THE GERSTENHABER BRACKET AS A SCHOUTEN BRACKET FOR POLYNOMIAL RINGS EXTENDED BY FINITE GROUPS

CRIS NEGRON AND SARAH WITHERSPOON

ABSTRACT. We apply new techniques to compute Gerstenhaber brackets on the Hochschild cohomology of a skew group algebra formed from a polynomial ring and a finite group (in characteristic 0). We show that the Gerstenhaber brackets can always be expressed in terms of Schouten brackets on polyvector fields. We obtain as consequences some conditions under which brackets are always 0, and show that the Hochschild cohomology is a graded Gerstenhaber algebra under the codimension grading, strengthening known results.

1. INTRODUCTION

We compute brackets on the Hochschild cohomology of a skew group algebra formed from a symmetric algebra (i.e. polynomial ring) and a finite group in characteristic 0. Our results strengthen those given by Halbout and Tang [8] and by Shepler and the second author [17]: The Hochschild cohomology decomposes as a direct sum indexed by conjugacy classes of the group. In [8, 17] the authors give formulas for Gerstenhaber brackets in terms of this decomposition, compute examples, and present vanishing results. Here we go further and show that brackets are always sums of projections of Schouten brackets onto these group components. As just one consequence, a bracket of two nonzero cohomology classes supported *off* the kernel of the group action is always 0 when their homological degrees are smallest possible. Our results complete the picture begun in [8, 17], facilitated here by new techniques from [13].

From a theoretical perspective, the category of modules over the skew group algebra under consideration here is equivalent to the category of equivariant quasi-coherent sheaves on the corresponding affine space. So the Hochschild cohomology of the skew group algebra, along with the Gerstenhaber bracket, is reflective of the deformation theory of this category. This relationship can be realized explicitly through the work of Lowen and Van den Bergh [11]. The Hochschild cohomology of the skew group algebra is also strongly related to Chen and Ruan's orbifold cohomology (see e.g. [2, 15]).

We briefly summarize our main results. If V is a finite dimensional vector space with an action of a finite group G, there is an induced action of G by automorphisms on the symmetric algebra $S(V)$, and one may form the skew group algebra (also known as a smash product or semidirect product) $S(V)\#G$. Its Hochschild cohomology $H :=$

$\mathrm{HH}^\bullet(S(V)\#G)$ is isomorphic to the G-invariant subspace of a direct sum $\oplus_{g\in G}H_g$, and the G-action permutes the components via the conjugation action of G on itself. See for example [4, 7, 14]; we give some details as needed in Section 4.1.

Each space H_g may be viewed in a canonical way as a subspace of $S(V)\otimes\bigwedge^\bullet V^*$, and we construct canonical projections $p_g : S(V)\otimes\bigwedge^\bullet V^* \to H_g$. Since the space $S(V)\otimes\bigwedge^\bullet V^*$ can be identified with the algebra of polyvector fields on affine space, it admits a canonical graded Lie bracket, namely the Schouten bracket (also known as the Schouten-Nijenhuis bracket), which we denote here by $\{\ ,\ \}$. By way of the inclusions $H_g \subset S(V)\otimes\bigwedge^\bullet V^*$ we may apply the Schouten bracket to elements of H_g, and hence to elements in the Hochschild cohomology $\mathrm{HH}^\bullet(S(V)\#G) = (\oplus_{g\in G}H_g)^G$. Our main theorem is:

Theorem 5.2.3. *Let $X = \sum_{g\in G}X_g$ and $Y = \sum_{h\in G}Y_h$ be classes in $\mathrm{HH}^\bullet(S(V)\#G)$ where $X_g \in H_g$, $Y_h \in H_h$. Their Gerstenhaber bracket is*

$$[X,Y] = \sum_{g,h\in G} p_{gh}\{X_g,Y_h\}.$$

This result was obtained by Halbout and Tang [8, Theorem 4.4, Corollary 4.11] in some special cases.

In the body of the text, we assign to each summand H_g its own copy of $S(V)\otimes\bigwedge^\bullet V^*$ and label this copy with a g. So we will write instead $S(V)\otimes\bigwedge^\bullet V^*g$, and write elements in $S(V)\otimes\bigwedge^\bullet V^*g$ as $X_g g$ instead of just X_g.

We note that all of the ingredients in the expressions $\sum p_{gh}\{X_g,Y_h\}$ are canonically defined. Thus we have closed-form expressions for Gerstenhaber brackets on classes in arbitrary degree. There are very few algebras for which we have such an understanding of the graded Lie structure on Hochschild cohomology, aside from smooth commutative algebras (over a given base field). For a smooth commutative algebra R there is the well known HKR isomorphism [9] between the Hochschild cohomology of R and polyvector fields on $\mathrm{Spec}(R)$, along with the Schouten bracket. The particular form of Theorem 5.2.3 is referential to this classic result. Having such a complete understanding of the Lie structure is useful, for example, in the production of L_∞-morphisms and formality results (see [6, 10]).

The proof of Theorem 5.2.3 uses the approach to Gerstenhaber brackets given in [13], in which we introduced new techniques that are particularly well-suited to computations. We summarize the necessary material from [13] in Section 2, explaining how to apply it to skew group algebras. In Sections 3 and 4, we develop further the theory needed to apply the techniques to the skew group algebra $S(V)\#G$ in particular.

We obtain a number of consequences of Theorem 5.2.3 in Sections 5 and 6. In Corollary 5.3.4 we recover [17, Corollary 7.4], stating that in case X,Y are supported entirely on group elements acting trivially on V, the Gerstenhaber bracket is simply the sum of the componentwise Schouten brackets. In Corollary 5.3.5 we recover [17, Proposition 8.4], giving some conditions on invariant subspaces under which the bracket $[X,Y]$ is known to vanish. Corollary 5.3.8 is another vanishing result that generalizes [17, Theorem 9.2] from degree 2 to arbitrary degree, stating that in case

X, Y are supported entirely off the kernel of the group action and their homological degrees are smallest possible, their Gerstenhaber bracket is 0. In Corollary 5.3.7 we also show that the Hochschild cohomology $\mathrm{HH}^\bullet(S(V)\#G)$ is a graded Gerstenhaber algebra with respect to a certain natural grading coming from the geometry of the fixed spaces V^g, which we refer to as the codimension grading. Section 6 consists of examples and a general explanation of (non)vanishing of the Gerstenhaber bracket for $S(V)\#G$, rephrasing some of the results of [17].

Let k be a field and $\otimes = \otimes_k$. For our main results, we assume the characteristic of k is 0, but this is not needed for the general techniques presented in Section 2. We adopt the convention that a group G will act on the left of an algebra, and on the right of functions from that algebra. Similarly, we will let G act on the left of the finite dimensional vector space $V \subset S(V)$ and on the right of its dual space V^*.

2. An alternate approach to the Lie bracket

Let A be an algebra over the field k. Let $B \to A$ denote the *bar resolution* of A as an A-bimodule:

$$\cdots \xrightarrow{\delta_3} A^{\otimes 4} \xrightarrow{\delta_2} A^{\otimes 3} \xrightarrow{\delta_1} A \otimes A \xrightarrow{\mu} A \to 0,$$

where μ denotes multiplication and

$$\delta_n(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n}(-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1} \tag{2.0.1}$$

for all $a_0, \ldots, a_{n+1} \in A$. The *Gerstenhaber bracket* of homogeneous functions $f, g \in \mathrm{Hom}_{A^e}(B, A)$ is defined to be

$$[f, g] = f \circ g - (-1)^{(|f|-1)(|g|-1)} g \circ f,$$

where the circle product $f \circ g$ is given by

$$(f \circ g)(a_1 \otimes \cdots \otimes a_{|f|+|g|-1})$$

$$= \sum_{j=1}^{|f|}(-1)^{(|g|-1)(j-1)} f(a_1 \otimes \cdots \otimes a_{j-1} \otimes g(a_j \otimes \cdots \otimes a_{j+|g|-1}) \otimes \cdots \otimes a_{|f|+|g|-1}),$$

for all $a_1, \ldots, a_{|f|+|g|-1} \in A$, and similarly $g \circ f$. This induces the bracket on Hochschild cohomology. (Here we have identified $\mathrm{Hom}_{A^e}(B, A)$ with $\mathrm{Hom}_k(A^{\otimes \bullet}, A)$.)

Typically when one computes brackets, one uses explicit chain maps between this bar resolution B and a more convenient one for computational purposes, navigating back and forth. Such chain maps are usually awkward, and this way can be inefficient and technically difficult. In this section, we first recall from [13] an alternate approach, for some types of algebras, introduced to avoid this trouble. Then we explain how to apply it to skew group algebras in particular.

2.1. **A collection of brackets.** Given a bimodule resolution $K \to A$ satisfying some conditions as detailed below, one can produce a number of coarse brackets $[\ ,\ ]_\phi$ on the complex $\mathrm{Hom}_{A^e}(K, A)$, each depending on a map ϕ. These brackets are coarse in the sense that they will not, in general, produce dg Lie algebra structures on the complex

$\mathrm{Hom}_{A^e}(K, A)$. They will, however, be good enough to compute the Gerstenhaber bracket on the cohomology $\mathrm{H}^\bullet(\mathrm{Hom}_{A^e}(K, A)) = \mathrm{HH}^\bullet(A)$. We have precisely:

Theorem 2.1.1 ([13, Theorem 3.2.5]). *Suppose $\mu : K \to A$ is a projective A-bimodule resolution of A satisfying the Hypotheses 2.1.2(a)–(c) below. Let $F_K : K \otimes_A K \to K$ be the chain map $F_K = \mu \otimes \mathrm{id}_K - \mathrm{id}_K \otimes \mu$. Then for any degree -1 bimodule map $\phi : K \otimes_A K \to K$ satisfying $d_K \phi + \phi d_{K \otimes_A K} = F_K$, there is a bilinear operation $[\;,\;]_\phi$ on $\mathrm{Hom}_{A^e}(K, A)$. Each operation $[\;,\;]_\phi$ satisfies the following properties.*

(1) *$[f, g]_\phi$ is a cocycle whenever f and g are cocycles in $\mathrm{Hom}_{A^e}(K, A)$.*
(2) *$[f, g]_\phi$ is a coboundary whenever, additionally, f or g is a coboundary.*
(3) *The induced operation on cohomology $\mathrm{H}^\bullet(\mathrm{Hom}_{A^e}(K, A)) = \mathrm{HH}^\bullet(A)$ is precisely the Gerstenhaber bracket: $[f, g]_\phi = [f, g]$ on cohomology.*

The bilinear map $[\;,\;]_\phi$ is defined by equations (2.1.3) and (2.1.4) below. By [13, Lemma 3.2.1], such maps ϕ satisfying the conditions in the theorem always exist. We call such a map ϕ a *contracting homotopy* for F_K.

There is a *diagonal map* $\Delta_B : B \to B \otimes_A B$ given by

$$\Delta_B(a_0 \otimes \ldots \otimes a_{n+1}) = \sum_{i=0}^{n} (a_0 \otimes \ldots \otimes a_i \otimes 1) \otimes_A (1 \otimes a_{i+1} \otimes \ldots \otimes a_{n+1})$$

for all $a_0, \ldots, a_{n+1} \in A$. The hypotheses on the projective A-bimodule resolution $K \to A$, to which the theorem above refers, are as follows.

Hypotheses 2.1.2. (a) K admits a chain embedding $\iota : K \to B$ that fits into a commuting diagram

$$K \xrightarrow{\;\iota\;} B$$
$$\searrow \quad \swarrow$$
$$A$$

(b) The embedding ι admits a retract π. That is, there is an chain map $\pi : B \to K$ such that $\pi\iota = \mathrm{id}_K$.
(c) The diagonal map $\Delta_B : B \to B \otimes_A B$ preserves K, and hence restricts to a diagonal map $\Delta_K : K \to K \otimes_A K$. Equivalently, $\Delta_B\iota = (\iota \otimes_A \iota)\Delta_K$.

Conditions (a) and (c) together can alternatively be stated as the condition that K is a dg coalgebra in the monoidal category A-bimod that admits an embedding into the bar resolution. We denote the coproducts of elements in K using Sweedler's notation, e.g.

$$\Delta(w) = w_1 \otimes w_2, \quad (\Delta \otimes_A \mathrm{id}_K)\Delta(w) = w_1 \otimes w_2 \otimes w_3,$$

for $w \in K$, with the implicit sum $\Delta(w) = \sum_{i_1, i_2} w_{i_1} \otimes w_{i_2}$ suppressed. Koszul resolutions of Koszul algebras, as well as related algebras such as universal enveloping algebras, Weyl algebras, and Clifford algebras, will fit into this framework [1, 12].

Given a resolution $K \to A$ satisfying Hypotheses 2.1.2(a)–(c) and a contracting homotopy ϕ for F_K, we construct the ϕ-bracket as follows: We first define the *ϕ-circle product* for functions f, g in $\mathrm{Hom}_{A^e}(K, A)$ by

$$(f \circ_\phi g)(w) = (-1)^{|w_1||g|} f\big(\phi(w_1 \otimes g(w_2) \otimes w_3)\big) \qquad (2.1.3)$$

for homogeneous w in K, and define the ϕ-*bracket* as the graded commutator

$$[f, g]_\phi = f \circ_\phi g - (-1)^{(|f|-1)(|g|-1)} g \circ_\phi f. \tag{2.1.4}$$

We will be interested in producing such brackets particularly for a skew group algebra formed from a symmetric algebra (i.e. polynomial ring) under a finite group action. We will start with the Koszul resolution K of the symmetric algebra itself and then construct a natural extension $\widetilde{K}$ to resolve the skew group algebra. We will define a contracting homotopy ϕ for F_K that extends to $\widetilde{K}$, and use it to compute Gerstenhaber brackets via Theorem 2.1.1. We describe these constructions next in the context of more general skew group algebras.

2.2. **Skew group algebras.** Let G be a finite group whose order is not divisible by the characteristic of the field k. Assume that G acts by automorphisms on the algebra A. Let $B = B(A)$ be the bar resolution of A, and let $K = K(A)$ be a projective A-bimodule resolution of A satisfying Hypotheses 2.1.2(a)–(c). Assume that G acts on K and on B, and this action commutes with the differentials and with the maps $\iota, \pi, \Delta_K, \Delta_B$. These assumptions all hold in the case that A is a Koszul algebra on which G acts by graded automorphisms and K is a Koszul resolution; in particular, π may be replaced by $\frac{1}{|G|} \sum_{g \in G} g\pi g^{-1}$ if it is not a priori G-linear.

Let $A\#G$ denote the *skew group algebra*, that is $A \otimes kG$ as a vector space, with multiplication defined by $(a \otimes g)(b \otimes h) = a(^gb) \otimes gh$ for all $a, b \in A$ and $g, h \in G$, where left superscript denotes the G-action. We will sometimes write $a\#g$ or simply ag in place of the element $a \otimes g$ of $A\#G$ when there can be no confusion.

Let $B(A\#G)$ denote the bar resolution of $A\#G$ as a k-algebra, and let $\widetilde{B}(A\#G)$ denote its bar resolution *over* kG:

$$\cdots \xrightarrow{\delta_3} (A\#G)^{\otimes_{kG}4} \xrightarrow{\delta_2} (A\#G)^{\otimes_{kG}3} \xrightarrow{\delta_1} (A\#G) \otimes_{kG} (A\#G) \xrightarrow{\mu} A\#G \to 0,$$

with differentials defined as in (2.0.1). Since kG is semisimple, this is a projective resolution of $A\#G$ as an $(A\#G)^e$-module. There is a vector space isomorphism

$$(A\#G)^{\otimes_{kG}i} \cong A^{\otimes i} \otimes kG$$

for each i, and from now on we will identify $\widetilde{B}_j(A\#G)$ with $A^{\otimes(j+2)} \otimes kG$ for each j, and the differentials of $\widetilde{B}$ with those of B tensored with the identity map on kG.

Similarly we wish to extend K to a projective resolution of $A\#G$ as an $(A\#G)^e$-module. Let $\widetilde{K} = \widetilde{K}(A\#G)$ denote the following complex:

$$\cdots \xrightarrow{d_3 \otimes \mathrm{id}_{kG}} K_2 \otimes kG \xrightarrow{d_2 \otimes \mathrm{id}_{kG}} K_1 \otimes kG \xrightarrow{d_1 \otimes \mathrm{id}_{kG}} K_0 \otimes kG \xrightarrow{\mu \otimes \mathrm{id}_{kG}} A\#G \to 0.$$

We give the terms of this complex the structure of $A\#G$-bimodules as follows:

$$\begin{aligned}
(a\#g)(x \otimes h) &= a(^gx) \otimes gh, \\
(x \otimes h)(a\#g) &= x(^ha) \otimes hg,
\end{aligned}$$

for all $a \in A$, $g, h \in G$, and $x \in K_i$. Then $\widetilde{K}$ is a projective resolution of $A\#G$ by $(A\#G)^e$-modules.

Next we will show that $\widetilde{K} \to A\#G$ satisfies Hypotheses 2.1.2(a)–(c) for the algebra $A\#G$. Let $\tilde{\iota} : \widetilde{K} \to B(A\#G)$ be the composition

$$\widetilde{K} \xrightarrow{\iota \otimes \mathrm{id}_{kG}} \widetilde{B}(A\#G) \xrightarrow{i} B(A\#G)$$

where $i(a_0 \otimes \cdots \otimes a_{j+1} \otimes g) = (a_0 \# 1) \otimes \cdots \otimes (a_j \# 1) \otimes (a_{j+1} \# g)$ for all $a_0, \ldots, a_{j+1} \in A$ and $g \in G$. Let $\tilde{\pi} : B(A\#G) \to \widetilde{K}$ be the composition

$$B(A\#G) \xrightarrow{p} \widetilde{B}(A\#G) \xrightarrow{\pi \otimes \mathrm{id}_{kG}} \widetilde{K}$$

where

$$p((a_0 \# g_0) \otimes (a_1 \# g_1) \otimes (a_2 \# g_2) \otimes \cdots \otimes (a_j \# g_j)) \tag{2.2.1}$$

$$= a_0 \otimes (^{g_0} a_1) \otimes (^{g_0 g_1} a_2) \otimes \cdots \otimes (^{g_0 g_1 \cdots g_{j-1}} a_j) \otimes g_0 g_1 \cdots g_j$$

for all $a_0, \ldots, a_j \in A$ and $g_0, \ldots, g_j \in G$. One can check that i and p are indeed chain maps. Then

$$\tilde{\pi}\tilde{\iota} = (\pi \otimes \mathrm{id}_{kG}) pi (\iota \otimes \mathrm{id}_{kG}) = \mathrm{id}_{\widetilde{K}}$$

since $pi = \mathrm{id}_{\widetilde{B}(A\#G)}$ by the definitions of these maps, and $\pi\iota = \mathrm{id}_K$ by Hypothesis 2.1.2(b) applied to K. Therefore Hypotheses 2.1.2(a) and (b) hold for $\widetilde{K}$.

Now let $\Delta_{\widetilde{K}} : \widetilde{K} \to \widetilde{K} \otimes_{A\#G} \widetilde{K}$ be defined by $\Delta_{\widetilde{K}} = \Delta_K \otimes \mathrm{id}_{kG}$, after identifying $\widetilde{K}_i \otimes_{A\#G} \widetilde{K}_j = (K_i \otimes kG) \otimes_{A\#G} (K_j \otimes kG)$ with $(K_i \otimes_A K_j) \otimes kG$. One may check that $\Delta_{\widetilde{K}}$ satisfies Hypothesis 2.1.2(c).

Let $\phi_K : K \otimes_A K \to K$ be a map satisfying $d_K \phi_K + \phi_K d_{K \otimes_A K} = F_K$ as in Theorem 2.1.1. Assume that ϕ_K is G-linear. Let

$$\phi_{\widetilde{K}} = \phi_K \otimes \mathrm{id}_{kG}, \tag{2.2.2}$$

a map from $\widetilde{K} \otimes_{A\#G} \widetilde{K}$ to $\widetilde{K}$, under the identification $\widetilde{K} \otimes_{A\#G} \widetilde{K} \cong (K \otimes_A K) \otimes kG$. Since ϕ_K is G-linear, this map $\phi_{\widetilde{K}}$ is an $A\#G$-bimodule map. Further,

$$d_{\widetilde{K}} \phi_{\widetilde{K}} + \phi_{\widetilde{K}} d_{\widetilde{K} \otimes_A \widetilde{K}} = (d_K \phi_K + \phi_K d_{K \otimes_A K}) \otimes \mathrm{id}_{kG} = F_K \otimes \mathrm{id}_{kG} = F_{\widetilde{K}}.$$

As a consequence, by Theorem 2.1.1, $\phi_{\widetilde{K}}$ may be used to define the Gerstenhaber bracket on the Hochschild cohomology of $A\#G$ via (2.1.3) and (2.1.4).

3. Symmetric algebras

Let V be a finite dimensional vector space over the field k of characteristic 0, and let $A = S(V)$, the symmetric algebra on V. In this section, we construct a map ϕ that will allow us to compute Gerstenhaber brackets on the Hochschild cohomology of the skew group algebra $A\#G$ arising from a representation of the finite group G on V, via Theorem 2.1.1 and equation (2.2.2).

3.1. **The Koszul resolution.** We will use a standard description of the Koszul resolution K of $A = S(V)$ as an A^e-bimodule, given as a subcomplex of the bar resolution $B = B(A)$: For all $v_1, \ldots, v_i \in V$, let

$$o(v_1, \ldots, v_i) = \sum_{\sigma \in S_i} \mathrm{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(i)}$$

in $V^{\otimes i} \subset A^{\otimes i}$. Sometimes we write instead

$$o(v_I) = o(v_1, \ldots, v_i),$$

where $I = \{1, \ldots, i\}$. Take $o(\varnothing) = 1$. Let $K_i = K_i(A)$ be the free A^e-submodule of $A^{\otimes(i+2)}$ with free basis all $1 \otimes o(v_1, \ldots, v_i) \otimes 1$ in $A^{\otimes(i+2)}$. We may in this way identify K_i with $A \otimes \bigwedge^i V \otimes A$ and K with $A \otimes \bigwedge^\bullet V \otimes A$. The differentials on the bar resolution, restricted to K, may be rewritten in terms of the chosen free basis of K as

$$d(1 \otimes o(v_1, \ldots, v_i) \otimes 1)$$

$$= \sum_{j=1}^{i} (-1)^{j-1} (v_j \otimes o(v_1, \ldots, \hat{v}_j, \ldots, v_i) \otimes 1 - 1 \otimes o(v_1, \ldots, \hat{v}_j, \ldots, v_i) \otimes v_j),$$

where $\hat{v}_j$ indicates that v_j has been deleted from the list of vectors. Note that the action of G preserves the vector subspace of K_i spanned by all $1 \otimes o(v_1, \ldots, v_i) \otimes 1$. There is a retract $\pi : B \to K$ that can be chosen to be G-linear. We will not need an explicit formula for π here.

The diagonal map $\Delta_K : K \to K \otimes_A K$ is given by

$$\Delta_K(1 \otimes o(v_I) \otimes 1) = \sum_{I_1, I_2} \pm (1 \otimes o(v_{I_1}) \otimes 1) \otimes (1 \otimes o(v_{I_2}) \otimes 1) \tag{3.1.1}$$

where the sum is indexed by all ordered disjoint subsets $I_1, I_2 \subset I$ with $I_1 \cup I_2 = I$, and $\pm$ is the sign of σ, the unique permutation for which $\{i_{\sigma(1)}, \ldots i_{\sigma(|I|)}\} = I_1 \cup I_2$ as an ordered set. Note that Δ_K is G-linear. Letting $\iota : K \to B$ be the embedding of K as a subcomplex of B, Hypotheses 2.1.2(a)–(c) now hold. We may thus use Theorem 2.1.1 and (2.2.2) to compute brackets on $\mathrm{HH}^\bullet(S(V)\#G)$ once we have a suitable map ϕ.

3.2. **An invariant map ϕ.** We will now define an A-bilinear map $\phi : K \otimes_A K \to K$ that will be G-linear and independent of choice of ordered basis of V. Compare with [13, Definition 4.1.3], where a simpler map ϕ was defined which however depends on such a choice. At first glance, the map ϕ below looks rather complicated, but in practice we find it easier to use when extended to a skew group algebra than explicit chain maps between bar and Koszul resolutions.

Definition 3.2.1. *Assume the characteristic of k is 0. Let $\phi : K \otimes_A K \to K$ be the A-bilinear map given on ordered monomials as follows:*

$$\phi_0(1 \otimes v_1 \cdots v_t \otimes 1) = \frac{1}{t!} \sum_{\substack{1 \le r \le t \\ \sigma \in \bar{S}_t}} v_{\sigma(1)} \cdots v_{\sigma(r-1)} \otimes o(v_{\sigma(r)}) \otimes v_{\sigma(r+1)} \cdots v_{\sigma(t)}$$

for all $v_1, \ldots, v_t \in V$. On $K_0 \otimes_A K_z$ and on $K_s \otimes_A K_0$ $(s, z > 0)$, let

$$\phi_z(1 \otimes v_1 \cdots v_t \otimes o(w_1, \ldots, w_z) \otimes 1)$$

$$= \frac{(-1)^z}{(t+z)!} \sum_{\substack{1 \le r \le t \\ \sigma \in \bar{S}_t}} \Big(\prod_{i=0}^{z-1} (r+i) \Big) v_{\sigma(1)} \cdots v_{\sigma(r-1)} \otimes o(w_1, \ldots, w_z, v_{\sigma(r)}) \otimes v_{\sigma(r+1)} \cdots v_{\sigma(t)},$$

$$\phi_s(1 \otimes o(u_1, \ldots, u_s) \otimes v_1 \cdots v_t \otimes 1)$$

$$= \frac{1}{(t+s)!} \sum_{\substack{1 \le r \le t \\ \sigma \in \bar{S}_t}} \Big(\prod_{i=1}^{s} (t-r+i) \Big) v_{\sigma(1)} \cdots v_{\sigma(r-2)} \otimes o(v_{\sigma(r)}, u_1, \ldots, u_s) \otimes v_{\sigma(r+1)} \cdots v_{\sigma(t)},$$

for all $u_1, \ldots, u_s, v_1, \ldots, v_t, w_1, \ldots, w_z \in V$. When $s > 0$ and $z > 0$, let

$$\phi_{s+z}(1 \otimes o(u_1, \ldots, u_s) \otimes v_1 \cdots v_t \otimes o(w_1, \ldots, w_z) \otimes 1)$$

$$= \sum_{\substack{1 \le r \le t \\ \sigma \in \bar{S}_t}} c_r^{s,t,z} v_{\sigma(1)} \cdots v_{\sigma(r-1)} \otimes o(w_1, \ldots, w_z, v_{\sigma(r)}, u_1, \ldots, u_s) \otimes v_{\sigma(r+1)} \cdots v_{\sigma(t)},$$

where $\quad c_r^{s,t,z} = \dfrac{(-1)^{sz+z}}{(s+t+z)!} \Big(\prod_{i=0}^{z-1} (r+i) \Big) \Big(\prod_{j=1}^{s} (t-r+j) \Big).$

To see that ϕ is well-defined, one can first construct the corresponding map from the tensor powers $(V^{\otimes s}) \otimes (V^{\otimes t}) \otimes (V^{\otimes z})$ (using the universal property of the tensor product, for example) then note that the given map is $S_s \times S_t \times S_z$-invariant and hence induces a well-defined map ϕ on the coinvariants $(\bigwedge^s V) \otimes S^t(V) \otimes (\bigwedge^z V)$. One can also see directly that ϕ is G-invariant. In fact, it is invariant under the action of the entire group $\mathrm{GL}(V)$. We next state that ϕ is a contracting homotopy for the map F_K defined in the statement of Theorem 2.1.1.

Lemma 3.2.2. *Let* $\phi : K \otimes_A K \to K$ *be the A-bilinear map of Definition 3.2.1. Then* $d_K \phi + \phi d_{K \otimes_A K} = F_K$.

Proof. In degree 0, we check:

$$(d\phi + \phi d)(1 \otimes v_1 \cdots v_t \otimes 1)$$

$$= d \left(\frac{1}{t!} \sum_{\substack{1 \le r \le t \\ \sigma \in \bar{S}_t}} v_{\sigma(1)} \cdots v_{\sigma(r-1)} \otimes o(v_{\sigma(r)}) \otimes v_{\sigma(r+1)} \cdots v_{\sigma(t)} \right)$$

$$= \frac{1}{t!} \sum_{\substack{1 \le r \le t \\ \sigma \in \bar{S}_t}} \big(v_{\sigma(1)} \cdots v_{\sigma(r)} \otimes v_{\sigma(r+1)} \cdots v_{\sigma(t)} - v_{\sigma(1)} \cdots v_{\sigma(r-1)} \otimes v_{\sigma(r)} \cdots v_{\sigma(t)} \big)$$

$$= \frac{1}{t!} \sum_{\sigma \in S_t} \big(v_{\sigma(1)} \cdots v_{\sigma(t)} \otimes 1 - 1 \otimes v_{\sigma(1)} \cdots v_{\sigma(t)} \big)$$

$$= v_1 \cdots v_t \otimes 1 - 1 \otimes v_1 \cdots v_t \quad = \quad F_K(1 \otimes v_1 \cdots v_t \otimes 1).$$

Other verifications are tedious, but similar. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4. ϕ-CIRCLE PRODUCT FORMULA AND PROJECTIONS ONTO GROUP COMPONENTS

We first recall some basic facts about the Hochschild cohomology of the skew group algebra $S(V)\#G$. The graded vector space structure of the cohomology is well-known, see for example [4, 7, 14]. We give some details here as will be needed for our bracket computations. We then derive a formula for ϕ-circle products (defined in (2.1.3)) on this Hochschild cohomology, and define projection operators needed for our main results. We assume from now on that the characteristic of k is 0. Then

$$\mathrm{HH}^\bullet(S(V)\#G) \cong \mathrm{HH}^\bullet(S(V), S(V)\#G)^G, \qquad (4.0.1)$$

where the superscript G denotes invariants of the action of G on Hochschild cohomology induced by its action on complexes (via the standard group action on tensor products and functions). This follows for example from [5]. We will focus our discussions and computations on $\mathrm{HH}^\bullet(S(V), S(V)\#G)$, and results for Hochschild cohomology $\mathrm{HH}^\bullet(S(V)\#G)$ will follow by restricting to its G-invariant subalgebra.

4.1. **Structure of the cohomology** $\mathrm{HH}^\bullet(S(V), S(V)\#G)$**.** Let $\{x_1, \ldots, x_n\}$ be a basis for V and $\{x_1^*, \ldots, x_n^*\}$ the dual basis for its dual space V^*. The Hochschild cohomology $\mathrm{HH}^\bullet(S(V), S(V)\#G)$ is computed as the homology of the complex

$$
\begin{aligned}
\mathrm{Hom}_{S(V)^e}(S(V) \otimes \textstyle\bigwedge^\bullet V \otimes S(V), S(V)\#G) \;&\cong\; \mathrm{Hom}_k(\textstyle\bigwedge^\bullet V, S(V)\#G) \\
&\cong\; \bigoplus_{g\in G} S(V) \otimes \textstyle\bigwedge^\bullet V^* g.
\end{aligned}
$$

The differential on the graded space $\bigoplus_{g\in G} S(V) \otimes \bigwedge^\bullet V^* g$ induced by the above sequence of isomorphisms is left multiplication by the diagonal matrix

$$E = \mathrm{diag}\{E_g\}_{g\in G}, \quad E_g = \sum_i (x_i - {}^g x_i)\partial_i,$$

where $\partial_i = 1 \otimes x_i^*$. So $E \cdot (\sum_g Y_g g) = \sum_g (E_g Y_g)g$. This complex breaks up into a sum of subcomplexes $(S(V) \otimes \bigwedge^\bullet V^* g, E_g)$, and we have

$$\mathrm{HH}^\bullet(S(V), S(V)\#G) = \bigoplus_{g\in G} \mathrm{H}^\bullet(S(V) \otimes \textstyle\bigwedge^\bullet V^* g).$$

We note that each E_g is independent of the choice of basis, since it is simply the image of $1g \in \mathrm{Hom}_k(\bigwedge^0 V, S(V)g)$ under the differential on $\mathrm{Hom}_k(\bigwedge^\bullet V, S(V)g)$.

The right G-action on $\mathrm{Hom}_{S(V)^e}(K, S(V)\#G)$ is given by $(f \cdot g)(x) = g^{-1} f({}^g x)g$, for $f \in \mathrm{Hom}_{S(V)^e}(K, S(V)\#G)$ and $g \in G$, giving it the structure of a G-complex. This translates to the action

$$\big((a \otimes f_1 \ldots f_l)g\big) \cdot h = \big(({}^{h^{-1}}a) \otimes f_1^h \ldots f_l^h\big)h^{-1}gh \qquad (4.1.1)$$

on $\bigoplus_{g\in G} S(V) \otimes \bigwedge^\bullet V^* g$, where $a \in S(V)$ and $f_i \in V^*$. It will be helpful to have the following general lemma.

Lemma 4.1.2. *Given any G-representation M, and element $g \in G$, there is a canonical complement to the g-invariant subspace M^g, given by $(M^g)^\perp = (1 - g) \cdot M$. This*

gives a splitting $M = M^g \oplus (1-g)M$ of M as a $\langle g \rangle$-representation. This decomposition satisfies

$$M^g = M^{g^{-1}} \quad \text{and} \quad (1-g)M = (1-g^{-1})M$$

and is compatible with the G-action in the sense that, for any $h \in G$,

$$h \cdot M^g = M^{hgh^{-1}} \quad \text{and} \quad h \cdot \big((1-g)M\big) = (1 - hgh^{-1})M.$$

Proof. The operation $(1-g) \cdot - : M \to M$ has kernel precisely M^g. Furthermore $M^g \cap (1-g) \cdot M = 0$, since for any invariant element $m - gm$ in $(1-g) \cdot M$ we will have

$$(m - gm) = \int_g (m - gm) = \int_g m - \int_g gm = \int_g m - \int_g m = 0,$$

where $\int_g = \frac{1}{|g|} \sum_{i=0}^{|g|-1} g^i$. We conclude that $M = M^g \oplus (1-g)M$ when M is finite dimensional, by a dimension count. When M is infinite dimensional, the result can be deduced from the fact that M is the union of its finite dimensional submodules.

The equality $M^g = M^{g^{-1}}$ is clear, and the equality $(1-g)M = (1-g^{-1})M$ follows from the fact that $M = -gM$. As for the compatibility claim, the identity $h \cdot M^g = M^{hgh^{-1}}$ is obvious, while the equality $h(1-g)M = (1 - hgh^{-1})M$ follows from the computation

$$h(1-g)M = h(1-g)h^{-1}M = (hh^{-1} - hgh^{-1})M = (1 - hgh^{-1})M.$$

$\square$

Let us take $\det_g^\perp$ to be the one dimensional $\langle g \rangle$-representation:

$$\det\nolimits_g^\perp = \bigwedge\nolimits^{\operatorname{codim}V^g}((1-g)V)^* g.$$

We then have the embedding

$$S(V^g) \otimes \bigwedge\nolimits^{\bullet - \operatorname{codim}V^g}(V^g)^* \det\nolimits_g^\perp \longrightarrow S(V) \otimes \bigwedge\nolimits^\bullet V^* g \qquad (4.1.3)$$

induced by the embedding of V^g as a subspace of V and a corresponding dual subspace embedding.

Lemma 4.1.4. *For each $g \in G$,*

$$E_g \cdot \big(S(V^g) \otimes \bigwedge\nolimits^{\bullet - \operatorname{codim}V^g}(V^g)^* \det\nolimits_g^\perp\big) = 0$$

and

$$(\operatorname{Im}(E_g \cdot -)) \cap \Big(S(V^g) \otimes \bigwedge\nolimits^{\bullet - \operatorname{codim}V^g}(V^g)^* \det\nolimits_g^\perp\Big) = 0.$$

That is to say, the subspaces $S(V^g) \otimes \bigwedge^{\bullet - \operatorname{codim}V^g}(V^g)^* \det_g^\perp$ consist entirely of cocycles and contain no nonzero coboundaries.

Proof. If we choose a basis $\{x_1, \ldots, x_n\}$ for V such that the first l elements are a basis for V^g, and the remaining are a basis for $(1-g)V$, then we have

$$E_g = \sum_{i=1}^n (x_i - {}^g x_i)\partial_i = \sum_{i>l}(x_i - {}^g x_i)\partial_i,$$

since $x_i = {}^g x_i$ for all $i \leq l$. So $E_g \det_g^\perp = 0$ and, if we let $\mathfrak{I}_g$ denote the ideal in $S(V)$ generated by $(1-g)V$, we have

$$E_g\left(S(V) \otimes \textstyle\bigwedge^\bullet V^* g\right) \subset \mathfrak{I}_g \otimes \textstyle\bigwedge^\bullet V^* g.$$

The second statement here implies that the proposed intersection is trivial. $\square$

By the above information, there is an induced map

$$S(V^g) \otimes \textstyle\bigwedge^{\bullet - \mathrm{codim} V^g}(V^g)^* \det_g^\perp \;\longrightarrow\; \mathrm{H}^\bullet\left(S(V) \otimes \textstyle\bigwedge^\bullet V^* g\right)$$

which is *injective*. The following is a rephrasing of Farinati's calculation [4].

Proposition 4.1.5. *The induced maps*

$$S(V^g) \otimes \textstyle\bigwedge^{\bullet - \mathrm{codim} V^g}(V^g)^* \det_g^\perp \;\longrightarrow\; \mathrm{H}^\bullet\left(S(V) \otimes \textstyle\bigwedge^\bullet V^* g\right)$$

are isomorphisms for each $g \in G$, and so there is an isomorphism

$$\bigoplus_{g \in G}\left(S(V^g) \otimes \textstyle\bigwedge^{\bullet - \mathrm{codim} V^g}(V^g)^* \det_g^\perp\right) \;\overset{\cong}{\longrightarrow}\; \mathrm{H}^\bullet\Big(\bigoplus_{g \in G}(S(V) \otimes \textstyle\bigwedge^\bullet V^* g)\Big). \qquad (4.1.6)$$

Recalling that the codomain of (4.1.6) is the cohomology $\mathrm{HH}^\bullet(S(V), S(V) \# G)$, the second portion of this proposition gives an identification

$$\bigoplus_{g \in G}\left(S(V^g) \otimes \textstyle\bigwedge^{\bullet - \mathrm{codim} V^g}(V^g)^* \det_g^\perp\right) = \mathrm{HH}^\bullet(S(V), S(V) \# G). \qquad (4.1.7)$$

In addition to providing this description of the cohomology, the embedding (4.1.3) is compatible with the G-action in the following sense.

Proposition 4.1.8.   (1) *For any $g, h \in G$ there is an equality*

$$\left(S(V^g) \otimes \textstyle\bigwedge^{\bullet - \mathrm{codim} V^g}(V^g)^* \det_g^\perp\right) \cdot h = S(V^{h^{-1}gh}) \otimes \textstyle\bigwedge^{\bullet - \mathrm{codim} V^{h^{-1}gh}}(V^{h^{-1}gh})^* \det_{h^{-1}gh}^\perp$$

$\quad$ *in $\bigoplus_g S(V) \otimes \textstyle\bigwedge^\bullet V^* g$.*

(2) *The sum $\bigoplus_g S(V^g) \otimes \textstyle\bigwedge^{\bullet - \mathrm{codim} V^g}(V^g)^* \det_g^\perp$ is a G-subcomplex of the sum $\bigoplus_g S(V) \otimes \textstyle\bigwedge^\bullet V^* g$.*

(3) *The isomorphism*

$$\bigoplus_{g \in G}\left(S(V^g) \otimes \textstyle\bigwedge^{\bullet - \mathrm{codim} V^g}(V^g)^* \det_g^\perp\right) \overset{\cong}{\longrightarrow} \mathrm{H}^\bullet\Big(\bigoplus_{g \in G}(S(V) \otimes \textstyle\bigwedge^\bullet V^* g)\Big)$$

$\quad$ *is one of graded G-modules.*

Proof. From Lemma 4.1.2, and the descriptions of $(V^g)^*$ and $((1-g)V)^*$ as those functions vanishing on $(1-g)V$ and V^g respectively, we have

$$(V^g)^* \cdot h = \{\text{functions vanishing on } h^{-1}(1-g)V\} = (V^{h^{-1}gh})^*$$

$$((1-g)V)^* \cdot h = \{\text{functions vanishing on } h^{-1}V^g\} = ((1 - h^{-1}gh)V)^*$$

and $h^{-1} \cdot S(V^g) = S(V^{h^{-1}gh})$. This, along with the description (4.1.1) gives the equality in (1). Statement (2) follows from (1), and (3) follows from (2) and Proposition 4.1.5. $\square$

4.2. **The ϕ-circle product formula.** We will compute first with the complex

$$\mathrm{Hom}_{S(V)^e}(K, S(V)\#G) = \bigoplus_{g \in G} S(V) \otimes \bigwedge^\bullet V^* g,$$

and then restrict to G-invariant elements to make conclusions about the cohomology $\mathrm{HH}^\bullet(S(V)\#G)$. Letting ϕ_K be the map of Definition 3.2.1 and $\phi_{\widetilde{K}} = \phi_K \otimes \mathrm{id}_{kG}$ as in equation (2.2.2), $\phi := \phi_{\widetilde{K}}$ gives rise to a perfectly good *bilinear operation*

$$Xg \circ_\phi Yh = \big(w \mapsto (-1)^{|w_1||Y|}Xg(\phi(w_1 \otimes Y(w_2) \otimes {}^h w_3))h\big),$$

which need not be a chain map. Here $X, Y \in S(V) \otimes \bigwedge^\bullet V^*$, and $w \in K$. We can also define the ϕ-bracket in the most naive manner as

$$[Xg, Yh]_\phi = Xg \circ_\phi Yh - (-1)^{(|X|-1)(|Y|-1)}Yh \circ_\phi Xg.$$

This operation, again, need not be well behaved at all on non-invariant functions, but it will be a bilinear map.

In the lemma below, we give a formula for the ϕ-circle product of special types of elements. Our formula may be compared with [8, Lemma 4.1] and [17, Theorem 7.2]. Due to the structure of the Hochschild cohomology of $S(V)\#G$ stated in Proposition 4.1.5, this formula will in fact suffice to compute all brackets. To do this, we will only need to consider G-invariant elements and compute relevant ϕ-circle products on summands representing elements in $\mathrm{HH}^\bullet(S(V), S(V)g) \times \mathrm{HH}^\bullet(S(V), S(V)h)$ for all pairs g, h in G. This we will do in Section 5.

Lemma 4.2.1. *Let $g, h \in G$, $p \geq 0$, and ω_h any generator for $\bigwedge^{\mathrm{codim}V^h}((1-h)V)^*$. Consider any $X_i \in S(V) \otimes V^*$, $\bar{Y} \in \bigwedge^p V^*\omega_h$, and $Y = v_1 \ldots v_t\bar{Y}$ for $v_i \in V$. Then*

$$(X_1 \ldots X_dg \circ_\phi Yh) =$$

$$\sum_{l,r,\sigma} \zeta_{r,|Y|}^{l,d,t} v_{\sigma(1)} \ldots v_{\sigma(r-1)} X_l(v_{\sigma(r)})^g v_{\sigma(r+1)} \ldots {}^g v_{\sigma(t)}(X_1 \ldots X_{l-1})\bar{Y}(X_{l+1} \ldots X_d)gh$$

where the sum is over $1 \leq l \leq d$, $1 \leq r \leq t$, $\sigma \in S_t$, and the coefficients $\zeta_{r,m}^{l,d,t}$ are the nonzero rational numbers

$$\zeta_{r,m}^{l,d,t} = (-1)^{(m-1)(l-1)}\frac{(r+l-2)!(t-r+d-l)!}{(r-1)!(t-r)!(d+t+l-1)!}.$$

Proof. Choose bases $y_1, \ldots, y_s$ of $(V^h)^\perp$ and $y_{s+1}, \ldots, y_n$ of V^h. Let $y_1^*, \ldots, y_n^*$ be the dual basis of V^*. Recall the notation $\partial_j = 1 \otimes y_j^*$. We may write each of the X_i as a sum of elements of the form $f\partial_l$, with $f \in S(V)$, and write $\bar{Y}$ as a sum of elements of the form $\partial_1 \ldots \partial_s\partial_J$, with J an ordered subset of $\{s+1, \ldots, n\}$. By expanding both sides of the proposed equality accordingly, one sees that it suffices to prove the result in the case in which the X_i are of the form $f_i\partial_{j_i}^*$ and $\bar{Y} = \partial_1 \ldots \partial_s\partial_J$. We are free to reorder indices if necessary so that $\bar{Y} = \partial_1 \ldots \partial_m$ with $m \geq s$.

Let us check the value of the ϕ-circle product when applied to monomials of the form $o(y_{j_{l+1}}, \ldots, y_{j_d}, y_1, \ldots, y_m, y_{j_1}, \ldots, y_{j_{l-1}})$, for arbitrary l. If any of the y_{j_i} are in $(V^h)^\perp$, or if any of the indices are repeated, then the given monomial will be 0. So

we may assume that each of the y_{j_i}, for $i \neq l$, are in V^h and that none of the indices are repeated.

Applying the definition (2.1.3) of the ϕ-circle product and the diagonal map (3.1.1), we find

$$((X_1 \cdots X_d g) \circ_\phi (Yh))\big(o(y_{j_{l+1}}, \ldots, y_{j_d}, y_1, \ldots, y_m, y_{j_1}, \ldots, y_{j_{l-1}})\big)$$
$$= (-1)^{(d-l)m}(X_1 \cdots X_d g)\big(\phi(o(y_{j_{l+1}}, \ldots, y_{j_d}) \otimes v_1 \cdots v_t \otimes {}^h o(y_{j_1}, \ldots, y_{j_{l-1}}))h\big).$$

Since all the y_{j_i} are in V^h, we may replace ${}^h o(y_{j_1}, \ldots, y_{j_{l-1}})$ by $o(y_{j_1}, \ldots, y_{j_{l-1}})$ in the above expression. Therefore the value of $(X_1 \cdots X_d g) \circ_\phi (Yh)$ on the given monomial is as follows, using Definition 3.2.1 of ϕ:

$$(-1)^{(d-l)m}(X_1 \cdots X_d g)\big(\phi(o(y_{j_{l+1}}, \ldots, y_{j_d}) \otimes v_1 \cdots v_t \otimes o(y_{j_1}, \ldots, y_{j_{l-1}})h\big)$$
$$= (-1)^{(d-l)m}(X_1 \cdots X_d g)\Big(\sum_{\substack{1 \leq r \leq t \\ \sigma \in \bar{S}_t}} c_r^{d-l,t,l-1} v_{\sigma(1)} \cdots v_{\sigma(r-1)} \otimes$$
$$o(y_{j_1}, \ldots, y_{j_{l-1}}, v_{\sigma(r)}, y_{j_{l+1}}, \ldots, y_{j_d}) \otimes v_{\sigma(r+1)} \cdots v_{\sigma(t)}h\Big)$$
$$= (-1)^{(d-l)m} \sum_{\substack{1 \leq r \leq t \\ \sigma \in \bar{S}_t}} c_r^{d-l,t,l-1} v_{\sigma(1)} \cdots v_{\sigma(r-1)} X_l(v_{\sigma(r)}) f_1 \cdots f_{l-1} f_{l+1} \cdots f_d g v_{\sigma(r+1)} \cdots v_{\sigma(t)}h.$$

The above argument can also be used to show that the circle product vanishes on all monomials $o(y_I)$ which are not of the form $\pm o(y_{j_1}, \ldots, y_{j_{l-1}}, y_1, \ldots, y_m, y_{j_{l+1}}, \ldots, y_{j_d})$ for some l. To obtain the equality of the theorem, we must reorder the factors in our initial argument:

$$o(y_{j_{l+1}}, \ldots, y_{j_d}, y_1, \ldots, y_m, y_{j_1}, \ldots, y_{j_{l-1}})$$
$$= (-1)^{m(l-1)+(d-l)(l-1)+(d-l)m} o(y_{j_1}, \ldots, y_{j_{l-1}}, y_1, \ldots, y_m, y_{j_{l+1}}, \ldots, y_{j_d}).$$

Multiply by this coefficient and compare values with those of the function in the statement of the theorem to see that they are the same. $\qquad\square$

4.3. Projections onto group components. For each $g \in G$, we will construct a chain retraction

$$p_g : S(V) \otimes \bigwedge{}^\bullet V^* g \longrightarrow S(V^g) \otimes \bigwedge{}^{\bullet - \operatorname{codim} V^g}(V^g)^* \det{}_g^\perp$$

onto the subspace $S(V^g) \otimes \bigwedge^{\bullet - \operatorname{codim} V^g}(V^g)^* \det_g^\perp$. (The differential on the codomain is taken to be 0.) Simply by virtue of being a retract of an injective quasi-isomorphism, each p_g will also be a quasi-isomorphism.

In the sections that follow we will often think of the p_g as quasi-isomorphisms from $S(V) \otimes \bigwedge^\bullet V^* g$ to itself, simply by composing with the embedding. We outline the construction of p_g below.

Construction of p_g. From the canonical decomposition $V = V^g \oplus (1-g)V$ we get an identification $S(V) = S(V^g \oplus (1-g)V)$ and canonical projection $p_g^1 : S(V) \to S(V^g)$. We also get a canonical decomposition of the dual space and its higher wedge powers,

$$V^* = (V^g)^* \oplus ((1-g)V)^* \quad \text{and} \quad \bigwedge{}^i V^* = \bigoplus_{i_1 + i_2 = i} (\bigwedge{}^{i_1}(V^g)^*) \wedge (\bigwedge{}^{i_2}((1-g)V)^*),$$

whence we have a second canonical projection

$$p_g^2 : \bigwedge{}^{\bullet} V^* \to \bigwedge{}^{\bullet - \operatorname{codim} V^g}(V^g)^* \wedge (\bigwedge{}^{\operatorname{codim} V^g}((1-g)V)^*) = \bigwedge{}^{\bullet - \operatorname{codim} V^g}(V^g)^* \operatorname{det}_g^{\perp}.$$

We now define p_g as the tensor product $p_g^1 \otimes p_g^2$,

$$p_g : S(V) \otimes \bigwedge{}^{\bullet} V^* g \;\longrightarrow\; S(V^g) \otimes \bigwedge{}^{\bullet - \operatorname{codim} V^g}(V^g)^* \operatorname{det}_g^{\perp}. \qquad (4.3.1)$$

It is apparent from the construction that each p_g restricts to the identity on the subspace $S(V^g) \otimes \bigwedge^{\bullet - \operatorname{codim} V^g}(V^g)^* \operatorname{det}_g^{\perp}$. Furthermore, since the ideal $\mathfrak{I}_g$ generated by $(1-g)V$ is precisely the kernel of p_g^1, and left multiplication by E_g has image in $\mathfrak{I}_g \otimes \bigwedge^{\bullet} V^*$, we see that $p_g(E_g \cdot -) = 0$. This is exactly the statement that p_g is a chain map. Note that, by Proposition 4.1.5, the projections will be quasi-isomorphisms.

Recall that, by Proposition 4.1.8, the subspace

$$\bigoplus_g S(V^g) \otimes \bigwedge{}^{\bullet - \operatorname{codim} V^g}(V^g)^* \operatorname{det}_g^{\perp} \;\subset\; \bigoplus_g S(V) \otimes \bigwedge{}^{\bullet} V^* g$$

is a G-subcomplex. These projections p_g are compatible with the G-action in the sense of

Lemma 4.3.2.   (1) *For any $Xg \in S(V) \otimes \bigwedge^{\bullet} V^* g$ the projections p_g and $p_{h^{-1}gh}$ satisfy the relation*

$$p_{h^{-1}gh}((Xg) \cdot h) = p_g(X_g) \cdot h.$$

(2) *The coproduct map*

$$p : \bigoplus_g S(V) \otimes \bigwedge{}^{\bullet} V^* g \;\longrightarrow\; \bigoplus_g S(V^g) \otimes \bigwedge{}^{\bullet - \operatorname{codim} V^g}(V^g)^* \operatorname{det}_g^{\perp},$$

i.e. the map $p = \operatorname{diag}\{p_g : g \in G\}$, is a G-linear quasi-isomorphism.

(3) *If a sum of elements $\sum_g X_g g$ is G-invariant then so is $\sum_g p_g(X_g g)$.*

Proof. As was the case in the proof of Proposition 4.1.8, (1) follows from the compatibilities of the decompositions $V = V^g \oplus (1-g)V$ with the G-action given in Lemma 4.1.2. Statements (2) and (3) follow from (1) and the fact that each p_g is a quasi-isomorphism. $\qquad \square$

In the following results, we take $\mathfrak{I}_g \subset S(V)$ to be the ideal generated by $(1-g)V$, as was done above. Given an ordered subset $I = \{i_1, \ldots, i_j\}$ of $\{1, \ldots, n\}$, let $\partial_I = \partial_{i_1} \cdots \partial_{i_j}$ in $S(V) \otimes \bigwedge^{\bullet} V^*$ where $\partial_i = 1 \otimes x_i^*$ as before.

Lemma 4.3.3. *Let ω_g be an arbitrary generator for $\bigwedge^{\operatorname{codim} V^g}((1-g)V)^*$. The kernel of the projection p_g defined in (4.3.1) is the sum*

$$\ker(p_g) = \mathfrak{I}_g \otimes \bigwedge{}^{\bullet} V^* g + S(V) \cdot \{\partial_I g : \omega_g \text{ does not divide } \partial_I\}.$$

If we choose bases $\{x_1, \ldots, x_l\}$ of V^g and $\{x_{l+1}, \ldots, x_n\}$ of $(1-g)V$, and take $\omega_g = \partial_{l+1} \cdots \partial_n$, the second set can be written as

$$S(V) \cdot \{\partial_I g : \{l+1, \ldots, n\} \text{ is not a subset of } I\}.$$

Proof. Note that, for p_g^1 and p_g^2 as in the above construction of p_g, we have

$$\ker(p_g^1) = \mathfrak{I}_g \quad \text{and} \quad \ker(p_g^2) = k\{\partial_I g : \omega_g \text{ does not divide } \partial_I\}$$

So the description of $\ker(p_g)$ follows from the fact that for any product of vector space maps $\sigma_1 \otimes \sigma_2 : W_1 \otimes W_2 \to U_1 \otimes U_2$, its kernel is the sum $\ker(\sigma_1) \otimes W_2 + W_1 \otimes \ker(\sigma_2)$. $\square$

5. Brackets

In this section we assume the characteristic of k is 0, and derive a general formula for brackets on Hochschild cohomology of $S(V)\#G$, using the ϕ-circle product formula of Lemma 4.2.1 and the projection maps (4.3.1). The Schouten bracket for the underlying symmetric algebra features prominently. We use our formula to obtain several conclusions about brackets, in particular some vanishing criteria.

We will use the notation and results of Section 4. In particular, we will express elements of the Hochschild cohomology $\mathrm{HH}^\bullet(S(V)\#G)$ as G-invariant elements of $\mathrm{HH}^\bullet(S(V), S(V)\#G)$ by (4.0.1), and we will use the identification of cohomology,

$$\bigoplus_{g \in G} S(V^g) \otimes \bigwedge{}^{\bullet-\mathrm{codim}V^g}(V^g)^*\mathrm{det}_g^\perp = \mathrm{HH}^\bullet(S(V), S(V)\#G)$$

given by (4.1.7). Elements of cohomology $\mathrm{HH}^\bullet(S(V), S(V)\#G)$ that are nonzero only in the component indexed by a unique g in G in the above sum are said to be *supported on g*. Elements that are nonzero only in components indexed by elements in the conjugacy class of g are said to be *supported on the conjugacy class of g*. The canonical projections $p_g : S(V) \otimes \bigwedge^\bullet V^* g \to S(V^g) \otimes \bigwedge^\bullet(V^g)^*\mathrm{det}_g^\perp g$, defined in (4.3.1), will appear in our expressions of brackets on Hochschild cohomology $\mathrm{HH}^\bullet(S(V)\#G)$ below.

5.1. **Preliminary information on the Schouten bracket and group actions.**
We let the bracket $\{X, Y\}$ denote the standard Schouten bracket on $S(V) \otimes \bigwedge^\bullet V^* \cong \bigwedge_{S(V)}^\bullet T$, where T denotes the global vector fields on $\mathrm{Spec}(S(V))$ (or rather, algebra derivations on $S(V)$). Gerstenhaber brackets in this case are precisely Schouten brackets, and we briefly verify that our approach does indeed give this expected result:

Lemma 5.1.1. *For any $X, Y \in S(V) \otimes \bigwedge^\bullet V^*$,*

$$[X, Y]_\phi = \{X, Y\}.$$

Proof. By construction, the restriction of ϕ to $K \otimes_{S(V)} K \subset \tilde{K} \otimes_{S(V)\#G} \tilde{K}$ provides a contracting homotopy for F_K. Therefore, for elements in $S(V) \otimes \bigwedge^\bullet V^*$, we will have $[X, Y]_\phi = [X, Y]_{\phi|K \otimes_{S(V)} K}$. So it does make sense to consider the proposed equality $[X, Y]_\phi = \{X, Y\}$ in $S(V) \otimes \bigwedge^\bullet V^* \cong \bigwedge_{S(V)}^\bullet T$.

It is shown in [13, §4] that some choice of contracting homotopy ψ is such that $[X, Y]_\psi$ is the Schouten bracket. By [13, Theorem 3.2.5], for any two choices of contracting homotopy, the difference in the associated brackets is a coboundary. Since the differential vanishes on $\mathrm{Hom}_{A^e}(K, S(V)) = S(V) \otimes \bigwedge^\bullet V^*$, we see that $[X, Y]_\phi = [X, Y]_\psi = \{X, Y\}$. $\square$

Under the identification $S(V) \otimes \bigwedge^\bullet V^* \cong \bigwedge^\bullet_{S(V)} T$, the right action of an element $g \in G$ on $S(V) \otimes V^*$ is identified with conjugation by the corresponding automorphism, $X \cdot g = X^g = g^{-1} X g$. On higher degree elements the action is given by the standard formula, $(X_1 \cdots X_l) \cdot g = (X_1 \cdots X_l)^g = X_1^g \cdots X_l^g$. Under the identification

$$\bigoplus_{g \in G} (S(V) \otimes \bigwedge^\bullet V^* g) \cong \bigoplus_{g \in G} (\bigwedge^\bullet_{S(V)} T g),$$

the action is given by $(Xg) \cdot h = X^h h^{-1} g h$, where we can view X either as an element in $S(V) \otimes V^*$ or as a polyvector field.

Lemma 5.1.2. *For any $X, Y \in S(V) \otimes \bigwedge^\bullet V^*$, and $g \in G$, we have $\{X, Y\}^g = \{X^g, Y^g\}$.*

Proof. We identify $S(V) \otimes \bigwedge^\bullet V^*$ with the global polyvector fields $\bigwedge^\bullet_{S(V)} T$. Then the lemma follows from the fact that the G-action is simply given by conjugating by the corresponding automorphism, the fact that the Schouten bracket is given by composition of vector fields on T, and the Gerstenhaber identity $\{X, Y_1 Y_2\} = \{X, Y_1\} Y_2 \pm Y_1 \{X, Y_2\}$. $\square$

Lemma 5.1.3. *For any G-invariant elements $\sum_g X_g g$ and $\sum_{h \in G} Y_h h$ in the sum $\bigoplus_{g \in G} (S(V) \otimes \bigwedge^\bullet V^* g)$, the element $\sum_{g, h \in G} \{X_g, Y_h\} g h$ is also G-invariant. Furthermore, for any such $\sum_g X_g g$ and $\sum_h Y_h h$, the element $\sum_{g, h \in G} p_{gh} \{X_g, Y_h\} g h$ will be a G-invariant cocycle.*

Proof. We know an element $\sum_g Z_g g$ will be invariant if and only if, for each $g, \sigma \in G$, $Z_g^\sigma = Z_{\sigma^{-1} g \sigma}$. So the X_g and Y_h have this property, and it follows that the sum $\sum_{\{g, h \in G: gh = \tau\}} \{X_g, Y_h\}$ will have this property for each $\tau \in G$ since

$$\sum_{gh=\tau} \{X_g, Y_h\}^\sigma = \sum_{gh=\tau} \{X_g^\sigma, Y_h^\sigma\} = \sum_{gh=\tau} \{X_{\sigma^{-1} g \sigma}, Y_{\sigma^{-1} h \sigma}\} = \sum_{\{g', h': g' h' = \sigma \tau \sigma^{-1}\}} \{X_{g'}, Y_{h'}\}.$$

The last statement now follows directly from Lemma 4.3.2(3). $\square$

5.2. ϕ-brackets as Schouten brackets.

Before we begin, it will be useful to have the following two lemmas. Recall that $\mathfrak{I}_g$ is the ideal generated by $(1 - g)V$ in $S(V)$.

Lemma 5.2.1. *For any vectors $u_{i_1}, \ldots, u_{i_\nu} \in V$ and $g \in G$, the element $^{(1-g)}(u_{i_1} \ldots u_{i_\nu})$ is in $\mathfrak{I}_g$.*

Proof. We proceed by induction on the number of vectors ν. When $\nu = 1$ the result is immediate. For $\nu > 1$ we have

$$
\begin{aligned}
^{(1-g)}(u_{i_1} \ldots u_{i_\nu}) \ &= u_{i_1} \ldots u_{i_\nu} - {}^g u_{i_1} \ldots {}^g u_{i_\nu} \\
&= (u_{i_1} \ldots u_{i_{\nu-1}} - {}^g u_{i_1} \ldots {}^g u_{i_{\nu-1}}) u_{i_\nu} + {}^g u_{i_1} \ldots {}^g u_{i_{\nu-1}} {}^{(1-g)} u_{i_\nu},
\end{aligned}
$$

which is now in $\mathfrak{I}_g$ by induction. $\square$

Lemma 5.2.2. *Suppose c and c' are invariant cocycles in $\bigoplus_g S(V) \otimes \bigwedge^\bullet V^* g$ that differ by a (possibly non-invariant) coboundary. Then c and c' differ by an invariant coboundary.*

Proof. Note that $c - c'$ is also invariant. Let b be any element with $d(b) = c - c'$, where d is the differential $d = E \cdot -$. Then we have

$$c - c' = d(b) \cdot \int_G = d(b \cdot \int_G),$$

where $\int_G$ is the standard integral $\frac{1}{|G|} \sum_{g \in G} g$. So $b \cdot \int_G$ provides the desired invariant bounding element. $\square$

We can now give a general formula for the Gerstenhaber bracket on $\mathrm{HH}^\bullet(S(V)\#G)$ in terms of Schouten brackets. One may compare with [8, Theorem 4.4, Corollary 4.11] where the authors give similar formulas under some conditions on the group G and its action on V.

Theorem 5.2.3. *Let $X = \sum_g X_g g$ and $Y = \sum_h Y_h h$ be G-invariant cocycles in $\oplus_{g \in G} S(V^g) \otimes \bigwedge^{\bullet - \mathrm{codim} V^g}(V^g)^* \det_g^\perp$. The sum $\sum_{g,h \in G} p_{gh}\{X_g, Y_h\}gh$ is a G-invariant cocycle and, considered as elements of the cohomology $\mathrm{HH}^\bullet(S(V)\#G)$,*

$$[X, Y] = \sum_{g,h \in G} p_{gh}\{X_g, Y_h\}gh.$$

Proof. By Lemma 5.2.2, it suffices to show that the equality holds up to arbitrary coboundaries. Note that

$$p[X, Y]_\phi = \sum_{g,h} p_{gh}(X_g g \circ_\phi Y_h h) - (-1)^{(|Y|-1)(|X|-1)} \sum_{g,h} p_{hg}(Y_h h \circ_\phi X_g g).$$

By considering the group automorphism

$$G \times G \to G \times G, \quad (g, h) \mapsto (g, ghg^{-1}),$$

and the equality $ghg^{-1}g = gh$, we see that we can reindex the second sum to obtain

$$\begin{aligned} p[X, Y]_\phi &= \sum_{g,h} p_{gh}(X_g g \circ_\phi Y_h h) \mp \sum_{g,h} p_{gh}(Y_{ghg^{-1}} ghg^{-1} \circ_\phi X_g g) \\ &= \sum_{g,h} \left(p_{gh}(X_g g \circ_\phi Y_h h) \mp p_{gh}(Y_{ghg^{-1}} ghg^{-1} \circ_\phi X_g g) \right). \end{aligned} \tag{5.2.4}$$

We claim that there is an equality

$$p_{gh}(X_g g \circ_\phi Y_h h) \mp p_{gh}(Y_{ghg^{-1}} ghg^{-1} \circ_\phi X_g g) = p_{gh}[X_g, Y_h]_\phi gh = p_{gh}\{X_g, Y_h\}gh \tag{5.2.5}$$

for each pair $g, h \in G$, where the second equality follows already by Lemma 5.1.1. If we can establish (5.2.5) then we are done, by the final expression in (5.2.4) and the fact that the difference $[X, Y]_\phi - p[X, Y]_\phi$ is a coboundary.

Let us fix elements $g, h \in G$. Write X_g as a sum of elements of the form $u_1 \cdots u_s \bar{X}$ with the $u_j \in V^g$, and Y_h as a sum of elements $v_1 \cdots v_t \bar{Y}$ with the $v_i \in V^h$. Here $\bar{X}, \bar{Y} \in \bigwedge^\bullet V^*$. By h-invariance there is an equality ${}^g v_i = {}^{gh} v_i$ for each i. For arbitrary Z in $S(V) \otimes \bigwedge^\bullet V^*$ and elements $a_i \in S(V)$, the projection $p_{gh}({}^{(1-gh)}(a_1 \ldots a_q)Zgh)$ vanishes by Lemma 4.3.3 and Lemma 5.2.1. So for any $r < t$ and $\sigma \in S_n$ there will be equalities

$$\begin{aligned} p_{gh}\big({}^g(v_{\sigma(r+1)} \ldots v_{\sigma(t)})Zgh\big) &= p_{gh}\big({}^{gh}(v_{\sigma(r+1)} \ldots v_{\sigma(t)})Zgh\big) \\ &= p_{gh}(v_{\sigma(r+1)} \ldots v_{\sigma(t)}Zgh). \end{aligned} \tag{5.2.6}$$

It now follows from the expression for the circle operation given in Lemma 4.2.1 that there is an equality

$$p_{gh}(X_g g \circ_\phi Y_h h) = p_{gh}(X_g \circ_\phi Y_h)gh. \tag{5.2.7}$$

This covers half of what we need.

We would like to show now

$$p_{gh}(Y_{ghg^{-1}}ghg^{-1} \circ_\phi X_g g) = p_{gh}(Y_h \circ_\phi X_g).$$

Simply replacing g and h with ghg^{-1} and g in (5.2.7), as well as X_g with $Y_{ghg^{-1}}$ and Y_h with X_g, gives

$$p_{gh}(Y_{ghg^{-1}}ghg^{-1} \circ_\phi X_g g) = p_{gh}(Y_{ghg^{-1}} \circ_\phi X_g)gh.$$

Now G-invariance of Y implies immediately $Y_{ghg^{-1}} = Y_h^{g^{-1}}$. So we have

$$p_{gh}(Y_{ghg^{-1}}ghg^{-1} \circ_\phi X_g g) = p_{gh}(Y_h^{g^{-1}} \circ_\phi X_g)gh,$$

whence we need to show $p_{gh}(Y_h^{g^{-1}} \circ_\phi X_g)gh = p_{gh}(Y_h \circ_\phi X_g)gh$.

Recall our expressions for Y_h and X_g from above, in terms of the v_i, u_j, $\bar{Y}$ and $\bar{X}$. In the notation of Lemma 4.3.3, we may assume that $\omega_g|\bar{X}$, and hence that $\omega_{g^{-1}}|\bar{X}$ by Lemma 4.1.2. We write $\bar{Y}$ as a sum of monomials $Y_1 \cdots Y_e$ for functions $Y_i \in V^*$. For each Y_i and u_j we have $Y_i^{g^{-1}}(u_j) = Y_i({}^{g^{-1}}u_j) = Y_i(u_j)$. We also have $Y_i^{g^{-1}}\bar{X} = Y_i\bar{X}$ since $\omega_{g^{-1}}|\bar{X}$. From these observations and the expression of Lemma 4.2.1, since Y_h is a sum of elements of the form $v_1 \cdots v_t\bar{Y}$, we deduce an equality

$$Y_h^{g^{-1}} \circ_\phi X_g = \sum {}^g(v_1 \dots v_t)(\bar{Y}^{g^{-1}}) \circ_\phi X_g = \sum {}^g(v_1 \dots v_t)\bar{Y} \circ_\phi X_g.$$

Finally, by the same argument given for the equality (5.2.6) we find also that

$$\sum p_{gh}\big({}^g(v_1 \dots v_t)\bar{Y} \circ_\phi X_g\big) = \sum p_{gh}\big((v_1 \dots v_t)\bar{Y} \circ_\phi X_g\big) = p_{gh}(Y_h \circ_\phi X_g).$$

Taking these two sequences of equalities together gives the desired equality

$$p_{gh}(Y_h^{g^{-1}} \circ_\phi X_g) = p_{gh}(Y_h \circ_\phi X_g),$$

establishes (5.2.5), and completes the proof. $\qquad\qquad\qquad\square$

5.3. Corollaries: The codimension grading and some general vanishing results.
We apply the formula in Theorem 5.2.3 to analyze distinct cases. In one case, we consider brackets with an element X supported on group elements that act trivially on V. In another case, we consider brackets of X and Y supported on elements that act nontrivially. We will have in this second case some general vanishing results. The following observation helps in organizing these cases.

Observation 5.3.1. The graded G-module

$$\bigoplus_{g \in G} S(V^g) \otimes \bigwedge^{\bullet - \operatorname{codim}V^g}(V^g)^* \det_g^\perp \tag{5.3.2}$$

decomposes as a direct sum of graded G-subspaces $\mathscr{D}(i)$, for $0 \leq i \leq \dim(V)$, as does its G-invariant subspace,

$$\left(\bigoplus_{g \in G} S(V^g) \otimes \bigwedge^{\bullet - \mathrm{codim} V^g} (V^g)^* \mathrm{det}_g^\perp \right)^G = \bigoplus_{0 \leq i \leq \dim(V)} \mathscr{D}(i)^G. \tag{5.3.3}$$

The subspace $\mathscr{D}(i)$ consists of all sums of elements supported on group elements g for which $\mathrm{codim} V^g = i$. We call the decomposition (5.3.3) the codimension grading for $\mathrm{HH}^\bullet(S(V) \# G)$.

Said another way, $\mathscr{D}(i)$ consists of all summands in (5.3.2) whose first nonzero cohomology class occurs in degree i. Note that classes in $\mathscr{D}(0)$ are supported on only those group elements which act trivially on V. The brackets between elements in $\mathscr{D}(0)^G$ will just be given by the Schouten brackets (cf. [17, Corollary 7.4]):

Corollary 5.3.4. *Let $X = \sum_g X_g g$ and $Y = \sum_h Y_h h$ be G-invariant cocycles in $\oplus_{g \in G} S(V^g) \otimes \bigwedge^{\bullet - \mathrm{codim} V^g} (V^g)^* \mathrm{det}_g^\perp$ that are supported on group elements acting trivially, i.e. $X, Y \in \mathscr{D}(0)$. Then in cohomology,*

$$[X, Y] = \sum_{g,h} \{X_g, Y_h\} gh.$$

Proof. In this case for each g, h with X_g and Y_h nonzero we will have $V^g = V^h = V^{gh} = V$ and $p_g = id$, $p_h = id$ and $p_{gh} = id$. The result now follows from Theorem 5.2.3. □

We refer directly to Theorem 5.2.3 for information on the bracket between cochains in $\mathscr{D}(0)^G$ and $\mathscr{D}(>0)^G$. We next give some conditions under which brackets are 0. The following corollary was first proved in [17] using different techniques.

Corollary 5.3.5 ([17, Proposition 8.4]). *Let $g, h \in G$ be such that $(V^g)^\perp \cap (V^h)^\perp$ is nonzero and is a kG-submodule of V. Let X, Y be G-invariant cocycles in the sum $\oplus_{g \in G} S(V^g) \otimes \bigwedge^{\bullet - \mathrm{codim} V^g} (V^g)^* \mathrm{det}_g^\perp$ supported on the conjugacy classes of g, h, respectively. Then*

$$[X, Y] = 0.$$

Proof. The hypotheses imply that $(V^{aga^{-1}})^\perp \cap (V^{bhb^{-1}})^\perp$ is nonzero for all $a, b \in G$. We will argue that $X_g g \circ_\phi Y_h h = 0$ at the chain level, and similar reasoning will apply to $X_{aga^{-1}} aga^{-1} \circ_\phi Y_{bhb^{-1}} bhb^{-1}$ and to $Y_{bhb^{-1}} bhb^{-1} \circ_\phi X_{aga^{-1}} aga^{-1}$. Consider the argument $o(y_{j_1}, \ldots, y_{j_{l-1}}, y_1, \ldots, y_m, y_{j_{l+1}}, \ldots, y_{j_d})$ in the proof of Lemma 4.2.1. This can be nonzero only in case $y_{j_l} \in (V^g)^\perp \cap (V^h)^\perp$, due to linear dependence of the vectors involved otherwise. Thus the only possible terms in the ϕ-circle product formula of Lemma 4.2.1 that could be nonzero are indexed by such l. However then $X_l(v_{\sigma(r)}) = 0$ for all r, σ, since $v_{\sigma(r)} \in V^h$, in the notation of the proof of Lemma 4.2.1. □

The following corollary was pointed out to us by Travis Schedler.

Corollary 5.3.6. *For classes $X \in \mathscr{D}(i)^G$ and $Y \in \mathscr{D}(j)^G$ we have $[X, Y] \in \mathscr{D}(i+j)^G$.*

Proof. As was argued in the proof of Corollary 5.3.5 we find that $X' \circ_\phi Y' = 0$ whenever

$$X' \in S(V^g) \otimes \bigwedge\nolimits^{\bullet - \mathrm{codim} V^g}(V^g)^* \mathrm{det}_g^\perp \quad \text{and} \quad Y' \in S(V^h) \otimes \bigwedge\nolimits^{\bullet - \mathrm{codim} V^h}(V^h)^* \mathrm{det}_h^\perp$$

and $(V^g)^\perp \cap (V^h)^\perp \neq 0$. That is to say, nonvanishing of the element $X' \circ_\phi Y'$ implies $(V^g)^\perp \cap (V^h)^\perp = 0$ and therefore $\mathrm{codim} V^{gh} = \mathrm{codim} V^g + \mathrm{codim} V^h$ [16, Lemma 2.1]. It follows that for (even non-invariant) classes $X = \sum_g X_g g \in \mathscr{D}(i)$ and $Y = \sum_h Y_h h \in \mathscr{D}(j)$ we have

$$[X, Y]_\phi = \sum_{g,h} [X_g g, Y_h h]_\phi \in \mathscr{D}(i + j).$$

$\square$

Corollary 5.3.7. *The Hochschild cohomology* $\mathrm{HH}^\bullet(S(V) \# G)$ *is a graded Gerstenhaber algebra with respect to the codimension grading.*

Proof. We just saw in Corollary 5.3.6 that the Gerstenhaber bracket respects the codimension grading. It has also already been established that the cup product respects the codimension grading [3], [16, Proposition 8.1, Theorem 9.2]. $\square$

The following corollary generalizes [17, Theorem 9.2], where it was proven in homological degree 2. The cocycles X, Y in the corollary are by hypothesis of smallest possible homological degree in their group components.

Corollary 5.3.8. *Take any G-invariant cocycles $X = \sum_{g \in G} X_g g$, $Y = \sum_{h \in G} Y_h h$ in $\oplus_{g \in G} S(V^g) \otimes \mathrm{det}_g^\perp$ that are supported on elements that act nontrivially on V. Then $[X, Y] = 0$ on cohomology.*

Proof. We have that X and Y are sums of invariant classes X' and Y' which are of minimal degrees in $\mathscr{D}(i)$ and $\mathscr{D}(j)$ respectively, i.e. degrees i and j. By Corollary 5.3.6 we have $[X', Y'] \in \mathscr{D}(i+j)$. But now since the Gerstenhaber bracket is homogeneous of degree -1 we have that $[X', Y']$ is of degree $i + j - 1$. Since there are no nonzero classes of degree $i + j - 1$ in $\mathscr{D}(i + j)$ we have that all $[X', Y'] = 0$ and hence $[X, Y] = 0$. $\square$

6. (Non)vanishing of brackets in the case $(V^g)^\perp \cap (V^h)^\perp = 0$

With Corollaries 5.3.5 and 5.3.8 we seem to be approaching a general result. Namely, that for any X and Y supported on group elements that act nontrivially we will have $[X, Y] = 0$. It is even known that such a vanishing result holds in degree 2 by [17, Theorem 9.2]. This is, however, not going to be the case in higher degrees. The result even fails to hold when we consider the bracket $[X, Y]$ of elements in degrees 2 and 3 (see Example 6.1.2 below). We give in this section a few examples to illustrate this nonvanishing, and (re)establish vanishing in degree 2.

6.1. **Some examples for which $(V^g)^\perp \cap (V^h)^\perp = 0$.** The following two examples illustrate, first, the essential role of taking invariants in establishing the degree 2 vanishing result of [17, Theorem 9.2] and, second, an obstruction to establishing a general vanishing result in the case $(V^g)^\perp \cap (V^h)^\perp = 0$.

Example 6.1.1. Take $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $V = k\{x_1, x_2, x_3\}$. Let g and h be the generators of the first and second copies of $\mathbb{Z}/2\mathbb{Z}$, and take the G-action on V defined by

$$g \cdot x_i = (-1)^{\delta_{i1}} x_i \quad \text{and} \quad h \cdot x_j = (-1)^{\delta_{j3}} x_j.$$

So $V^g = k\{x_2, x_3\}$, $(1-g)V = kx_1$, $V^h = k\{x_1, x_2\}$, $(1-h)V = kx_3$. Obviously, $(1-g)V \cap (1-h)V = 0$. We also have respective generators

$$\omega_g = \partial_1, \quad \omega_h = \partial_3, \quad \text{and} \quad \omega_{gh} = \partial_1 \partial_3$$

of the highest wedge powers of $\big((1-g)V\big)^*$, $\big((1-h)V\big)^*$, and $\big((1-gh)V\big)^*$ respectively. Thus $\det_g^\perp = k\omega_g g$, $\det_h^\perp = k\omega_h h$, and $\det_{gh}^\perp = k\omega_{gh}gh$.

Consider the degree 2 cochains

$$X = \omega_g \partial_2 g \in S(V^g) \otimes (V^g)^* \det_g^\perp$$

and

$$Y = x_2 \omega_h \partial_2 h \in S(V^h) \otimes (V^h)^* \det_h^\perp.$$

Then one can easily see, directly from Lemma 4.2.1, that applying the bilinear operation $[\ ,\ ]_\phi$ produces

$$[X, Y]_\phi = \omega_g \omega_h \partial_2 gh = \omega_{gh} \partial_2 gh \in S(V^{gh}) \otimes (V^{gh})^* \det_{gh}^\perp.$$

This would appear to contradict the degree 2 vanishing result of [17], but it actually does not! The point is that neither X nor Y is invariant. In fact, $X \cdot \int_G = Y \cdot \int_G = 0$.

Example 6.1.2. Let $G = \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/M\mathbb{Z}$ for integers $N, M > 1$. We assume $k = \bar{k}$, or that $M = N = 2$. Let σ and τ be the generators of $\mathbb{Z}/N\mathbb{Z}$ and $\mathbb{Z}/M\mathbb{Z}$ respectively. Take $W = k\{x_1, x_2, x_3, x_4, x_5\}$ and embed G in $GL(W)$ by identifying σ and τ with the diagonal matrices

$$\sigma = \mathrm{diag}\{\zeta, \zeta^{-1}, 1, 1, 1\}, \quad \tau = \mathrm{diag}\{1, 1, 1, \vartheta^{-1}, \vartheta\},$$

where ζ and ϑ are primitive Nth and Mth roots of 1 in k.

We have

$$(1 - \sigma) = \mathrm{diag}\{(1-\zeta), (1-\zeta^{-1}), 0, 0, 0\}, \quad (1 - \tau) = \mathrm{diag}\{0, 0, 0, (1-\vartheta^{-1}), (1-\vartheta)\},$$

and these are both rank 2 matrices. More specifically, $(1 - \sigma)W = k\{x_1, x_2\}$ and $(1 - \tau)W = k\{x_4, x_5\}$. So $(W^\sigma)^\perp \cap (W^\tau)^\perp = 0$. Similarly we have

$$\sigma\tau = \mathrm{diag}\{\zeta, \zeta^{-1}, 1, \vartheta^{-1}, \vartheta\}, \quad (1 - \sigma\tau) = \mathrm{diag}\{(1-\zeta), (1-\zeta^{-1}), 0, (1-\vartheta^{-1}), (1-\vartheta)\},$$

$$(1 - \sigma\tau)W = k\{x_1, x_2, x_4, x_5\}.$$

We take

$$\omega_\sigma = \partial_1 \partial_2, \quad \omega_\tau = \partial_4 \partial_5, \quad \omega_{\sigma\tau} = \partial_1 \partial_2 \partial_4 \partial_5,$$

and $X = \omega_\sigma \partial_3 \sigma$, $Y = x_3 \omega_\tau \tau$. In this case X and Y are G-invariant, and hence represent classes in $\mathrm{HH}^\bullet(S(V)\#G)$. One then produces via Theorem 5.2.3 the nonvanishing Gerstenhaber bracket

$$[X, Y] = \omega_{\sigma\tau} \sigma\tau \in \mathrm{HH}^4(S(V)\#G).$$

This example can be generalized easily to produce nonzero brackets in higher degree.

6.2. **Vanishing of brackets in degree 2.** Consider the subcomplex

$$\mathscr{D}(1) \subset \bigoplus_g S(V^g) \otimes \textstyle\bigwedge^{\bullet - \mathrm{codim} V^g}(V^g)^* \det{}_g^{\perp}$$

consisting of all summands corresponding to group elements g with $\mathrm{codim} V^g = 1$. This subcomplex is stable under the G-action. It was seen already in [4, 17] that $\mathscr{D}(1)^G = 0$. So actually, after we take invariance, we have

$$\mathrm{HH}^{\bullet}(S(V)\#G) = \big(\bigoplus_g S(V^g) \otimes \textstyle\bigwedge^{\bullet - \mathrm{codim} V^g}(V^g)^* \det{}_g^{\perp} \big)^G = \mathscr{D}(0)^G \oplus \mathscr{D}(> 1)^G.$$

where in $\mathscr{D}(> 1)$ we have all summands corresponding to g with $\mathrm{codim} V^g > 1$.

It follows that, after we take invariants and restrict ourselves to considering only elements in degree 2, the only situations that can occur when taking brackets in $\mathscr{D}(> 0)^G = \mathscr{D}(> 1)^G$ are covered by Corollary 5.3.8. Hence when we apply the Gerstenhaber bracket we get

$$\left[\big(\mathscr{D}(> 0)^2\big)^G, \big(\mathscr{D}(> 0)^2\big)^G \right] = 0.$$

This rephrases the argument given in [17, Theorem 9.2].

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DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MA 02139, USA
    E-mail address: negronc@mit.edu

DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX 77843, USA
    E-mail address: sjw@math.tamu.edu