#\Gamma$ is a quasi-isomorphism of $(A\#\Gamma)^e$-complexes.*

Proof. The fact that $\tau\otimes\xi^{\uparrow}$ is a quasi-isomorphism follows from the facts that both τ and $\xi^{\uparrow}$ are quasi-isomorphisms, and that the tensor product of any two quasi-isomoprhisms (over a field) is yet another quasi-isomorphism. It is trivial to check that $\tau\otimes\xi^{\uparrow}$ respects the left A-action and right Γ-action. One can simply use the definition of the A and Γ-actions on $K\#L^{\uparrow}$ given in Definition 3.2. For the left Γ-action we have, for any $x\in K$, $l\in L^{\uparrow}$, and $\gamma\in\Gamma$,

$$\begin{array}{ll}(\tau\otimes\xi^{\uparrow})(\gamma\cdot(x\otimes l)) & =\tau(\gamma_1x)\xi^{\uparrow}(\gamma_2l)\\ & ={}^{\gamma_1}\tau(x)\gamma_2\xi^{\uparrow}(l)\\ & =\gamma(\tau(x)\xi^{\uparrow}(l))\\ & =\gamma\cdot\left((\tau\otimes\xi^{\uparrow})(x\otimes l)\right).\end{array}$$

So $\tau\otimes\xi^{\uparrow}$ is a map of left Γ-complexes.

Recall that, by Proposition 2.8, the quasi-isomorphism $\xi^\uparrow : L^\uparrow \to \Gamma$ is one of Hopf bimodules. Hence $l_{-1} \otimes \xi^\uparrow(l_0) = \xi^\uparrow(l)_1 \otimes \xi^\uparrow(l)_2$ for all $l \in L^\uparrow$. Thus, for any $a \in A$ we have

$$\begin{aligned}
(\tau \otimes \xi^\uparrow)((x \otimes l) \cdot a) \quad &= \tau(x({}^{l_{-1}}a))\xi^\uparrow(l_0) \\
&= \tau(x({}^{\xi^\uparrow(l)_1}a))\xi^\uparrow(l)_2 \\
&= \tau(x)({}^{\xi^\uparrow(l)_1}a\xi^\uparrow(l)_2) \qquad (A\text{-linearity of } \tau) \\
&= (\tau(x)\xi^\uparrow(l))a \\
&= \left((\tau \otimes \xi^\uparrow)(x \otimes l)\right) a.
\end{aligned}$$

This verifies that $\tau \otimes \xi^\uparrow$ is a map of right A-complexes and completes the proof that $\tau \otimes \xi^\uparrow$ is a quasi-isomorphism of $(A\#\Gamma)^e$-complexes. □

Theorem 3.5. *Let K be a bimodule resolution of A satisfying conditions (I) and (II), and let L be any projective resolution of the trivial right Γ-module $\mathbf{k}$. Then the smash product $K\#L^\uparrow$ is a projective $A\#\Gamma$-bimodule resolution of $A\#\Gamma$.*

The proof of the theorem will be clear from the following lemma.

Lemma 3.6. *Let M be a Γ-equivariant bimodule which is free on a base space $\bar{M} \subset M$ satisfying $\Gamma\bar{M} = \bar{M}$, and suppose N is a projective right Γ-module. Then the smash product module $M\#N^\uparrow$ is projective over $A\#\Gamma$.*

Proof. Suppose that N is free on some base $\bar{N} \subset N$. Then we have (bi)module isomorphisms $A \otimes \bar{M} \otimes A \cong M$ and $\bar{N} \otimes \Gamma \cong N$ given by restricting the action maps $A \otimes M \otimes A \to M$ and $N \otimes \Gamma \to N$. We will also have $\Gamma \otimes \bar{N} \otimes \Gamma \cong N^\uparrow$. Now the restriction of the action map on $M\#N \cong (A \otimes \bar{M} \otimes A)\#(\Gamma \otimes \bar{N} \otimes \Gamma)$ provides an isomorphism

$$A\#\Gamma \otimes (\bar{M} \otimes \bar{N}) \otimes A\#\Gamma \to M\#N^\uparrow$$

with inverse

$$\begin{gathered}
M\#N^\uparrow \cong (A \otimes \bar{M} \otimes A)\#(\Gamma \otimes \bar{N} \otimes \Gamma) \to A\#\Gamma \otimes (\bar{M} \otimes \bar{N}) \otimes A\#\Gamma \\
(a \otimes m \otimes a') \otimes (\gamma \otimes n \otimes \gamma') \mapsto a\gamma_2 \otimes (S^{-1}(\gamma_1)m \otimes n) \otimes {}^{S(\gamma_3)}a'\gamma'.
\end{gathered}$$

So the smash product is a free $A\#\Gamma$-bimodule.

In the case that N is not free, we know that N is a summand of some free module $\mathcal{N}$. This will imply that $N^\uparrow$ is a summand of $\mathcal{N}^\uparrow$ as a Hopf bimodule. It follows that $M\#N^\uparrow$ is a summand of the free module $M\#\mathcal{N}^\uparrow$, and hence projective. □

Proof of Theorem 3.5. We already know that there is a quasi-isomorphism of $(A\#\Gamma)^e$-complexes $K\#L^\uparrow \to A\#\Gamma$, by Lemma 3.4. So we need only show that the smash product complex is projective in each degree. We have chosen K so that each K^i is an equivariant bimodule satisfying the hypotheses of Lemma 3.6, and each L^j is projective by choice. So each $K^i\#(L^j)^\uparrow = K^i\#(L^\uparrow)^j$ is projective by Lemma 3.6. Now, projectivity of the smash product $K\#L^\uparrow$ in each degree follows from the fact that each $(K\#L^\uparrow)^n$ is a finite sum of projective modules $K^i\#(L^\uparrow)^j$. □

Remark 3.7. The resolution of $A\#\Gamma$ constructed above is one of a number resolutions that have appeared in the literature. In [7], Guccione and Guccione provide a resolution X of the smash $A\#\Gamma$ which is the tensor product of the bar resolution of A with the bar resolution of Γ, along with some explicit differential. In the case that Γ is a group algebra, Shepler and Witherspoon have provided a class of resolutions of the smash product [16, Section 4]. Our resolution $K\#L^\uparrow$ is a member of their class of resolutions (up to isomorphism). The reader should be aware that the construction given in [16] is somewhat different than the one given here.

4. Hochschild cochains as derived invariants

Let Γ be a Hopf algebra and A be a Γ-module algebra.

Definition 4.1. Let M be a complex of $A\#\Gamma$-bimodules and let X be a complex of Γ-equivariant A-bimodules. We define a right Γ-module structure on the set of homs $\mathrm{Hom}_{A^e}(X, M)$ by the formula

$$f \cdot \gamma(x) := S(\gamma_1)f(\gamma_2 x)\gamma_3,$$

where $f \in \mathrm{Hom}_{A^e}(X, M)$, $\gamma \in \Gamma$, and $x \in X$.

This action was also considered in [7], and similar actions have appeared throughout the literature (see for example [10, Section 5]). The first portion of the action, $S(\gamma_1)f(\gamma_2 x)$, assures that $f\cdot\gamma$ preserves left A-linearity. The additional right action is necessary to preserve right A-linearity.

Lemma 4.2. *Let M be a complex of $A\#\Gamma$-bimodules and X be a complex of Γ-equivariant A-bimodules. The Γ-module structure on $\mathrm{Hom}_{A^e}(X, M)$ given in Definition 4.1 is compatible with the differential on the hom complex. That is to say, $\mathrm{Hom}_{A^e}(X, -)$ is a functor from $A\#\Gamma$-complexes to Γ-complexes.*

Proof. Recall that the differential on the hom complex is given by $d : f \mapsto d_M f \pm f d_X$. So Γ-linearity of the differential on the hom complex follows by Γ-linearity of d_M and d_X. □

Let L be a projective resolution of the trivial right Γ-module $\mathbf{k}$, and K be a bimodule resolution of A satisfying conditions (I) and (II) of the previous section. For a complex of $A\#\Gamma$-bimodules M, any map $\theta \in \mathrm{Hom}_{\mathbf{k}}(K\#L^\uparrow, M)$, and any $l \in L^\uparrow$, we let $\theta(-\otimes l)$ denote the $\mathbf{k}$-linear map

$$\begin{gathered} K \to M \\ x \mapsto (-1)^{|x||l|}\theta((l_{-1}x)\otimes l_0). \end{gathered}$$

Before giving the main theorem of this section let us highlight some points of interest. First, note that there is an embedding of chain complexes $L \to L^\uparrow$, $\ell \mapsto \ell \otimes_\Gamma 1$. This map becomes Γ-linear if we take the codomain to be $L^\uparrow$ with the adjoint action. It is via this map that we view L as a subcomplex in $L^\uparrow$. Second, note that for any $l \in L \subset L^\uparrow$ we have $\rho(l) = 1 \otimes l$. Therefore, for all $l \in L \subset L^\uparrow$, $\theta(-\otimes l)$ is just the map $x \mapsto (-1)^{|x||l|}\theta(x\otimes l)$.

Theorem 4.3. *Let L be a projective resolution of the trivial right Γ-module $\mathbf{k}$, and K be a bimodule resolution of A satisfying conditions (I) and (II). Then for any complex M of $A\#\Gamma$-bimodules the map*

$$\begin{gathered} \Xi : \mathrm{Hom}_{(A\#\Gamma)^e}(K\#L^\uparrow, M) \to \mathrm{Hom}_\Gamma(L, \mathrm{Hom}_{A^e}(K, M)) \\ \theta \mapsto (l \mapsto \theta(-\otimes l)) \end{gathered}$$

is a natural isomorphism of chain complexes.

In light of Theorem 3.5, we are claiming that there is an explicit natural isomorphism of derived functors

$$\mathrm{RHom}_{(A\#\Gamma)^e}(A\#\Gamma, -) \overset{\cong}{\to} \mathrm{RHom}_\Gamma(\mathbf{k}, \mathrm{RHom}_{A^e}(A, -)).$$

Proof. To distinguish between the action of Γ on L as a subcomplex in $L^\uparrow$, and the action of Γ on L itself, we will denote the action of Γ on $L^\uparrow$ by juxtaposition, and the action of Γ on L by a dot $\cdot$. So, for $l \in L \subset L^\uparrow$ and $\gamma \in \Gamma$, we have

$$l\cdot\gamma = S(\gamma_1)l\gamma_2.$$

It is straightforward to check that Ξ is a map of chain complexes, and we omit the computation. We need to check that, for each θ, the map $\Xi(\theta)$ is a right Γ-linear, that each $\theta(-\otimes l)$ is A^e-linear, and that Ξ is bijective.

Fix a homogeneous $A\#\Gamma$-bimodule map $\theta : K\#L^\uparrow \to M$. Since the coaction on $L^\uparrow$ restricts to a trivial coaction on L, the map $\theta(-\otimes l) : K \to M$ is seen to be A^e-linear for any $l \in L$. Furthermore, for any $\gamma \in \Gamma$, $l \in L$, and $x \in K$, Γ-linearity of θ on the left and right gives the sequence of equalities

$$\begin{aligned} \theta(-\otimes l\cdot\gamma)(x) &= (-1)^{|l||x|}\theta(x\otimes l\cdot\gamma) \\ &= (-1)^{|l||x|}\theta(x\otimes S(\gamma_1)l\gamma_2) \\ &= (-1)^{|l||x|}\theta((S(\gamma_2)\gamma_3 x)\otimes S(\gamma_1)l\gamma_4) \\ &= (-1)^{|l||x|}S(\gamma_1)\theta((\gamma_2 x)\otimes l)\gamma_3 \\ &= (\theta(-\otimes l)\cdot\gamma)(x). \end{aligned}$$

So we see that $\Xi(\theta)$ is in fact a right Γ-linear map $L \to \mathrm{Hom}_{A^e}(K, M)$.

To see that Ξ is an isomorphism we provide an explicit inverse. By a computation similar to (2.1), one can check that $L^\uparrow$ is generated as a right Γ-complex by the subcomplex $L \subset L^\uparrow$. Using this fact, we define, for any Γ-linear map

$$\chi : L \to \mathrm{Hom}_{A^e}(K, M),$$

a graded vector space map $\Phi(\chi) : K\#L^\uparrow \to M$. For $l \in L$, $\gamma \in \Gamma$, and $x \in K$ take

$$\Phi(\chi)(x\otimes l\gamma) := (-1)^{|x||l|}\chi(l)(x)\gamma. \tag{4.1}$$

Let us assume for the moment that $\Phi(\chi)$ is well defined. We will return to this point at the end of the proof.

The fact that $\Phi(\chi)$ is left A-linear and right Γ-linear is clear. Right A-linearity follows from right A-linearity of $\chi(l)$ and the fact that coaction on $L^\uparrow$ restricts to a trivial coaction on L. For left Γ-linearity, let $x \in K$, $l \in L \subset L^\uparrow$, and $\gamma \in \Gamma$. We have

$$\begin{aligned}
\Phi(\chi)(\gamma(x\otimes l)) &= \Phi(\chi)(\gamma_1 x \otimes \gamma_2 l)\\
&= \Phi(\chi)\big(\gamma_1 x \otimes (l\cdot S^{-1}(\gamma_3))\gamma_2\big)\\
&= (-1)^{|x||l|}\chi(l\cdot S^{-1}(\gamma_3))(\gamma_1 x)\gamma_2\\
&= (-1)^{|x||l|}\gamma_5\chi(l)(S^{-1}(\gamma_4)\gamma_1 x)S^{-1}(\gamma_3)\gamma_2 \quad (\Gamma\text{-linearity of } \chi)\\
&= (-1)^{|x||l|}\gamma_3\chi(l)(S^{-1}(\gamma_2)\gamma_1 x)\\
&= (-1)^{|x||l|}\gamma\chi(l)(x)\\
&= \gamma\Phi(\chi)(x\otimes l).
\end{aligned}$$

We can use right Γ-linearity of $\Phi(\chi)$ to extend the above computation to all of $K\#L^\uparrow = K\otimes L\Gamma$. Whence we see that $\Phi(\chi)$ is a $A\#\Gamma$-bimodule map for arbitrary $\chi : L \to \mathrm{Hom}_{A^e}(K, M)$. The equalities $\Phi(\Xi(\theta)) = \theta$ and $\Xi(\Phi(\chi)) = \chi$ follow by construction. So $\Phi = \Xi^{-1}$ and $\Xi = \Phi^{-1}$.

Now, let us deal with the question of whether or not $\Phi(\chi)$ is well defined. In the case that L is free on a subspace $\bar{L}$, we will have $L^\uparrow = \Gamma\otimes\bar{L}\otimes\Gamma$. We can then define the $\mathbf{k}$-linear map $K\#L^\uparrow = K\otimes\Gamma\otimes\bar{L}\otimes\Gamma \to M$ on monomials by

$$x\otimes\gamma\otimes\bar{l}\otimes\gamma' \mapsto x\otimes\bar{l}\otimes S^{-1}(\gamma_2)\otimes\gamma_1\otimes\gamma' \mapsto (-1)^{|x||\bar{l}|}\chi(\bar{l}\cdot S^{-1}(\gamma_2))(x)\gamma_1\gamma'.$$

In the case that $\gamma\otimes\gamma' = S(\gamma_1)\otimes\gamma_2$, i.e. in the case that $\gamma\otimes\bar{l}\otimes\gamma'$ is in $L\subset L^\uparrow$, the image of this map is $(-1)^{|x||\bar{l}|}\chi(\bar{l}\cdot\gamma)(x)$. So we see that we have recovered $\Phi(\chi)$ as defined at (4.1), and it follows that $\Phi(\chi)$ is well defined. We can deal with the general case, in which L is simply projective in each degree, by noting that L will be a summand of a free resolution. ☐

Corollary 4.4. *Let L and K be as in Theorem 4.3, and M be a $A\#\Gamma$-bimodule. Then we have a graded isomorphism*

$$\mathrm{HH}(A\#\Gamma, M) \cong \mathrm{H}\big(\mathrm{Hom}_\Gamma(L, \mathrm{Hom}_{A^e}(K, M))\big).$$

Proof. This follows from Theorem 4.3 and the fact that the Hochschild cohomology is given by the homology of the complex $\mathrm{Hom}_{(A\#\Gamma)^e}(K\#L^\uparrow, M)$, since $K\#L^\uparrow$ is a projective bimodule resolution of the smash product $A\#\Gamma$. ☐

We can, in fact, replace our resolution K with any equivariant A^e-projective resolution of A. Let $P\to A$ be any Γ-equivariant A-bimodule resolution of A which is projective over A^e. By a straightforward process, we can produce an equivariant complex Q admitting equivariant quasi-isomorphisms $P\to Q$ and $K\to Q$. First, take d^0 to be the coproduct map $K^0\oplus P^0\to A$. Then we construct Q inductively as the complex

$$Q = \cdots \to K^2\oplus(A\otimes\ker d^1\otimes A)\oplus P^2 \xrightarrow{d^2} K^1\oplus(A\otimes\ker d^0\otimes A)\oplus P^1 \xrightarrow{d^1} K^0\oplus P^0\to 0,$$

where Γ acts diagonally on the summands $(A\otimes\ker d^i\otimes A)$. Note that Q is a complex of Γ-equivariant bimodules, and that each Q^i is projective over A^e. The map $d^0 : Q\to A$ is a quasi-isomorphism by construction. Whence we see that the two inclusions $i_K : K\to Q$ and $i_P : P\to Q$ are equivariant quasi-isomorphisms. Taking

$$\mathscr{X} = \mathrm{Hom}_\Gamma(L, \mathrm{Hom}_{A^e}(Q, M))$$

then gives the following corollary.

Corollary 4.5. *Let L and K be as in Theorem 4.3, and M be a complex of $A\#\Gamma$-bimodules. Let $P\to A$ be an equivariant bimodule resolution of A which is projective over A^e. The complex $\mathscr{X}$ admits quasi-isomorphisms*

$$\mathrm{Hom}_{(A\#\Gamma)^e}(K\#L^\uparrow, M) \xleftarrow{\sim} \mathscr{X} \xrightarrow{\sim} \mathrm{Hom}_\Gamma(L, \mathrm{Hom}_{A^e}(P, M)).$$

Proof. Since L is a bounded above complex of projectives, the functor $\mathrm{Hom}_\Gamma(L, -)$ preserves quasi-isomorphisms. Whence the proposed quasi-isomorphisms can be given by

$$\mathrm{Hom}_\Gamma(L, \mathrm{Hom}_{A^e}(Q, M)) \xrightarrow{(i_K^*)_*} \mathrm{Hom}_\Gamma(L, \mathrm{Hom}_{A^e}(K, M)) \cong \mathrm{Hom}_{(A\#\Gamma)^e}(K\#L^\uparrow, M)$$

and

$$\mathrm{Hom}_\Gamma(L, \mathrm{Hom}_{A^e}(Q, M)) \xrightarrow{(i_P^*)_*} \mathrm{Hom}_\Gamma(L, \mathrm{Hom}_{A^e}(P, M)).$$

□

For L and K as above, and any $A\#\Gamma$-bimodule M, the complex $\mathrm{Hom}_\Gamma(L, \mathrm{Hom}_{A^e}(K, M))$ is the total complex of the first quadrant double complex

$$\begin{tikzcd}
 & \vdots & \vdots & \\
0 \arrow[r] & \mathrm{Hom}(L^0, \mathrm{Hom}(K^{-1}, M)) \arrow[r, "{d_L}^*"] \arrow[u] & \mathrm{Hom}(L^{-1}, \mathrm{Hom}(K^{-1}, M)) \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & \mathrm{Hom}(L^0, \mathrm{Hom}(K^0, M)) \arrow[r, "{d_L}^*"] \arrow[u, "\pm(d_K^*)_*"] & \mathrm{Hom}(L^{-1}, \mathrm{Hom}(K^0, M)) \arrow[r] \arrow[u, "\pm(d_K^*)_*"] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & .
\end{tikzcd} \tag{4.2}$$

It follows that there are two spectral sequences converging to the Hochschild cohomology of $A\#\Gamma$ with coefficients in M. Filtering by the degree on L produces a spectral sequence

$$E_2 = \mathrm{Ext}_\Gamma(\mathbf{k}, \mathrm{HH}(A\#\Gamma, M)).$$

The existence of this spectral sequence is well known. It first appeared in the work of Stefan as a Grothendieck spectral sequence in the setting of a Hopf Galois extension [18], and then in a paper by Guccione and Guccione [7, Corollary 3.2.3]. Since these results are well established, we do not elaborate on the details here. We will show in Section 6 that both of these spectral sequences can be used to calculate the cup product on Hochschild cohomology when appropriate. All necessary details will be given there.

Notation 4.6. The filtration induced by the degree of L on the cohomology

$$\mathrm{HH}(A\#\Gamma, M) = \mathrm{H}(\mathrm{Hom}_\Gamma(L, \mathrm{Hom}_{A^e}(K, M)))$$

will be denoted F^Γ. The filtration induced by the degree on K will be denoted F^A. The associated graded spaces with respect to these filtrations will be denoted

$$\mathrm{gr}_\Gamma \mathrm{HH}(A\#\Gamma, M) = \bigoplus_i \frac{F_i^\Gamma\big(\mathrm{HH}(A\#\Gamma, M)\big)}{F_{i-1}^\Gamma\big(\mathrm{HH}(A\#\Gamma, M)\big)}$$

and

$$\mathrm{gr}_A \mathrm{HH}(A\#\Gamma, M) = \bigoplus_i \frac{F_i^A\big(\mathrm{HH}(A\#\Gamma, M)\big)}{F_{i-1}^A\big(\mathrm{HH}(A\#\Gamma, M)\big)}$$

respectively.

5. Reminder of the cup products on Hochschild cohomology and derived invariant algebras

The following general approach to the cup product on Hochschild cohomology follows [17]. Let R be any algebra and let B be an algebra extension of R, i.e. an algebra equipped with an algebra map $R \to B$. Let P be a projective R-bimodule resolution of R with quasi-isomorphism $\varphi : P \to R$. Then $P \otimes_R P$ is also a projective resolution of R with quasi-isomorphism $\varphi \otimes_R \varphi : P \otimes_R P \to R$. Whence there exists a quasi-isomorphism $\omega : P \to P \otimes_R P$ which fits into a diagram

$$\begin{tikzcd}
P \arrow[rr, "\omega"] \arrow[dr, "\varphi"'] & & P \otimes_R P \arrow[dl, "\varphi \otimes_R \varphi"] \\
 & R &
\end{tikzcd} \tag{5.1}$$

and is unique up to homotopy. From this we get a product map

$$\begin{gathered}\mathrm{Hom}_{R^e}(P,B)\otimes \mathrm{Hom}_{R^e}(P,B)\to \mathrm{Hom}_{R^e}(P,B)\\ f\otimes g\mapsto \mu_B(f\otimes_R g)\omega,\end{gathered}$$

and subsequent dg algebra structure on $\mathrm{Hom}_{R^e}(P,B)$. One can check that any choice of ω results in the same product on the cohomology $\mathrm{HH}(R,B)$. We call this product the *cup product*. Note that the dg algebra $\mathrm{Hom}_{R^e}(P,B)$ need not be associative, but it will be associative up to a homotopy.

Suppose now that Γ is a Hopf algebra and L is a projective resolution of $\mathbf{k}_\Gamma=\Gamma/\ker\epsilon$. Let $\mathscr{B}$ be a right Γ-module dg algebra. (We do not require that $\mathscr{B}$ is strictly associative.) Since $\Gamma\otimes\Gamma$ is free over Γ, the diagonal action on $L\otimes L$ makes it into a projective resolution of $\mathbf{k}$ as well. So, again, we have a quasi-isomorphism $\sigma: L\to L\otimes L$ which is unique up to homotopy and fits into a diagram analogous to (5.1). Hence, we get a similarly defined product on the derived invariants

$$\begin{gathered}\mathrm{Hom}_\Gamma(L,\mathscr{B})\otimes \mathrm{Hom}_\Gamma(L,\mathscr{B})\to \mathrm{Hom}_\Gamma(L,\mathscr{B})\\ f\otimes g\mapsto \mu_{\mathscr{B}}(f\otimes g)\sigma.\end{gathered}$$

This product is unique on cohomology and gives $\mathrm{Hom}_\Gamma(L,\mathscr{B})$ the structure of a (not-necessarily-associative) dg algebra.

6. Hochschild cohomology as a derived invariant algebra

Let Γ be a Hopf algebra and A be a Γ-module algebra. We also fix a bimodule resolution $\tau: K\to A$ which satisfies conditions (I) and (II) of Section 3, and a projective resolution $\xi: L\to\mathbf{k}$ of the trivial right Γ-module. From here on out we assume K also satisfies

(III) there is a quasi-isomorphism $\omega: K\to K\otimes_A K$ of complexes of Γ-equivariant A-bimodules.

As was stated in the previous section, there will always be some quasi-isomorphism ω of A^e-complexes. The content of condition (III) is that we may choose ω to be Γ-linear.

In the case of the bar resolution

$$BA=\cdots\to A\otimes A^{\otimes 2}\otimes A\to A\otimes A\otimes A\to A\otimes A\to 0$$

the map ω is given by

$$\omega: b\otimes a_1\otimes\ldots\otimes a_n\otimes b'\mapsto \sum_{0\le i\le n}(b\otimes a_1\otimes\ldots\otimes a_i\otimes 1)\otimes_A(1\otimes a_{i+1}\otimes\ldots\otimes a_n\otimes b') \qquad (6.1)$$

We will denote the image of ω using a Sweedler's type notation, as if ω were a comultiplication. Specifically, on elements we take $\omega_1(x)\otimes_A\omega_2(x)=\omega(x)$, with the sum suppressed. In this notation, Γ-linearity of ω is equivalent to the equality $\omega_1(\gamma x)\otimes_A\omega_2(\gamma x)=\gamma_1\omega_1(x)\otimes_A\gamma_2\omega_2(x)$, for all $\gamma\in\Gamma$ and $x\in K$.

Let us also fix a quasi-isomorphism $\sigma: L\to L\otimes L$. As with ω and K, we denote the image of $l\in L$ under σ by $\sigma_1(l)\otimes\sigma_2(l)$. In this notation Γ-linearity appears as $\sigma_1(l\cdot\gamma)\otimes\sigma_2(l\cdot\gamma)=\sigma_1(l)\cdot\gamma_1\otimes\sigma_2(l)\cdot\gamma_2$.

Proposition 6.1. *For any algebra extension B of $A\#\Gamma$, the complex $\mathrm{Hom}_{A^e}(K,B)$, with the product of Section 5 and Γ-action of Definition 4.1, is a right Γ-module dg algebra.*

Proof. Let us denote the multiplication on $\mathrm{Hom}_{A^e}(K,B)$ by juxtaposition. We need to show that for functions $f,g\in\mathrm{Hom}_{A^e}(K,M)$, and $\gamma\in\Gamma$, the formula $(fg)\cdot\gamma=(f\cdot\gamma_1)(g\cdot\gamma_2)$ holds. Let us simply check on elements. We have, for any $x\in K$,

$$\begin{aligned}((fg)\cdot\gamma)(x)&=S(\gamma_1)(fg)(\gamma_2 x)\gamma_3\\ &=\pm S(\gamma_1)f(\gamma_2\omega_1(x))g(\gamma_3\omega_2(x))\gamma_4 && \text{(by Γ-linearity of ω)}\\ &=\pm\big(S(\gamma_1)f(\gamma_2\omega_1(x))\gamma_3\big)\big(S(\gamma_4)g(\gamma_5\omega_2(x))\gamma_6\big)\\ &=(f\cdot\gamma_1)(\omega_1(x))(g\cdot\gamma_2)(\omega_2(x))\\ &=\big((f\cdot\gamma_1)(g\cdot\gamma_2)\big)(x).\end{aligned}$$

□

According to this proposition, and the material of Section 5, the double complex $\mathrm{Hom}_\Gamma(L,\mathrm{Hom}_{A^e}(K,B))$ will now cary a natural dg algebra structure.

We now seek to extend the diagonal map σ on L to a diagonal map on the induced complex $L^\uparrow$. One can verify that the obvious map $L\otimes L\to L^\uparrow\otimes_\Gamma L^\uparrow$ is an embedding, since the statement holds when L is free.

In this way we view $L \otimes L$ as a subcomplex of $L^\uparrow \otimes_\Gamma L^\uparrow$. The complex $L^\uparrow \otimes_\Gamma L^\uparrow$ is taken to be a Γ-comodule under the standard tensor Γ-comodule structure $l \otimes_\Gamma l' \mapsto (l_{-1}l'_{-1}) \otimes (l_0 \otimes_\Gamma l'_0)$. Since $L^\uparrow$ is itself a Hopf bimodule over Γ, this coaction gives $L^\uparrow \otimes_\Gamma L^\uparrow$ the structure of a Hopf bimodule as well. Before giving the next result we also note that, on elements, commutativity of the diagram

$$\begin{tikzcd} L \arrow[rr, "\sigma"] \arrow[dr, "\xi"'] & & L \otimes L \arrow[dl, "\xi\otimes\xi"] \\ & \mathbf{k} & \end{tikzcd}$$

produces the equality $\xi(l) = \xi(\sigma_1(l))\xi(\sigma_2(l))$ for each $l \in L$.

Lemma 6.2. *The map $\sigma : L \to L \otimes L \subset L^\uparrow \otimes_\Gamma L^\uparrow$ extends uniquely to a quasi-isomorphism of chain complexes of Hopf-bimodules $\sigma^\uparrow : L^\uparrow \to L^\uparrow \otimes_\Gamma L^\uparrow$.*

Proof. Let $\cdot$ denote the right action of Γ on L and juxtaposition denote the action of Γ on the bimodule $L^\uparrow$. Take $l \in L \subset L^\uparrow$ and $\gamma, \gamma' \in \Gamma$. We extend σ to all of $L^\uparrow$ according to the formula

$$\sigma^\uparrow(\gamma l \gamma') := \gamma\sigma_1(l) \otimes_\Gamma \sigma_2(l)\gamma'.$$

Recall that $l \cdot \gamma = S(\gamma_1) l \gamma_2$. To see that $\sigma^\uparrow$ is well defined, we first define the Γ^e-linear map

$$\Sigma : L \otimes \Gamma^e = \Gamma \otimes L \otimes \Gamma \to L^\uparrow \otimes_\Gamma L^\uparrow$$

by the same formula $\Sigma(\gamma \otimes l \otimes \gamma') = \gamma\sigma_1(l) \otimes_\Gamma \sigma_2(l)\gamma'$. The computation

$$\begin{aligned} \Sigma(S(\gamma_1) \otimes l \otimes \gamma_2) \quad &= S(\gamma_1)\sigma_1(l) \otimes_\Gamma \sigma_2(l)\gamma_2 \\ &= S(\gamma_1)\sigma_1(l)\gamma_2 \otimes_\Gamma S(\gamma_3)\sigma_2(l)\gamma_4 \\ &= \big(\sigma_1(l) \cdot \gamma_1\big) \otimes_\Gamma \big(\sigma_2(l) \cdot \gamma_2\big) \\ &= \sigma_1(l \cdot \gamma) \otimes_\Gamma \sigma_2(l \cdot \gamma) \qquad \text{(by Γ-linearity of σ)} \\ &= \Sigma(1 \otimes (l \cdot \gamma) \otimes 1) \end{aligned}$$

shows that $\sigma^\uparrow$ respects the necessary relations to induce a map on the quotient $L^\uparrow = L \otimes_\Gamma \Gamma^e$. This recovers our original map $\sigma^\uparrow$, and shows that it is in fact well defined.

Recall that $\sigma : L \to L \otimes L$ was chosen so that $\xi(l) = \xi(\sigma_1(l))\xi(\sigma_2(l))$ for all $l \in L$, and that $\xi^\uparrow : L^\uparrow \to \Gamma$ is defined by $\gamma l \gamma' \mapsto \gamma\xi(l)\gamma'$. So we will have the commutative diagram

$$\begin{tikzcd} L^\uparrow \arrow[rr, "\sigma^\uparrow"] \arrow[dr, "\xi^\uparrow"'] & & L^\uparrow \otimes_\Gamma L^\uparrow \arrow[dl, "\xi^\uparrow\otimes_\Gamma\xi^\uparrow"] \\ & \Gamma & \end{tikzcd}.$$

The fact that $\sigma^\uparrow$ is a quasi-isomorphism follows from commutativity of the above diagram and the fact that $\xi^\uparrow$ and $\xi^\uparrow \otimes_\Gamma \xi^\uparrow$ are quasi-isomorphisms. As for colinearity of $\sigma^\uparrow$, by the definition of the coaction on the induced complex $L^\uparrow$ given in Definition 2.4, we have

$$\begin{aligned} (\sigma^\uparrow(\gamma l \gamma'))_{-1} \otimes (\sigma^\uparrow(\gamma l \gamma'))_0 \quad &= \big(\gamma\sigma_1(l) \otimes \sigma_2(l)\gamma'\big)_{-1} \otimes \big(\gamma\sigma_1(l) \otimes \sigma_2(l)\gamma'\big)_0 \\ &= (\gamma_1\gamma'_1) \otimes (\gamma_2\sigma_1(l) \otimes \sigma_2(l)\gamma'_2) \\ &= (\gamma_1\gamma'_1) \otimes \sigma^\uparrow(\gamma_2 l \gamma'_2) \\ &= (\gamma l \gamma')_{-1} \otimes \sigma^\uparrow((\gamma l \gamma)_0). \end{aligned}$$

$\square$

Now we have a quasi-isomorphism $\omega : K \to K \otimes_A K$ and have produced a quasi-isomorphism $\sigma^\uparrow : L^\uparrow \to L^\uparrow \otimes_\Gamma L^\uparrow$ from the given map $\sigma : L \to L \otimes L$. We would like to use this information, along with some twisting, to produce an explicit quasi-isomorphism

$$K\#L^\uparrow \to (K\#L^\uparrow) \otimes_{A\#\Gamma} (K\#L^\uparrow).$$

The next lemma offers the "twisting" portion of the proposed construction.

Lemma 6.3. *The isomorphism of* $\mathbf{k}$*-complexes*

$$(K \otimes_A K) \otimes (L \otimes L) \to (K \otimes L) \otimes_A (K \otimes L)$$
$$(x \otimes_A y) \otimes (l \otimes l') \mapsto (-1)^{|l||y|}(x \otimes l) \otimes_A (y \otimes l')$$

extends uniquely to an isomorphism $\phi : (K \otimes_A K)\#(L^\uparrow \otimes_\Gamma L^\uparrow) \to (K\#L^\uparrow) \otimes_{A\#\Gamma} (K\#L^\uparrow)$ *of complexes of* $A\#\Gamma$*-bimodules.*

Proof. The map ϕ is given by

$$\phi : (x \otimes_A y) \otimes (l \otimes_\Gamma l') \mapsto (-1)^{|l||y|}(x \otimes l_0) \otimes_{A\#\Gamma} (S^{-1}(l_{-1})y \otimes l'),$$

for $x, y \in K$, $l, l' \in L^\uparrow$. The fact that ϕ is well defined follows by standard manipulations, which we do not reproduce here. The fact that ϕ is a chain map can be verified by using the Γ-linearity and Γ-colinearity of the differentials on K and $L^\uparrow$ respectively.

In order to show that ϕ is an $A\#\Gamma$-bimodule map, the only non-trivial things to check are left Γ-linearity and right A-linearity. For left Γ-linearity we have, for any $\gamma \in \Gamma$,

$$\begin{aligned}
\phi(\gamma((x \otimes_A y) \otimes (l \otimes_\Gamma l'))) &= \phi((\gamma_1 x \otimes_A \gamma_2 y) \otimes (\gamma_3 l \otimes_\Gamma l')) \\
&= \pm(\gamma_1 x \otimes \gamma_4 l_0) \otimes_{A\#\Gamma} (S^{-1}(\gamma_3 l_{-1})\gamma_2 y \otimes l') \\
&= \pm(\gamma_1 x \otimes \gamma_4 l_0) \otimes_{A\#\Gamma} (S^{-1}(l_{-1})S^{-1}(\gamma_3)\gamma_2 y \otimes l') \\
&= \pm(\gamma_1 x \otimes \gamma_2 l_0) \otimes_{A\#\Gamma} (S^{-1}(l_{-1})y \otimes l') \\
&= \pm\gamma\left((x \otimes l_0) \otimes_{A\#\Gamma} (S^{-1}(l_{-1})y \otimes l')\right) \\
&= \gamma\phi((x \otimes_A y) \otimes (l \otimes_\Gamma l')).
\end{aligned}$$

For right A-linearity we have, for any $a \in A$,

$$\begin{aligned}
\phi(((x \otimes_A y) \otimes (l \otimes_\Gamma l'))a) &= \phi((x \otimes_A y({}^{l_{-1}l'_{-1}}a)) \otimes (l_0 \otimes_\Gamma l'_0)) \\
&= \pm(x \otimes l_0) \otimes_{A\#\Gamma} (S^{-1}(l_{-1})(y({}^{l_{-2}l'_{-1}}a)) \otimes l'_0) \\
&= \pm(x \otimes l_0) \otimes_{A\#\Gamma} ((S^{-1}(l_{-1})y)({}^{S^{-1}(l_{-2})l_{-3}l'_{-1}}a)) \otimes l'_0) \\
&= \pm(x \otimes l_0) \otimes_{A\#\Gamma} ((S^{-1}(l_{-1})y)({}^{l'_{-1}}a)) \otimes l'_0) \\
&= \pm\left((x \otimes l_0) \otimes_{A\#\Gamma} (S^{-1}(l_{-1})y \otimes l'_0)\right) a \\
&= \phi((x \otimes_A y) \otimes (l \otimes_\Gamma l'))a.
\end{aligned}$$

The inverse to ϕ is the map

$$(x \otimes l) \otimes_{A\#\Gamma} (y \otimes l') \mapsto (-1)^{|l||y|}(x \otimes_A l_{-1}y) \otimes (l_0 \otimes_\Gamma l').$$

□

Proposition 6.4. *Let* ϕ *be the isomorphism of Lemma 6.3, and* $\sigma^\uparrow : L^\uparrow \to L^\uparrow \otimes_\Gamma L^\uparrow$ *be the quasi-isomorphism of Lemma 6.2. Then the map*

$$\phi(\omega \otimes \sigma^\uparrow) : K\#L^\uparrow \to K\#L^\uparrow \otimes_{A\#\Gamma} K\#L^\uparrow$$

is a quasi-isomorphism of $(A\#\Gamma)^e$*-complexes.*

Proof. Since $\sigma^\uparrow$ is a map of complexes of Hopf bimodules by Lemma 6.2, and ω is a map of complexes of equivariant A-bimodules by choice, $\omega \otimes \sigma^\uparrow$ is $(A\#\Gamma)^e$-linear, by Lemma 3.3. Also, the product map $\omega \otimes \sigma^\uparrow$ is a quasi-isomorphism since ω and $\sigma^\uparrow$ are themselves quasi-isomorphisms. Now, since ϕ is an isomorphism of $(A\#\Gamma)^e$-complexes, the claim follows. □

For any algebra extension $A\#\Gamma \to B$, we define the cup product on $\mathrm{Hom}_{(A\#\Gamma)^e}(K\#L^\uparrow, B)$ by way of the diagonal map $K\#L^\uparrow \to K\#L^\uparrow \otimes_{A\#\Gamma} K\#L^\uparrow$ given in Proposition 6.4.

The following are the **hypotheses for Theorem 6.5**: K is a bimodule resolution of A equipped with a diagonal map $\omega : K \to K \otimes_A K$ satisfying conditions (I)-(III), and L is a projective resolution of the trivial right Γ-module $\mathbf{k}$ with a quasi-isomorphism $\sigma : L \to L \otimes L$. We give $K\#L^\uparrow$ the diagonal quasi-isomorphism of Proposition 6.4.

Theorem 6.5. *For any algebra extension* B *of the smash product* $A\#\Gamma$*, the isomorphism*

$$\Xi : \mathrm{Hom}_{(A\#\Gamma)^e}(K\#L^\uparrow, B) \overset{\cong}{\to} \mathrm{Hom}_\Gamma(L, \mathrm{Hom}_{A^e}(K, B))$$

of Theorem 4.3 is one of (not-necessarily-associative) dg algebras.

Let us note that, if K and L are chosen appropriately, the dg algebra $\mathrm{Hom}_\Gamma(L, \mathrm{Hom}_{A^e}(K,B))$ will be associative. It follows, by the theorem, that $\mathrm{Hom}_{(A\#\Gamma)^e}(K\#L^\uparrow, B)$ will also be associative in this case. For example, one can always take K to be the bar resolution BA of A and L to be the bar resolution $\mathbf{k}\otimes_\Gamma B\Gamma$ of $\mathbf{k}$ to get this property.

Proof. We want to verify commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Hom}(K\#L^\uparrow, B)\otimes \mathrm{Hom}(K\#L^\uparrow, B) \arrow[r, "\Xi\otimes\Xi"] \arrow[d, "mult"] & \mathrm{Hom}(L, \mathrm{Hom}(K,B))\otimes \mathrm{Hom}_\Gamma(L, \mathrm{Hom}(K,B)) \arrow[d, "mult"] \\
\mathrm{Hom}(K\#L^\uparrow, B) \arrow[r, "\Xi"] & \mathrm{Hom}(L, \mathrm{Hom}(K,B)).
\end{tikzcd} \tag{6.2}$$

There are three multiplications we need to deal with here. For the purpose of this proof we will denote the products on

$$\mathrm{Hom}_{(A\#\Gamma)^e}(K\#L^\uparrow, B),\ \mathrm{Hom}(L, \mathrm{Hom}_{A^e}(K,B)),\ \text{and } \mathrm{Hom}_{A^e}(K,B)$$

by a dot $\cdot$, an asterisk $*$, and juxtaposition respectively. Let θ and θ' be functions in $\mathrm{Hom}_{(A\#\Gamma)^e}(K\#L^\uparrow, A\#\Gamma)$ and fix arbitrary $x\in K$ and $l\in L\subset L^\uparrow$.

Following around the top of (6.2) sends $\theta\otimes\theta'$ to the function $\Xi(\theta)*\Xi(\theta')\in \mathrm{Hom}_\Gamma(L, \mathrm{Hom}_{A^e}(K,B))$. This function sends $l\in L$ to the map

$$(-1)^{|\theta'||\sigma_1(l)|}\theta(-\otimes\sigma_1(l))\theta'(-\otimes\sigma_2(l))$$

in $\mathrm{Hom}_{A^e}(K,B)$, where $\theta(-\otimes\sigma_1(l))$ and $\theta'(-\otimes\sigma_2(l))$ are as defined in the paragraphs preceding Theorem 4.3. Since the coaction on $L^\uparrow$ restricts to the trivial coaction on L, the above function evaluated at x is the element

$$(-1)^\epsilon\theta\left(\omega_1(x)\otimes\sigma_1(l)\right)\theta'\left(\omega_2(x)\otimes\sigma_2(l)\right)\in B,$$

where

$$\begin{aligned}
\epsilon &= |\theta'||\sigma_1(l)| + |\omega_1(x)|(|\theta'| + |\sigma_2(l)| + |\sigma_1(l)|) + |\omega_2(x)||\sigma_2(l)| \\
&= |\theta'||\sigma_1(l)| + |\omega_1(x)|(|\theta'| + |l|) + |\omega_2(x)||\sigma_2(l)|. \\
&= |\theta'|(|\sigma_1(l)| + |\omega_1(x)|) + |\omega_1(x)||l| + |\omega_2(x)||\sigma_2(l)|.
\end{aligned}$$

Following around the bottom row sends $\theta\otimes\theta'$ to the function $\Xi(\theta\cdot\theta')\in \mathrm{Hom}_\Gamma(L, \mathrm{Hom}_{A^e}(K,B))$, which takes our element $l\in L$ to $(\theta\cdot\theta')(-\otimes l)\in \mathrm{Hom}_{A^e}(K,B)$. Evaluating at $x\in K$ produces the element $(-1)^{|x||l|}(\theta\cdot\theta')(x\otimes l)\in B$. Recalling the diagonal map on $K\#L^\uparrow$ given in Proposition 6.4, the formula for $\phi|(K\otimes_A K)\otimes(L\otimes L)$ given in Lemma 6.3, and the fact that $\sigma^\uparrow|L=\sigma$, we have the equality

$$(-1)^{|x||l|}(\theta\cdot\theta')(x\otimes l) = (-1)^{\epsilon'}\theta\left(\omega_1(x)\otimes\sigma_1(l)\right)\theta'\left(\omega_2(x)\otimes\sigma_2(l)\right),$$

where

$$\begin{aligned}
\epsilon' &= |x||l| + |\sigma_1(l)||\omega_2(x)| + |\theta'|(|\omega_1(x)| + |\sigma_1(l)|) \\
&= (|\omega_1(x)| + |\omega_2(x)|)|l| + |\sigma_1(l)||\omega_2(x)| + |\theta'|(|\omega_1(x)| + |\sigma_1(l)|) \\
&= |\omega_1(x)||l| + |\omega_2(x)||l| + |\omega_2(x)||\sigma_1(l)| + |\theta'|(|\omega_1(x)| + |\sigma_1(l)|) \\
&= |\omega_1(x)||l| + |\omega_2(x)|(|\sigma_1(l)| + |\sigma_2(l)|) + |\omega_2(x)||\sigma_1(l)| + |\theta'|(|\omega_1(x)| + |\sigma_1(l)|) \\
&\equiv |\omega_1(x)||l| + |\omega_2(x)||\sigma_2(l)| + |\theta'|(|\omega_1(x)| + |\sigma_1(l)|) \mod 2 \\
&= \epsilon.
\end{aligned}$$

So following around the top or bottom of (6.2) produces the same function. □

Corollary 6.6. *Let B be an algebra extension of $A\#\Gamma$, and take K and L as in Theorem 6.5. Then there is an isomorphism of algebras*

$$\mathrm{HH}(A\#\Gamma, B)\cong \mathrm{H}\left(\mathrm{Hom}_\Gamma(L, \mathrm{Hom}_{A^e}(K,B))\right).$$

Proof. This is an immediate consequence of Theorem 6.5 and the fact that

$$\mathrm{HH}(A\#\Gamma, B) = \mathrm{H}\big(\mathrm{Hom}_{(A\#\Gamma)^e}(K\#L^\uparrow, M)\big)$$

as an algebra. □

As was the case with Theorem 4.3, we can drop condition (II) on K.

Corollary 6.7. *Let K, L, and B be as in Theorem 6.5. Let $P \to A$ be a Γ-equivariant bimodule resolution of A which is projective over A^e in each degree. Suppose additionally that P admits a diagonal quasi-isomorphism $P \to P \otimes_A P$ which is Γ-linear. Then there is third dg algebra $\mathscr{A}$ which admits quasi-isomorphisms*

$$\mathrm{Hom}_{(A\#\Gamma)^e}(K\#L^{\uparrow}, B) \overset{\sim}{\to} \mathscr{A} \overset{\sim}{\leftarrow} \mathrm{Hom}_\Gamma(L, \mathrm{Hom}_{A^e}(P, B)) \tag{6.3}$$

which are all algebra maps up to a homotopy.

Let η denote any of the maps of (6.3). The main point is that for any cycles f and g in the domain, the difference $\eta(fg) - \eta(f)\eta(g)$ will be a boundary. So all of the maps of (6.3) become algebra isomorphisms on homology. The proof of this result is a bit of distraction, and has been relegated to the appendix.

As was discussed in the introduction, a spectral sequence E is called multiplicative if it comes equipped with a bigraded products $E_r^{pq} \otimes E_r^{p'q'} \to E_r^{(p+p')(q+q')}$, for each r, such that each differential $d_r : E_r \to E_r$ is a graded derivation and each isomorphism $E_{r+1} \cong H(E_r)$ is one of algebras. The spectral sequence associated to any filtered dg algebra will be multiplicative, for example. For any multiplicative spectral sequence E, the limiting term E_∞ has the natural structure of a bigraded algebra [19, Multiplicative Structures 5.4.8]. We say that a multiplicative spectral sequence converges to a graded algebra H if H carries an additional filtration and we have an isomorphism of bigraded algebras $E_\infty = \mathrm{grH}$.

Recall the filtrations F^A and F^Γ on $\mathrm{HH}(A\#\Gamma, B)$ given in Notation 4.6. Since the multiplication on the double complex $\mathrm{Hom}_\Gamma(L, \mathrm{Hom}_{A^e}(K, B))$ is bigraded, both the row and column filtrations (i.e. the filtrations induced by the degrees on K and L) give it the structure of a filtered dg algebra. It follows that both of the associated spectral sequences are multiplicative. It also follows that F^A and F^Γ are algebra filtrations on the Hochschild cohomology.

Corollary 6.8. *For any algebra extension B of the smash product $A\#\Gamma$, there are two multiplicative spectral sequences*

$$E_2 = \mathrm{Ext}_\Gamma(\mathbf{k}, \mathrm{HH}(A, B)) \Rightarrow \mathrm{HH}(A\#\Gamma, B)$$

and

$${}'E_1 = \mathrm{Ext}_\Gamma(\mathbf{k}, \mathrm{RHom}_{A\text{-bimod}}(A, B)) \Rightarrow \mathrm{HH}(A\#\Gamma, B)$$

which converge to the Hochschild cohomology as an algebra.

Proof. These spectral sequences are induced by the row and column filtrations on the (first quadrant) double complex $\mathrm{Hom}_\Gamma(L, \mathrm{Hom}_{A^e}(K, B))$. Since the product on $\mathrm{Hom}_\Gamma(L, \mathrm{Hom}_{A^e}(K, B))$ respects both of these filtrations, and its homology is the Hochschild cohomology ring $\mathrm{HH}(A\#\Gamma, B)$, both of the spectral sequences are multiplicative and converge to the Hochschild cohomology.

Filtering by the degree on K produces the spectral sequence with ${}'E_1 = \mathrm{Ext}_\Gamma(\mathbf{k}, \mathrm{RHom}_{A^e}(A, B))$, where we take $\mathrm{RHom}_{A^e}(A, B) = \mathrm{Hom}_{A^e}(K, B)$. Filtering by the degree on L produces a spectral sequence with $E_1 = \mathrm{Hom}_\Gamma(L, \mathrm{HH}(A, B))$, since each $\mathrm{Hom}_\Gamma(L^i, -)$ is exact and hence commutes with homology. Since the differentials on E_1 are given by d_L^*, it follows that the E_2 term is as described. □

In the language of Notation 4.6, the E_∞-terms of these spectral sequences are the bigraded algebras $\mathrm{gr}_\Gamma \mathrm{HH}(A\#\Gamma, B)$ and $\mathrm{gr}_A \mathrm{HH}(A\#\Gamma, B)$ respectively.

Corollary 6.9. *If the global dimension of Γ is ≤ 1 then we have an isomorphism of algebras*

$$\mathrm{gr}_\Gamma \mathrm{HH}(A\#\Gamma, B) \cong \mathrm{Ext}_\Gamma(\mathbf{k}, \mathrm{HH}(A, B)).$$

Proof. In this case $E_2 = E_\infty$. □

7. Algebras of extensions as derived invariant algebras

The main purpose of this section is to give some multiplicative spectral sequences converging to the algebra $\mathrm{Ext}_{A\#\Gamma}(M, M)$, for any $A\#\Gamma$-module M. As was the case with Hochschild cohomology, we will derive our spectral sequences from some explicit isomorphism at the level of cochains. Some related spectral sequences for groups of extensions, without multiplicative structures, were given in [7, Section 3.2.6].

For any algebra R and R-modules M and N, there is a natural bimodule structure on $\mathrm{Hom}_{\mathbf{k}}(M, N)$ induced by the left R-actions on M and N. In the case that $M = N$, the bimodule structure is induced by the associated representation $R \to \mathrm{End}_{\mathbf{k}}(M)$. Since the representation $R \to \mathrm{End}_{\mathbf{k}}(M)$ is an algebra map, the endomorphism algebra of any R-module has the structure of an algebra extension of R.

The Yoneda product on $\mathrm{Ext}_R(M, M)$ is defined in the following manner: first take a projective resolution $Q \to M$, then define $\mathrm{Ext}_R(M, M)$ as the homology algebra of the endomorphism dg algebra $\mathrm{End}_R(Q)$. An alternate approach to the Yoneda product will be given in Corollary 7.3. Recall that, for any bimodule resolution P of R, the tensor product $P \otimes_R M$ provides a projective resolution of M. This fact is a consequence of Künneth's spectral sequence.

Proposition 7.1 ([19, Lemma 9.1.9])**.** *Let R be any algebra, P be a projective R-bimodule resolution of R, and M and N be any modules over R. The* $\otimes$-Hom *adjunction gives an isomorphism of complexes*

$$\mathrm{Hom}_{R^e}(P, \mathrm{Hom}_{\mathbf{k}}(M, N)) \overset{\cong}{\to} \mathrm{Hom}_R(P \otimes_R M, N).$$

In the case that $M = N$, and P is the bar resolution $P = BR$, there is a quasi-isomorphisms of dg algebras

$$\mathrm{Hom}_{R^e}(P, \mathrm{End}_{\mathbf{k}}(M)) \overset{\sim}{\to} \mathrm{End}_R(P \otimes_R M).$$

Strictly speaking, only the first portion of this proposition is given in Weibel's text. The compatibility of the cup product with the Yoneda product, in the case that $M = N$, should certainly also be well known. One simply verifies that the map sending $f \in \mathrm{Hom}_B(P, \mathrm{End}_{\mathbf{k}}(M))$ to the function

$$\begin{aligned} &r \otimes r_1 \otimes \ldots \otimes r_n \otimes m \\ &\quad \mapsto (-1)^{|f|(n-|f|)} r \otimes r_1 \otimes \ldots \otimes r_{n-|f|} \otimes f(1 \otimes r_{|n|-|f|+1} \otimes \ldots \otimes r_n \otimes 1)(m) \end{aligned}$$

in $\mathrm{End}_R(P \otimes_R M)$ is a morphism dg algebras. The fact that the proposed map is a quasi-isomorphism follows by commutativity of the diagram

$$\begin{tikzcd} & \mathrm{End}_R(P \otimes_R M) \arrow[dr, "\sim"] & \\ \mathrm{Hom}_{R^e}(P, \mathrm{End}_{\mathbf{k}}(M)) \arrow[ur] \arrow[rr, "\cong"] & & \mathrm{Hom}_R(P \otimes_R M, M). \end{tikzcd}$$

Corollary 7.2. *There is a canonical isomorphism of graded vector spaces* $\mathrm{HH}(R, \mathrm{Hom}_{\mathbf{k}}(M, N)) \cong \mathrm{Ext}_R(M, N)$ *and isomorphism of graded algebras* $\mathrm{HH}(R, \mathrm{End}_{\mathbf{k}}(M)) \cong \mathrm{Ext}_R(M, M)$*.*

Note that, for any projective bimodule resolution $P \to R$ with diagonal quasi-isomorphism $\omega : P \to P \otimes_R P$, the dg algebra structure on $\mathrm{Hom}_{R^e}(P, \mathrm{End}(M))$ induces a dg algebra structure on the complex $\mathrm{Hom}_R(P \otimes_R M, M)$ by way of the adjunction isomorphism of Proposition 7.1. For functions $f, g \in \mathrm{Hom}_R(P \otimes_R M, M)$, the product $fg \in \mathrm{Hom}_R(P \otimes_R M, M)$ will be given by

$$fg : x \otimes_R m \mapsto (-1)^{|g||\omega_1(x)|} f(\omega_1(x) \otimes_A g(\omega_2(x) \otimes_A m)),$$

where the notation $\omega(x) = \omega_1(x) \otimes_A \omega_2(x)$ is as in Section 6.

Corollary 7.3. *Let M be any R-module, P be any projective R-bimodule resolution over R, and $\omega : P \to P \otimes_R P$ be any R^e-linear quasi-isomorphism. Give* $\mathrm{Hom}_{R^e}(P \otimes_R M, M)$ *the dg algebra structure outlined above. Then*

$$\mathrm{Ext}_R(M, M) \cong \mathrm{H}(\mathrm{Hom}_R(P \otimes_R M, M))$$

as an algebra.

Proof. The algebra structure on $\mathrm{Hom}_{R^e}(P, \mathrm{End}(M))$ is defined so that the isomorphism $\mathrm{Hom}_{R^e}(P, \mathrm{End}(M)) \overset{\cong}{\to} \mathrm{Hom}_R(P \otimes_R M, M)$ is one of dg algebras. In the case that P is the bar resolution, there is an isomorphism of algebras

$$\mathrm{H}(\mathrm{Hom}_R(P \otimes_R M, M)) \cong \mathrm{H}(\mathrm{Hom}_{R^e}(P, \mathrm{End}(M))) \cong \mathrm{Ext}_R(M, M)$$

by Proposition 7.1. The result now follows from the fact that the cup product on $\mathrm{H}(\mathrm{Hom}_{R^e}(P, \mathrm{End}(M))) = \mathrm{HH}(R, \mathrm{End}_{\mathbf{k}}(M))$ can be computed using any resolution and any quasi-isomorphism $\omega : P \to P \otimes_A P$. $\square$

Let us return to our analysis of the cohomology of smash products. We fix a Hopf algebra Γ and Γ-module algebra A. Before giving the main results let us clarify a possible point of confusion.

For any $A\#\Gamma$-modules M and N there is a standard way to endow $\mathrm{RHom}_A(M,N)$ with a right Γ-module structure. One simply takes a $A\#\Gamma$-complex Q which is a projective resolution of M over A and defines the action on $\mathrm{RHom}_A(M,N) = \mathrm{Hom}_A(Q,N)$ by

$$f \cdot \gamma(q) := S(\gamma_1) f(\gamma_2 q).$$

We would like to know that this Γ-module structures agree with the Γ-module structure on $\mathrm{RHom}_{A^e}(A, \mathrm{End}_{\mathbf{k}}(M,N))$ given in Section 4.

Lemma 7.4. *Let M and N be modules over $A\#\Gamma$ and K be a projective bimodule resolution of A satisfying conditions (I) and (II). Then the complex $K \otimes_A M$ is a $A\#\Gamma$-complex under the diagonal Γ-action and the quasi-isomorphism $K \otimes_A M \to M$ is Γ-linear. Furthermore, the isomorphism*

$$\mathrm{Hom}_{A^e}(K, \mathrm{Hom}_{\mathbf{k}}(M,N)) \overset{\cong}{\to} \mathrm{Hom}_A(K \otimes_A M, N).$$

is one of complexes of right Γ-modules.

Since this lemma is not essential to the remainder of the paper, the proof is deferred to the appendix. We now give the main results of the section.

Theorem 7.5. *Let M and N be $A\#\Gamma$-modules, L be a projective resolution of the trivial right Γ-module $\mathbf{k}$, and K be a projective A-bimodule resolution of A satisfying (I) and (II). Then there is an isomorphism of chain complexes*

$$\mathrm{Hom}_{A\#\Gamma}(K\#L^{\uparrow} \otimes_{A\#\Gamma} M, N) \overset{\cong}{\to} \mathrm{Hom}_{\Gamma}(L, \mathrm{Hom}_A(K \otimes_A M, N)).$$

When the hypotheses of Theorem 6.5 are satisfied, the isomorphism

$$\mathrm{Hom}_{A\#\Gamma}(K\#L^{\uparrow} \otimes_{A\#\Gamma} M, M) \overset{\cong}{\to} \mathrm{Hom}_{\Gamma}(L, \mathrm{Hom}_A(K \otimes_A M, M))$$

is one of (non-necessarily-associative) dg algebras.

In the above statement, $\mathrm{Hom}_{A\#\Gamma}(K\#L^{\uparrow} \otimes_{A\#\Gamma} M, M)$ and $\mathrm{Hom}_A(K \otimes_A M, M)$ are supposed to have the algebra structures of Corollary 7.3.

Proof. From Proposition 7.1, Theorem 4.3, and the previous lemma, we get a sequence of isomorphisms of chain complexes

$$\begin{aligned} \mathrm{Hom}_{A\#\Gamma}(K\#L^{\uparrow} \otimes_{A\#\Gamma} M, N) &\cong \mathrm{Hom}_{(A\#\Gamma)^e}(K\#L^{\uparrow}, \mathrm{Hom}_{\mathbf{k}}(M,N)) \\ &\cong \mathrm{Hom}_{\Gamma}(L, \mathrm{Hom}_{A^e}(K, \mathrm{Hom}_{\mathbf{k}}(M,N)) \\ &\cong \mathrm{Hom}_{\Gamma}(L, \mathrm{Hom}_A(K \otimes_A M, N)). \end{aligned}$$

Suppose now that $N = M$ and that the hypotheses of Theorem 6.5 are met. Then the second isomorphism

$$\mathrm{Hom}_{(A\#\Gamma)^e}(K\#L^{\uparrow}, \mathrm{Hom}_{\mathbf{k}}(M,N)) \cong \mathrm{Hom}_{\Gamma}(L, \mathrm{Hom}_{A^e}(K, \mathrm{Hom}_{\mathbf{k}}(M,N))$$

is one of dg algebras by Theorem 6.5. The isomorphisms

$$\mathrm{Hom}_{A\#\Gamma}(K\#L^{\uparrow} \otimes_{A\#\Gamma} M, N) \cong \mathrm{Hom}_{(A\#\Gamma)^e}(K\#L^{\uparrow}, \mathrm{Hom}_{\mathbf{k}}(M,N))$$

and

$$\mathrm{Hom}_{A^e}(K, \mathrm{Hom}_{\mathbf{k}}(M,N)) \cong \mathrm{Hom}_A(K \otimes_A M, N)$$

are isomorphisms of dg algebras by the definition of the multiplicative structures considered in Corollary 7.3. It follows that the final isomorphism

$$\mathrm{Hom}_{\Gamma}(L, \mathrm{Hom}_{A^e}(K, \mathrm{Hom}_{\mathbf{k}}(M,N)) \cong \mathrm{Hom}_{\Gamma}(L, \mathrm{Hom}_A(K \otimes_A M, N))$$

is one of dg algebras as well. Taking this all together gives the proposed result. ☐

In the complex $\mathrm{Hom}_{\Gamma}(L, \mathrm{Hom}_A(K \otimes_A M, N))$, we may replace $K \otimes_A M$ with any Γ-linear A-projective resolution $Q \to M$. The proof of this fact is the similar to the one given for Corollary 4.5. In the entire statement of Theorem 7.5 we could have also replaces K with any A^e-projective Γ-equivariant resolution $P \to A$ with an equivariant diagonal, by Corollary 6.7.

Corollary 7.6. *Let M and N be modules over $A\#\Gamma$, and K and L be as in Theorem 6.5. Then there is an isomorphism of graded vector spaces*

$$\mathrm{Ext}_{A\#\Gamma}(M,N) \cong \mathrm{H}\big(\mathrm{Hom}_\Gamma(L, \mathrm{Hom}_A(K \otimes_A M, N))\big)$$

and an isomorphism of graded algebras

$$\mathrm{Ext}_{A\#\Gamma}(M,M) \cong \mathrm{H}\big(\mathrm{Hom}_\Gamma(L, \mathrm{Hom}_A(K \otimes_A M, M))\big).$$

Proof. This follows by the isomorphism of dg algebras

$$\mathrm{Hom}_{A\#\Gamma}(K\#L^\uparrow \otimes_{A\#\Gamma} M, M) \xrightarrow{\cong} \mathrm{Hom}_\Gamma(L, \mathrm{Hom}_A(K \otimes_A M, M))$$

of Theorem 7.5 and the fact that

$$\mathrm{Ext}_{A\#\Gamma}(M,M) = \mathrm{H}\big(\mathrm{Hom}_{A\#\Gamma}(K\#L^\uparrow \otimes_{A\#\Gamma} M, M)\big)$$

as an algebra, by Corollary 7.3. □

Of course, from the Theorem we can also derive some multaplicative spectral sequences converging to the Ext algebra $\mathrm{Ext}_{A\#\Gamma}(M,M)$. We define the filtrations F^Γ and F^A on each $\mathrm{Ext}_A(M,M)$ in the same manner as was done at Notation 4.6.

Corollary 7.7. *For any $A\#\Gamma$-module M, there are two multiplicative spectral sequences*

$$E_2 = \mathrm{Ext}_{\Gamma\text{-mod}}(\mathbf{k}, \mathrm{Ext}_{A\text{-mod}}(M,M)) \Rightarrow \mathrm{Ext}_{A\#\Gamma\text{-mod}}(M,M)$$

and

$$'E_1 = \mathrm{Ext}_{\Gamma\text{-mod}}(\mathbf{k}, \mathrm{RHom}_{A\text{-mod}}(M,M)) \Rightarrow \mathrm{Ext}_{A\#\Gamma\text{-mod}}(M,M)$$

which converge to $\mathrm{Ext}_{A\#\Gamma\text{-mod}}(M,M)$ as an algebra. In the case that the global dimension of Γ is ≤ 1 we have

$$\mathrm{gr}_\Gamma \mathrm{Ext}_{A\#\Gamma}(M,M) = \mathrm{Ext}_\Gamma(\mathbf{k}, \mathrm{Ext}_A(M,M))$$

as an algebra.

Proof. The two spectral sequences arise from considering the row and column filtrations on the double complex $\mathrm{Hom}_\Gamma(L, \mathrm{Hom}_A(K \otimes_A M, M))$. The details are the same as those given in the proofs of Corollaries 6.8 and 6.9. □

This corollary can be seen as a generalization of the Lyndon-Hochschild-Serre spectral sequence in the following sense: if N and G are groups, and G acts on N by automorphisms, we get an action of $\Gamma = \mathbf{k}G$ on $A = \mathbf{k}N$. We then have $A\#\Gamma = \mathbf{k}(N \rtimes G)$, where $N \rtimes G$ denotes the semi-direct product. When $M = \mathbf{k}$ is the trivial $N \rtimes G$ module, the multiplicative spectral sequence

$$\mathrm{Ext}_\Gamma(\mathbf{k}, \mathrm{Ext}_A(M,M)) = \mathrm{H}(G, \mathrm{H}(N, \mathbf{k})) \Rightarrow \mathrm{Ext}_{A\#\Gamma}(\mathbf{k},\mathbf{k}) = \mathrm{H}(N \rtimes G, \mathbf{k})$$

of Corollary 7.7 is simply the Lyndon-Hochschild-Serre spectral sequence.

Appendix

Proof of Corollary 6.7. Let $\mathscr{P} \to P$ be a projective resolution of P as a complex in $\mathrm{EQ}_\Gamma A^e$-mod. Since the restriction functor $\mathrm{EQ}_\Gamma A^e$-mod$\to A^e$-mod preserves projectives, $\mathscr{P}$ will be a projective bimodule resolution of A as well. The tensor product of the quasi-isomorphism $\mathscr{P} \to P$ then produces a quasi-isomorphism $\mathscr{P} \otimes_A \mathscr{P} \to P \otimes_A P$ and we get a diagram

$$\begin{tikzcd} \mathscr{P} \otimes_A \mathscr{P} \arrow[r] & P \otimes_A P \\ \mathscr{P} \arrow[u] \arrow[r] \arrow[ur, phantom, "h \circlearrowleft"] & P, \arrow[u] \end{tikzcd}$$

which commutes up to a Γ-equivariant homotopy $h : \mathscr{P} \to P \otimes_A P[1]$. The existence of this homotopy follow by projectivity of $\mathscr{P}$ and the fact that the diagram commutes on homology. We will then have a diagram

$$\begin{tikzcd}
\mathrm{Hom}_{A^e}(P,B) \otimes \mathrm{Hom}_{A^e}(P,B) \arrow[r] \arrow[d] \arrow[dd, dotted, bend right=60, "mult"'] & \mathrm{Hom}_{A^e}(\mathscr{P},B) \otimes \mathrm{Hom}_{A^e}(\mathscr{P},B) \arrow[d] \arrow[dd, dotted, bend left=60, "mult"] \\
\mathrm{Hom}_{A^e}(P \otimes_A P, B) \arrow[r] \arrow[d] \arrow[dr, phantom, "h^* \circlearrowleft"] & \mathrm{Hom}_{A^e}(\mathscr{P} \otimes_A \mathscr{P}, B) \arrow[d] \\
\mathrm{Hom}_{A^e}(P,B) \arrow[r] & \mathrm{Hom}_{A^e}(\mathscr{P},B),
\end{tikzcd}$$

where the top square commutes and the bottom square commutes up to the Γ-linear homotopy

$$h^* : \mathrm{Hom}_{A^e}(P \otimes_A P, B) \to \mathrm{Hom}_{A^e}(\mathscr{P}, B)[1].$$

To be clear, the top most vertical maps take a product of functions $f \otimes g$ to the function sending a monomial $x \otimes_A y$ in $P \otimes_A P$, or $\mathscr{P} \otimes_A \mathscr{P}$, to $(-1)^{|x||g|} f(x)g(y)$. It follows that the map $\mathrm{Hom}_{A^e}(P,B) \to \mathrm{Hom}_{A^e}(\mathscr{P},B)$ is an algebra map, up to a homotopy, as is the induced map

$$\mathrm{Hom}_\Gamma(L, \mathrm{Hom}_{A^e}(P,B)) \to \mathrm{Hom}_\Gamma(L, \mathrm{Hom}_{A^e}(\mathscr{P},B)).$$

Since P and $\mathscr{P}$ are projective A-bimodule resolutions of A, both of these maps are also seen to be quasi-isomorphisms.

Repeat the process with K to get a resolution $\mathscr{K} \to K$ and quasi-isomorphism

$$\mathrm{Hom}_\Gamma(L, \mathrm{Hom}_{A^e}(K,B)) \to \mathrm{Hom}_\Gamma(L, \mathrm{Hom}_{A^e}(\mathscr{K},B))$$

which is an algebra map up to a homotopy. Finally, since both $\mathscr{K}$ and $\mathscr{P}$ are projective resolutions of A in $\mathrm{EQ}_\Gamma A^e$, there is a Γ-equivariant quasi-isomorphism $\mathscr{K} \to \mathscr{P}$. We repeat the above argument a third time to deduce a quasi-isomorphism

$$\mathrm{Hom}_\Gamma(L, \mathrm{Hom}_{A^e}(\mathscr{P},B)) \to \mathrm{Hom}_\Gamma(L, \mathrm{Hom}_{A^e}(\mathscr{K},B))$$

which is an algebra map up to a homotopy. Taking $\mathscr{A} = \mathrm{Hom}_\Gamma(L, \mathrm{Hom}_{A^e}(\mathscr{K},B))$ then provides the desired result. □

Proof of Lemma 7.4. The first claim follows from the fact that K is a left $A\#\Gamma$-complex itself. Now, for any f in $\mathrm{Hom}_A(K \otimes_A M, N)$ let f_{Hom} denote its image in $\mathrm{Hom}_{A^e}(K, \mathrm{Hom}_{\mathbf{k}}(M,N))$ under the adjunction isomorphism of Proposition 7.1. Then we have, for any $\gamma \in \Gamma$, $x \in K$, and $m \in M$,

$$\begin{aligned}
\big((f\gamma)_{\mathrm{Hom}}(x)\big)(m) \quad &= f\gamma(x \otimes_A m) \\
&= S(\gamma_1) f(\gamma_2 x \otimes_A \gamma_3 m) \\
&= S(\gamma_1) f_{\mathrm{Hom}}(\gamma_2 x)(\gamma_3 m) \\
&= \big(S(\gamma_1) f_{\mathrm{Hom}}(\gamma_2 x)\gamma_3\big)(m) \\
&= (f_{\mathrm{Hom}}\gamma(x))(m).
\end{aligned}$$

So $f\gamma$ maps to $f_{\mathrm{Hom}}\gamma$ under the adjunction isomorphism of the proof of Theorem 7.1 and, consequently, the isomorphism

$$\mathrm{RHom}_{A^e}(A, \mathrm{Hom}_{\mathbf{k}}(M,N)) \overset{\cong}{\to} \mathrm{RHom}_A(M,N)$$

is Γ-linear. □

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