REVISITING THE STEINBERG REPRESENTATION AT ARBITRARY ROOTS OF 1

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ABSTRACT. We consider quantum group representations for a semisimple algebraic group G at a complex root of unity q. Here q is allowed to be of any order. We revisit some fundamental results of Parshall-Wang and Andersen-Polo-Wen from the 90's. In particular, we show that the category $\mathrm{Rep}(G_q)$ of quantum group representations has enough projectives and injectives, and that a G_q-representation is projective (resp. injective) if and only if its restriction to the small quantum group is projective (resp. injective). Our results reduce to an analysis of the Steinberg representation in the simply-connected setting, and are well-known at odd order q via works of the aforementioned authors. The details at arbitrary q have, to our knowledge, not appeared in the literature up to this point.

1. INTRODUCTION

Let k be an algebraically closed field of characteristic 0. For a semisimple algebraic group G and a choice of nonzero parameter q in k, we have the associated category of quantum group representations $\mathrm{Rep}(G_q)$. Equivalently, we consider the category of integrable, appropriately graded representations for Lusztig's divided power quantum enveloping algebra [43]. We are especially interested in the case where q is a root of unity.

The main point of this paper is to recover some well-known phenomena for quantum groups, but now at arbitrary quantum parameters. Specifically, we want to study projective and injective representations in $\mathrm{Rep}(G_q)$, restrictions of such representations to the small quantum group, and the behaviors of induction from the small quantum group to the big quantum group. Our analyses hinge on a study of the Steinberg representation in $\mathrm{Rep}(G_q)$ for simply-connected G.

To place things in their proper historical context; in the late 80's and early 90's there were a number of papers which established the fundamental behaviors of quantum group representations at *odd* order quantum parameters, or parameters of odd prime power order. One can see, for example, works of Andersen-Polo-Wen, Parshall-Wang, and Lusztig from this time period [41, 49, 4, 5, 2].

In the current mathematical and physical landscape however, we find ourselves in need of a deeper understanding of quantum group representations, specifically at *even* order parameters. This is due to recent developments in logarithmic conformal field theory and, what one might call, non-compact topological field theory. From a mathematical perspective, the logarithmic, or non-compact modifier indicates a non-semisimplicity in the theory. For appearances of quantum groups in such conformal and topological settings one can see the essentially-random samplings [23, 50, 28, 10, 26, 24, 55, 31] and [12, 13, 32, 52, 16, 11], respectively. For a specific

point of reference, one can consider the type A claims from the work [13] and the implications such claims have for quantum group representations at a $2p$-th root of 1 (see also [35, 29, 25]).

In the present paper we recover many of the results from the foundational texts [41, 49, 4, 5, 2], now at completely general q and completely general G. To begin, we have the following.

Theorem (10.12, 11.3). *Consider a semisimple algebraic group G at an arbitrary root of unity q.*

(1) *The category $\mathrm{Rep}(G_q)$ has enough projectives and injectives, as does its subcategory $\mathrm{rep}(G_q)$ of finite-dimensional representations.*

(2) *An object in $\mathrm{rep}(G_q)$ is projective (resp. injective) if and only if it is projective (resp. injective) in $\mathrm{Rep}(G_q)$.*

(3) *$\mathrm{Rep}(G_q)$ is Frobenius, in the sense that its projectives and injectives agree.*

Furthermore, when G is simply-connected, the Steinberg representation $\mathrm{St} = L(\rho_l)$ is projective and injective in $\mathrm{Rep}(G_q)$.

Here the Steinberg representation St is the distinguished simple representation of highest weight $\rho_l = \frac{1}{2}\sum_{\gamma \in \Phi^+}(l_\gamma - 1)\gamma$, where $l_\gamma = \mathrm{ord}\, q^{(\gamma,\gamma)}$. This representation has proved fantastically useful in studies of both quantum and modular representation theory, and our work is no exception. Indeed, one establishes points (1)–(3) above by first considering the behaviors of the Steinberg representation in the simply-connected setting.

We note that the results of Theorems 10.12 and 11.3 first appeared in works of Parshall-Wang and Andersen-Polo-Wen [49, 4], at odd order q, and are deducible by experts at arbitrary q [3, 1].

Now, at this point, we would like to make a seemingly benign assertion. Namely, we propose that the Steinberg representation restricts to a simultaneously projective, injective, and simple representation over the small quantum group. However, both for this statement and for the mathematics which precedes it, one first has to decide what the small quantum group actually is, or should be, at arbitrary q.

Addressing this point is another implicit goal of this work, as well as its predecessor [47] and successor [46].

The answer to this question is a little complicated, as there doesn't seem to be one small quantum group, but a family of small quantum groups which are parametrized by central subgroups in G, or more precisely by central subgroups in Lusztig's dual group G^* from [44]. Here we work with the smallest version of the small quantum group, which we refer to as the "smallest quantum algebra"[1]

$$\bar{u}_q = \bar{u}(G_q).$$

At odd order q and simply-connected G the algebra $\bar{u}_q$ is just Lusztig's original finite-dimensional Hopf algebra from [42, 43]. At even order q, $\bar{u}_q$ is some variant of Arkhipov and Gaitsgory's small quantum group from [7], and across all q one recovers the representation category $\mathrm{Rep}(\bar{u}_q)$ uniformly as the fiber of $\mathrm{Rep}(G_q)$ along quantum Frobenius. (See Section 1.1 for more details on this point.)

In terms of its formal structures, there is a surjective map from the quantum function algebra to the dual $\mathscr{O}(G_q) \to \bar{u}_q^*$ which makes $\bar{u}_q^*$ into a right $\mathscr{O}(G_q)$-module

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[1]Most of our results for $\bar{u}_q$ will hold for any reasonable version of the small quantum group, at arbitrary q. See Section 1.1 for some discussion.

coalgebra. Equivalently, at the level of categories, $\mathrm{Rep}(\bar{u}_q)$ has the structure of a module category over $\mathrm{Rep}(G_q)$ and the restriction map

$$\mathrm{res}_q^- : \mathrm{Rep}(G_q) \to \mathrm{Rep}(\bar{u}_q)$$

has the structure of a $\mathrm{Rep}(G_q)$-module category functor. The kernel of this functor is precisely the image of Lusztig's quantum Frobenius $\mathrm{Fr} : \mathrm{Rep}(G_\varepsilon^*) \to \mathrm{Rep}(G_q)$.

Descriptions of $\bar{u}_q$ in terms of its generators, dimension, triangular decomposition, braid group action, etc., can be found in Sections 5 and 6 below. Relations with quantum Frobenius are discussed in Section 7.

Our choice of $\bar{u}_q$, as the "cokernel" of Lusztig's quantum Frobenius functor, facilitates a Steinberg decomposition for simple representations over G_q.

Theorem (8.7/8.8). *Suppose G is simply-connected. Then every simple representation $L(\lambda)$ in $\mathrm{Rep}(G_q)$ admits a unique decomposition*

$$L(\lambda) = L(\lambda') \otimes L(\lambda''),$$

where $L(\lambda')$ is a simple representation which is in the image of quantum Frobenius and $L(\lambda'')$ is a simple representation which has simple restriction to $\mathrm{Rep}(\bar{u}_q)$.

This result is known at odd order parameters through work of Lusztig [41], and has its origins in modular representation theory [54]. We also have the following, which generalizes results from Andersen-Polo-Wen [5].

Theorem (10.12/11.3). *An object in $\mathrm{Rep}(G_q)$ is projective (equivalently injective) if and only if its restriction to $\mathrm{Rep}(\bar{u}_q)$ is projective (equivalently injective).*

In establishing the findings of Theorems 10.12 and 11.3 we also show that the induction functor from $\mathrm{Rep}(\bar{u}_q)$ to $\mathrm{Rep}(G_q)$ is exact and faithful. This result appears in Theorem 10.4 below.

For a more elaborate historical accounting of the findings herein, and their many predecessors in the literature, we invite the reader to see Section 11.3 at the conclusion of the text. We conclude the introduction with a short discussion of the smallest quantum algebra and its relation to "the" small quantum group, as constructed in the works [14, 27, 47, 31, 46] for example.

1.1. **The smallest quantum algebra vs. the small quantum group.** The small quantum group for G at q, whatever it might be [7, 14, 27, 30, 47, 46], is represented by an algebra u_q which fits into an exact sequence

$$k \to \bar{u}_q \to u_q \to k\Sigma \to k.$$

Here u_q is a flat extension of $\bar{u}_q$, Σ is a finite abelian group, and the final map induces an isomorphism $k \otimes_{\bar{u}_q} u_q = u_q \otimes_{\bar{u}_q} k \xrightarrow{\sim} k\Sigma$. From this sequence one sees that the representation theory, and the homological algebra, for u_q are generically identified with that of $\bar{u}_q$. For example, a u_q-representation is injective if and only if it has injective restriction to $\bar{u}_q$. In this way the conclusions of Theorem 11.3 transfer immediately from $\bar{u}_q$ to u_q. However, highly refined structural results, such as the Steinberg decomposition from [41] or Theorem 8.7, only hold for $\bar{u}_q$.

From a categorical perspective, the small quantum group represents the fiber

$$\mathrm{Vect} \otimes_{\mathrm{Rep}(\check{G})} \mathrm{Rep}(G_q) \xrightarrow{\sim} \mathrm{Rep}(u_q) \tag{1}$$

of $\mathrm{Rep}(G_q)$ along a central map $\mathbf{Fr} : \mathrm{Rep}(\check{G}) \to \mathrm{Rep}(G_q)$ which identifies the Tannakian center in $\mathrm{Rep}(G_q)$ with representations of a dual group $\check{G}$ [7, 30, 46].

This fiber construction annihilates the Tannakian center in $\mathrm{Rep}(G_q)$, and realizes $\mathrm{Rep}(u_q)$ as the unique (super-)modularization of the braided category of G_q-representations. This makes u_q a natural object to study from the perspective of quantum algebra or quantum topology.

On the other hand, representations for the smallest quantum algebra are identified with the fiber along Luztig's usual quantum Frobenius

$$\mathrm{Vect} \otimes_{\mathrm{Rep}(G_\varepsilon^*)} \mathrm{Rep}(G_q) \xrightarrow{\sim} \mathrm{Rep}(\bar{u}_q) \tag{2}$$

[7, 46]. Since $\mathrm{Rep}(G_\varepsilon^*)$ is generally non-central in $\mathrm{Rep}(G_q)$, and often not even symmetric, this fiber is most naturally understood as a right module category over $\mathrm{Rep}(G_q)$.[2]

It is at the level (2) that we recover classical structure theorems which "factor" the representation theory of $\mathrm{Rep}(G_q)$ between that of $\mathrm{Rep}(G_\varepsilon^*)$ and $\mathrm{Rep}(\bar{u}_q)$, as in [41, 49, 4, 5].

Contents

2. Preliminaries on categories and Hopf algebras

We fix an algebraically closed field k of characteristic 0. All categories are k-linear categories, algebras are k-algebras, coalgebras are k-coalgebras, etc. We let Vect denote the tensor category of arbitrary k-vector spaces.

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[2]We could have taken the fiber on the right, and had an action of $\mathrm{Rep}(G_q)$ on the left, but it doesn't matter. These two constructions are equivalent.

2.1. **Representations and corepresentations.** Let B be an algebra. By a B-module we always mean a left B-module. We let $\mathrm{Rep}(B)$ denote the category of all B-modules V which are the unions of their finite-dimensional submodules. Dually, for any coalgebra C, by a C-comodule we always mean a right C-comodule. We let $\mathrm{Corep}(C)$ denote the category of arbitrary C-comodules, and note that any comodule is already the union of its finite-dimensional subcomodules (see for example [45]).

2.2. **Tensor categories and module categories.** A tensor category is an abelian, compactly generated monoidal category $\mathscr{A}$ whose compact and rigid objects agree. We require additionally that the subcategory $\mathscr{A}^c$ of compact/rigid objects in $\mathscr{A}$ is an essentially small, locally finite abelian subcategory [21, Definition 1.8.1], that the unit object in $\mathscr{A}$ is simple, and that the tensor product commutes with colimits in each factor. These generation restrictions imply, for example, that any tensor category $\mathscr{A}$ is a Grothendieck abelian category [36, Theorem 8.6.5]. Since tensoring with any rigid object is exact [21, proof of Proposition 4.2.1], and filtered colimits are exact in any Grothendieck category, it follows that the product $\otimes$ on any tensor category is exact in each factor.

The following lemma allows us to do homological algebra in any tensor category $\mathscr{A}$, in the expected ways.

Lemma 2.1. *Any tensor category $\mathscr{A}$ has enough injectives. Furthermore, for any injective I and arbitrary V in $\mathscr{A}$, the products $I \otimes V$ and $V \otimes I$ are also injective.*

Proof. Any Grothendieck abelian category has enough injectives [36, Theorem 9.6.2], so that the first statement follows. Now, for rigid V and arbitrary injective I, we have natural identifications

$$\mathrm{Hom}_{\mathscr{A}}(-, V \otimes I) = \mathrm{Hom}_{\mathscr{A}}(V^* \otimes -, I) \quad \text{and} \quad \mathrm{Hom}_{\mathscr{A}}(-, I \otimes V) = \mathrm{Hom}_{\mathscr{A}}(- \otimes {}^*V, I).$$

Via exactness of the product functor $\otimes$, it follows that $V \otimes I$ and $I \otimes V$ are injective. Finally, since $\mathscr{A}$ is generated by compact objects, a version of Baer's criterion implies that filtered colimits of injectives in $\mathscr{A}$ remain injective. This implies that $V \otimes I$ and $I \otimes V$ are injective at arbitrary V. $\square$

A (right) module category over a tensor category $\mathscr{A}$ is an abelian, compactly generated category $\mathscr{M}$ which is equipped with an associative action of $\mathscr{A}$ on the right of $\mathscr{M}$, in the precise sense of [21, Definition 7.1.1]. We require that the subcategory of compact objects $\mathscr{M}^c$ is an essentially small, locally finite, abelian subcategory in $\mathscr{M}$ and that the action map $\otimes : \mathscr{M} \times \mathscr{A} \to \mathscr{M}$ commutes with arbitrary colimits in each factor.

Lemma 2.2. *Let $\mathscr{M}$ be a module category over a tensor category $\mathscr{A}$. Then $\mathscr{M}$ has enough injectives and, for an arbitrary object V in $\mathscr{A}$ and injective I in $\mathscr{M}$, the product $I \otimes V$ is injective in $\mathscr{M}$.*

Proof. Similar to the proof of Lemma 2.1. $\square$

In practice, the action map $\otimes : \mathscr{M} \times \mathscr{A} \to \mathscr{M}$ will be exact in both factors.

Lemma 2.3. *For any module category $\mathscr{M}$ over a tensor category $\mathscr{A}$, the action map $\otimes : \mathscr{M} \times \mathscr{A} \to \mathscr{A}$ is exact in the $\mathscr{A}$ variable. Furthermore, the action map $\otimes$ is exact in the $\mathscr{M}$ variable if and only if the product $M \otimes V$ of any nonzero objects M in $\mathscr{M}$ and V in $\mathscr{A}$ remains nonzero in $\mathscr{M}$.*

We omit the proof, since the result is not used in this text.

Call a module category $\mathscr{M}$ *pointed* if $\mathscr{M}$ comes equipped with a distinguished compact simple object $\mathbf{1}_{\mathscr{M}}$. Equivalently, a pointed module category is a module category $\mathscr{M}$ which comes equipped with a choice of a right $\mathscr{A}$-module map

$$u_{\mathscr{M}} : \mathscr{A} \to \mathscr{M}$$

which sends $\mathbf{1}_{\mathscr{A}}$ to a compact simple object in $\mathscr{M}$.

Remark 2.4. By restricting to the compacts we recover the standard notions of tensor categories and module categories, as considered in [21]. We only work in the cocomplete setting because it is more convenient.

Remark 2.5. The faithfullness condition of Lemma 2.3 holds for free whenever $\mathscr{A}$ admits a compact injective object. We do not know if it holds in general, cf. [21, Definition 7.3.1].

2.3. **Tannakian reconstruction.** Suppose that $\mathscr{A}$ is a tensor category which comes equipped with a tensor functor to vector spaces, $F : \mathscr{A} \to \mathrm{Vect}$. From such a pair $(\mathscr{A}, F)$ one constructs a uniquely associated Hopf algebra A and an equivalence of tensor categories $\mathrm{Corep}(A) \xrightarrow{\sim} \mathscr{A}$ which fits into a diagram

[51, Theorems 2.2.8, 2.4.2]. Indeed, this algebra is given by the so-called coendomorphisms

$$A = \mathrm{Coend}(F) = \mathrm{Hom}_{\mathrm{Cont}/k}(\mathrm{End}_k(F), k).$$

We refer the reader to [51] for a detailed presentation of this topic.

2.4. **Coideal subalgebras and module coalgebras.** Let A be a Hopf algebra. A right coideal subalgebra B in A is a subalgebra for which $\Delta(B) \subseteq B \otimes A$. A quotient module coalgebra $A \to C$ is a coalgebra quotient whose kernel is a right ideal in A. In this way C inherits a right A-action for which $\Delta(c \cdot a) = c_1 \cdot a_1 \otimes c_2 \cdot a_2$. These structures, coideal subalgebras and quotient module coalgebras, are dual to each other when A is finite-dimensional.

At a categorical level, representations for a coideal subalgebra $\mathrm{Rep}(B)$ form a right $\mathrm{Rep}(A)$-module category, and corepresentations for a quotient module coalgebra $\mathrm{Corep}(C)$ form a right $\mathrm{Corep}(A)$-module category. Furthermore, these module categories are pointed via the trivial representation for B, and the trivial corepresentation for C, respectively.

2.5. **Normality for pointed module categories.** Let $\mathscr{M} = (\mathscr{M}, \mathbf{1})$ be a pointed module category over a tensor category $\mathscr{A}$. For the sake of specificity, let us say that $\mathscr{A}$ acts on the right of $\mathscr{M}$.

Call an object M in $\mathscr{M}$ trivial if M admits a surjection $\mathbf{1}^{\oplus I} \to M$ from some additive power of the unit.

Lemma 2.6.   (1) *Any trivial object is isomorphic to a (possibly infinite) additive power of $\mathbf{1}$.*
    (2) *Any quotient of a trivial object in $\mathscr{M}$ is trivial.*

(3) *Any subobject of a trivial object is trivial.*

(4) *Any object M in $\mathscr{M}$ has a unique maximal trivial subobject* $\mathrm{triv}(M) \subseteq M$.

Proof. (1)–(3) All follow from the fact that any quotient of a semisimple object in an abelian category is semisimple, as is any subobject. See for example [37, Remark 2.2, Theorem 2.4]. (4) Take $\mathrm{triv}(M) = \mathrm{im}\left(\oplus_{f \in \mathrm{Hom}_{\mathscr{M}}(\mathbf{1}, M)} \mathbf{1} \to M \right)$. $\qquad\square$

We consider the structure map $u_{\mathscr{M}} : \mathscr{A} \to \mathscr{M}$, and define the kernel of $u_{\mathscr{M}}$ to be

$$\ker(u_{\mathscr{M}}) = \left\{ \begin{array}{c} \text{The full subcategory of } W \text{ in } \mathscr{A} \\ \text{for which } u_{\mathscr{M}}(W) \text{ is trivial in } \mathscr{M} \end{array} \right\}.$$

At this point we have standard notions of normality for tensor functors [9, Definition 3.4] (cf. [20]), which we now extend to the setting of pointed module categories.

Definition 2.7. For a pointed module category $\mathscr{M}$ over $\mathscr{A}$, we say the structure map $u_{\mathscr{M}} : \mathscr{A} \to \mathscr{M}$ is normal if the following properties hold:

(a) $u_{\mathscr{M}}$ is exact.

(b) The kernel of $u_{\mathscr{M}}$ is a tensor subcategory in $\mathscr{A}$.

(c) For any object V in $\mathscr{A}$ there exists a subobject $\mathrm{triv}_{\mathscr{M}}(V) \subseteq V$ for which $u(\mathrm{triv}_{\mathscr{M}}(V)) = \mathrm{triv}(u_{\mathscr{M}}(V))$.

We recall that exactness of the functor $u_{\mathscr{M}} = \mathbf{1}_{\mathscr{M}} \otimes -$ holds for free in most practical situations. See Lemma 2.3. Such exactness implies uniqueness of the subobject $\mathrm{triv}_{\mathscr{M}}(V)$ in V, and functoriality of the construction $V \mapsto \mathrm{triv}_{\mathscr{M}}(V)$.

3. Quantum group basics

Recall that k is an algebraically closed field of characteristic 0.

3.1. **Data for semisimple algebraic groups.** Fix a choice of a semisimple algebraic group G over k, along with a choice of maximal torus $T \subseteq G$. For such a pairing of G and T we have the associated root lattice Q, weight lattice P, and character lattice X. These lattices fit into a sequence of inclusions $Q \subseteq X \subseteq P$. We let Φ denote the collection of roots for G and fix a choice of simple roots Δ in Φ. This choice of Δ determines a collection Φ^+ of positive roots in Φ.

We let $(-, -) : X \times X \to \mathbb{Q}$ denote the unique scaling of the Killing form so that $(\gamma, \gamma) = 2$ at any short root γ. We have

$$(\gamma, \gamma) = d_\gamma \cdot 2$$

at a general root γ, where $d_\gamma \in \{1, 2, 3\}$. Let $\mathscr{W} = \langle \sigma_\gamma : \gamma \in \Phi \rangle$ denote the Weyl group for G.

Throughout this work G is always a semisimple algebraic group with specified data as above, and $\mathfrak{g} = \mathrm{Lie}(G)$. We call G *almost-simple* if its Dynkin diagram is connected, and the lacing number for an almost-simple algebraic group is defined as

$$\text{lacing number for G} := \max\left\{ -\frac{2(\alpha, \beta)}{(\alpha, \alpha)} : \alpha, \beta \in \Delta, \ \alpha \neq \beta \right\} = \max\{d_\gamma : \gamma \in \Phi\}.$$

This number is 1, 2, and 3, in types $\{A, D, E\}$, $\{B, C, F\}$, and G_2 respectively.

3.2. **Data for quantum groups.** Consider the collection of connected compo-
nents $\pi_0(\Delta)$ in the Dynkin diagram for G, and for each $i \in \pi_0(\Delta)$ let Δ_i denote
the associated component in Δ. Each component Δ_i determines an almost-simple
subgroup G_i in G, and the decomposition $\Delta = \Delta_1 \amalg \cdots \amalg \Delta_t$ corresponds to the
decomposition of G into almost-simple factors

$$G = G_1 \times \cdots \times G_t.$$

The category of quantum group representations is specified by the data of a
semisimple algebraic group G, with fixed maximal torus and simple roots Δ, and
a choice of symmetric bicharacter on the weight lattice

$$q : P \times P \to k^\times$$

which exponentiates the Killing form. Specifically, we require that the form q
is invariant under the action of the Weyl group and that $q(\lambda, \mu) = 1$ whenever
$(\lambda, \mu) = 0$. We call q *torsion* if q takes values in the roots of unity $\mathrm{tors}(k^\times)$.

Such a form q determines unique scalars q_i over each connected component $i \in$
$\pi_0(\Delta)$ under which

$$q(\alpha, \lambda) = q_i^{(\alpha, \lambda)} \text{ whenever } \alpha \in \Delta_i, \ \lambda \in P.$$

Indeed, if we let $\omega_\alpha \in P$ denote the fundamental weight associated to a given simple
root $\alpha \in \Delta$, and define

$$q_\alpha = q(\alpha, \omega_\alpha) \text{ for simple } \alpha,$$

we may take $q_i = q_\alpha$ at any short root α in Δ_i.

We refer to such a bilinear form q as a quantum parameter *for G. We refer to
the units q_i over the various components $i \in \pi_0(\Delta)$ as the* scalar parameters *for G
at q.* When we speak of a quantum group at a n-th "roots of unity", we mean a
pairing of G and q for which the associated scalar parameters q_i are all primitive
n-th roots of 1.

The values q_α determine a unique function $q_* : \Phi \to k^\times$ which is invariant under
the action of the Weyl group and which takes the prescribed values q_α on simple
roots. Explicitly, $q_\gamma = q_i^{d_\gamma}$ for each γ in the subsystem $\Phi_i = \Phi \cap \mathbb{Z}\Delta_i$.

Supposing the form q is torsion, we define

$$l_\gamma = \mathrm{ord}\big(q(\gamma, \gamma)\big) = \mathrm{ord}(q^2(\gamma, -)) \text{ for each } \gamma \in \Phi, \tag{3}$$

and for a given component $i \in \pi_0(\Delta)$ we take

$$l_i = l_\alpha \text{ where } \alpha \text{ is any short root in } \Delta_i.$$

Throughout this text, all quantum parameters q are assumed to be torsion!

3.3. **The quantum enveloping algebra.** Suppose first that G is almost-simple,
i.e. that the Dynkin diagram for G is connected. We consider Lusztig's integral
divided power algebra $U_v = U_v(\mathfrak{g})$ over $\mathbb{Z}[v, v^{-1}]$ [43, §1.3, Theorem 6.7], and define
U_q as the base change of U_v along the algebra map $\mathbb{Z}[v, v^{-1}] \to k$, $v \mapsto q_i$, where i
labels the unique component in the Dynkin diagram $\Delta_i = \Delta$. We might also write
$U_q = U_{q_i}$ in this case.

When G has multiple almost-simple factors $G = G_1 \times \cdots \times G_t$ we take

$$U_q = U_{q_1}(\mathfrak{g}_1) \otimes_k \ldots \otimes_k U_{q_t}(\mathfrak{g}_t).$$

As one expects, $\mathfrak{g}_i = \mathrm{Lie}(G_i)$ in the above expression and the parameters q_i are as in Section 3.2.

We recall that U_q has generators

$$E_\alpha^{(n)}, \ F_\alpha^{(n)}, \ K_\alpha, \text{ for simple } \alpha \in \Delta \text{ and } n \in \mathbb{Z}_{\geq 0},$$

and we have the distinguished toral elements

$$\left[\begin{array}{c} K_\alpha; 0 \\ l_\alpha \end{array} \right] = \prod_{i=1}^{l_\alpha} \left. \frac{K_\alpha v^{-i+1} - K_\alpha^{-1} v^{i-1}}{v^i - v^{-i}} \right|_{v = q_\alpha}$$

which also appear with some frequency. These generators satisfy the usual quantum Serre relations and we have a Hopf structure on U_q which satisfies

$$\Delta(E_\alpha) = E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha, \ \ \Delta(F_\alpha) = F_\alpha \otimes K_\alpha^{-1} + 1 \otimes F_\alpha, \ \ \Delta(K_\alpha) = K_\alpha \otimes K_\alpha$$

[43, §1.1]. We also consider the positive and negative subalgebras U_q^+ and U_q^- in U_q which are generated by the divided powers of the simple root vectors $E_\alpha^{(n)}$ and $F_\alpha^{(n)}$ respectively.

3.4. **Vector space bases.** We enumerate the simple roots $\Delta = \{\alpha_1, \ldots, \alpha_{\mathrm{rk}(G)}\}$ for G and, for a reduced expression of the longest element in the Weyl group $w_0 = s_r \ldots s_1$, we obtain an enumeration of the positive roots as

$$\Phi^+ = \{\gamma_1, \ldots, \gamma_r\}, \ \ \gamma_i = s_t \ldots s_{i+1}(\alpha_i).$$

As shown in [42, 43, 44], the simple reflections $s_\alpha \in \mathscr{W}$ lift to algebra automorphisms $T_\alpha : U_q \to U_q$ which satisfy the relations of the type $\mathfrak{g}$ braid group $B_\mathfrak{g}$ [44, §37.1, Ch 39]. Via applications of these braid group operators we obtain associated root vectors

$$E_{\gamma_i} = T_t \ldots T_{i+1}(E_i) \ \in \ U_q^+$$

for each $\gamma \in \Phi^+$. We obtain corresponding negative root vectors $F_\gamma = \phi(E_\gamma)$ in U_q^- via an application of the anti-algebra isomorphism $\phi : U_{q^{-1}} \to U_q$ which exchanges E_α and F_α, and inverts K_α. We then have Lusztig's bases for $U_q^\pm$ via ordered monomials in the roots vectors

$$\{E_{\gamma_1}^{(m_1)} \ldots E_{\gamma_r}^{(m_r)} : m : \Phi^+ \to \mathbb{Z}_{\geq 0}\} \ \text{ and } \ \{F_{\gamma_1}^{(m_1)} \ldots F_{\gamma_r}^{(m_r)} : m : \Phi^+ \to \mathbb{Z}_{\geq 0}\}$$

[44, Proposition 41.1.3] [43, Proposition 6.7].

Lemma 3.1. *Consider a semisimple algebraic group G at a torsion parameter q, and a positive root $\gamma \in \Phi^+$. We have $l_\gamma = 1$ if and only if E_γ and F_γ are nonnilpotent, and otherwise*

$$l_\gamma = \mathrm{nil.deg}(E_\gamma) = \mathrm{nil.deg}(F_\gamma).$$

Here by $\mathrm{nil.deg}(x)$ we mean the smallest postive integer m at which $x^m = 0$.

Proof. Follows from the expressions

$$E_\gamma^m = [m]_{q_\gamma}! \cdot E_\gamma^{(m)}, \ \ F_\gamma^m = [m]_{q_\gamma}! \cdot F_\gamma^{(m)},$$

and the fact that the q_γ-factorial $[m]_{q_\gamma}!$ [44, §1.3.3] vanishes if and only if $m \geq l_\gamma > 1$. $\square$

Remark 3.2. In this text we employ the specific braid group operators $T_\alpha := T''_{\alpha,1}$ from [44, §37.1].

3.5. **Quantum group representations.** We let $\mathrm{Rep}(G_q)$ denote the tensor category of character graded representations for the quantum enveloping algebra U_q, in the precise sense of [47, § 2.4] (cf. [4, 5]). So, a given G_q-representation V is graded by the character lattice X for G, the divided powers $E_\alpha^{(m)}$ and $F_\alpha^{(m)}$ act on V by homogeneous endomorphisms of degrees $m \cdot \alpha$ and $-m \cdot \alpha$ respectively, and the toral elements in U_q act as the semisimple operators

$$K_\alpha \cdot v = q(\alpha, \lambda)\, v \ \text{ and } \ \left[\begin{array}{c} K_\alpha; 0 \\ l_\alpha \end{array} \right] \cdot v = \left[\begin{array}{c} \langle \alpha, \lambda \rangle \\ l_\alpha \end{array} \right]_{q_\alpha} v, \ \text{ for } v \in V_\lambda.$$

The category $\mathrm{Rep}(G_q)$ is, equivalently, the category of integrable, unital representations for Lusztig's modified algebra $\mathrm{Rep}(G_q) = \mathrm{Rep}(\dot{U}_q)$ [44, § 31.2].

Remark 3.3. Up to isomorphism, the algebra U_q only depends on the restriction of the form q to $Q \times Q$, while the definition of the category $\mathrm{Rep}(G_q)$ depends on the restriction of q to $Q \times X$.

The restricted form $q|_{X \times X}$ provides solutions to certain equations which one needs to define the R-matrix (braiding) for $\mathrm{Rep}(G_q)$. The existence of an extension of this form to the entire weight lattice allows us to extract the scalar parameters q_i from q, and also provides a simply-connected form G_q^{sc} for any G_q.

Remark 3.4. Our use of bilinear forms, rather than roots of unity, loosely follows the presentation of [30].

3.6. **Quantum function algebras.** The quantum function algebra is, by definition, the Hopf algebra one reconstructs from the category $\mathrm{Rep}(G_q)$ and its forgetful functor to vector spaces,

$$\mathscr{O}(G_q) := \mathrm{Coend}(forget : \mathrm{Rep}(G_q) \to \mathrm{Vect}).$$

Again we refer the reader to [51] for a thorough discussion of Tannakian reconstruction.

3.7. **Dominant weights and simple representations.** The following result is proved in [41, Proposition 6.4] at odd order q, and one can use the formula [44, Lemma 32.1.2] to extend the arguments of [41] to general q. See alternatively [44, Propositions 31.2.7, 31.3.2].

Proposition 3.5 ([41, Proposition 6.4]). *For each dominant weight $\lambda \in X^+$ there is a unique simple object $L(\lambda)$ in $\mathrm{Rep}(G_q)$ which is of highest weight λ. Furthermore, the assignment*

$$X^+ \to \mathrm{Irrep}(G_q), \ \ \lambda \mapsto L(\lambda),$$

is a bijection.

3.8. **Collected notations.** Fix a semisimple algebraic group G with a choice of maximal torus and simple roots. We gather all of the notations from this section.

- Φ denotes the roots for G, Δ denotes the chosen base in Φ, and Φ^+ denotes the corresponding subset of positive roots in Φ. We identify Δ with the Dynkin diagram for G, via an abuse of notation.
- Q, P, and X denote the root, weight, and character lattices for G.
- $(-, -)$ denotes the normalized Killing form on P, so that $(\alpha, \alpha) = 2$ at any short root α. At a general root $\gamma \in \Phi$ we let d_γ denote the relative length $d_\gamma = (\gamma, \gamma)/2$.

- For any $i \in \pi_0(\Delta)$, $\Delta_i \subset \Delta$ denotes the corresponding connected component in the Dynkin diagram. We let $G_i \subseteq G$ denote the associated almost-simple factor in G.
- q denotes a quantum parameter for G. In particular, q is a symmetric form on the weight lattice which exponentiates the Killing form. We always assume q is torsion, i.e. takes values in the roots of unity $\mathrm{tors}(k^\times)$.
- For any simple root α, with corresponding fundamental weight ω_α, we take $q_\alpha = q(\alpha, \omega_\alpha)$. For a general positive root γ we take $q_\gamma = q_\alpha$, where α is any simple root which is in the Weyl group orbit of γ.
- For any $i \in \pi_0(\Delta)$, we take $q_i = q_\alpha$ where α is any short root in Δ_i. We note that q_i does not independent of the choice of α. These q_i are the *scalar parameters* associated to G at q.
- For any $\gamma \in \Phi^+$ we take

$$l_\gamma = \mathrm{ord}(q(\gamma, \gamma)) = \mathrm{ord}(q^2(\gamma, -)) = \begin{cases} \mathrm{nil.deg}(E_\gamma) & \text{if } E_\gamma \text{ is nilpotent} \\ 1 & \text{otherwise.} \end{cases}$$

Let us also list the quantum algebras which we'll encounter in the text, though one might ignore the interruption upon first reading.

- U_q denotes Lusztig's divided power quantum enveloping algebra [42, 43].
- $\dot{U}_q$ denotes Lusztig's modified enveloping algebra from [44, Ch 31].
- $\mathfrak{v}_q$ denotes the distinguished finite-dimensional Hopf subalgebra in U_q from [43, §8.2], and $\dot{\mathfrak{u}}_q$ denotes the modified analog of $\mathfrak{v}_q$, as in [44, §36.2.1]. See also Sections 5.1, 6.1, 9.1.
- $\bar{u}_q$ denotes the "smallest quantum algebra" from Sections 5.2, 6.3. There is also an intermediate Hopf algebra v_q which appears in these sections, and which serves as a link between $\mathfrak{v}_q$ and $\bar{u}_q$.

As defined above, $\mathrm{Rep}(G_q)$ is the category of character graded representations for the divided power algebra U_q.

4. Quantum Frobenius

We recall Lusztig's construction of the quantum Frobenius functor from [44]. This functor specifies a dual group G^* to G, a dual parameter ε to q, and an exact monoidal embedding $\mathrm{Rep}(G^*_\varepsilon) \to \mathrm{Rep}(G_q)$. The associated scalar parameters ε_i for ε take values ± 1, so that G^*_ε has representation theory which is approximately equivalent to the classical representation theory for G^*.

The quantum Frobenius functor plays a fundamental role in the analyses of quantum group representations that follow.

4.1. **Quantum groups at quasi-classical parameters.**

Proposition 4.1 ([44, Proposition 33.2.3]). *Let H be any semisimple algebraic group and ε be a quantum parameter with all $\varepsilon_i = \pm 1$. There is a linear equivalence of categories $\mathrm{Rep}(H_\varepsilon) \xrightarrow{\sim} \mathrm{Rep}(H)$. In particular, $\mathrm{Rep}(H_\varepsilon)$ is semisimple.*

Note that we have not claimed that $\mathrm{Rep}(H_\varepsilon)$ and $\mathrm{Rep}(H)$ are equivalent as tensor categories, at general ε.

4.2. **"Dual" lattices.** We take

$$X^* := \{\lambda \in X : q(\alpha, \lambda)^2 = 1 \text{ for all simple } \alpha \text{ with } l_\alpha > 1\}$$

$$= \big\{\lambda \in X : (\alpha, \lambda) \in l_i\mathbb{Z} \text{ for each } \alpha \in \Delta_i \text{ and } i \in \pi_0(\Delta)\big\}.$$

For the equality of these two expressions, note that $q(\alpha, \lambda)^2 = q_i^{2(\alpha,\lambda)}$ whenever $\alpha \in \Delta_i$, so that $q(\alpha, \lambda)^2 = 1$ if and only if $(\alpha, \lambda) \in l_i\mathbb{Z}$. For simple roots with $l_\alpha = 1$ such vanishing of $q(\alpha, \lambda)^2$ holds for free, at all λ, by the definition of l_α.

We also consider the "l-dualized" root lattice

$$lQ := \mathbb{Z} \cdot \{l_\alpha\alpha : \alpha \text{ simple}\}.$$

The following is essentially stated in [44].

Lemma 4.2. *lQ is contained in X^*, and both of the sublattices lQ and X^* are stable under the action of the Weyl group on P.*

Proof. For each pair of simple roots α and β we have

$$q(\alpha, l_\beta\beta)^2 = q(\alpha, \beta)^{2l_\beta} = 1,$$

by the definition of l_β. This establishes the containment $lQ \subseteq X^*$.

As for stability under the action of the Weyl group, for the simple reflection σ_α associated to $\alpha \in \Delta_i$, and $\lambda \in X^*$, we have

$$\sigma_\alpha(\lambda) = \lambda - \frac{(\alpha, \lambda)}{d_\alpha}\alpha. \tag{4}$$

Each pairing (α, λ) lies in $d_\alpha\mathbb{Z}$ and, by the definition of X^*, in $l_i\mathbb{Z}$ as well.

When $d_\alpha \mid l_i$ we have $l_\alpha = l_i/d_\alpha$ so that $(\alpha, \lambda)/d_\alpha$ is seen to lie in $l_\alpha\mathbb{Z}$. When $d_\alpha \nmid l_i$ we have $l_\alpha = l_i$ and $(\alpha, \lambda) \in \text{lcm}(d_\alpha, l_i)\mathbb{Z} = d_\alpha l_i\mathbb{Z} = d_\alpha l_\alpha\mathbb{Z}$, giving again $(\alpha, \lambda)/d_\alpha \in l_\alpha\mathbb{Z}$. So from the expression (4) we find

$$\sigma_\alpha(\lambda) \in \lambda + l_\alpha\mathbb{Z} \cdot \alpha \subseteq \lambda + lQ. \tag{5}$$

This expression lies in lQ when λ lies in lQ, and lies in X^* when λ lies in X^*. So we observe stability under the action of the Weyl group. $\square$

4.3. **Quasi-classical representations in** $\text{Rep}(G_q)$**.**

Proposition 4.3 ([44, Proposition 35.3.2])**.** *A simple representation $L(\lambda)$ in $\text{Rep}(G_q)$ is annihilated by all simple root vectors E_α and F_α with $l_\alpha > 1$ if and only if $\lambda \in X^*$. Furthermore, in this case $L(\lambda)$ is graded by the sublattice X^* in X.*

Proof. If $\lambda \in X^*$ then $L(\lambda)$ has grading in X^*, and is annihilated by all such E_α and F_α, directly by [44, Proposition 35.3.1 (a) and (b)]. If $\lambda \notin X^*$ consider a simple root α with $l_\alpha > 1$ and $q(\alpha, \lambda)^2 \neq 1$, and $v \in L(\lambda)$ of highest weight to find

$$[E_\alpha, F_\alpha] \cdot v = \frac{q(\alpha, \lambda) - q(\alpha, \lambda)^{-1}}{q_\alpha - q_\alpha^{-1}} \cdot v = q(\alpha, \lambda)^{-1}\left(\frac{q(\alpha, \lambda)^2 - 1}{q_\alpha - q_\alpha^{-1}}\right) \cdot v \neq 0.$$

This equation implies $F_\alpha \cdot v \neq 0$ and $E_\alpha(F_\alpha \cdot v) \neq 0$. Hence $L(\lambda)$ is not annihilated by the prescribed root vectors. $\square$

Lemma 4.4. *For a G_q-representation V, the following are equivalent:*

(a) *V is annihilated by all root vectors E_α and F_α, with α simple and $l_\alpha > 1$.*
(b) *V is annihilated by all root vectors E_γ and F_γ, with $\gamma \in \Phi^+$ and $l_\gamma > 1$.*

Proof. The implication (b) $\Rightarrow$ (a) is trivial. So we consider (a) $\Rightarrow$ (b). Let I be the ideal in U_q generated by the simple root vectors E_α and F_α with $l_\alpha > 1$. It suffices to show that I is stable under the actions of the simple braid group operators T_β. Indeed for any positive root γ with $l_\gamma > 1$ we can find a corresponding simple root α, with $l_\alpha > 1$, and a braid group operator T for which $T(E_\alpha) = E_\gamma$ and $T(F_\alpha) = F_\gamma$. However, such stability is clear from the explicit expressions of the T_β given in [44, §37.1.3]. $\qquad\square$

We call a representation V in $\mathrm{Rep}(G_q)$ *quasi-classical* if V is annihilated by all simple E_α and F_α with $l_\alpha > 1$. Note that this class of representations is closed under taking subobjects and quotients. We let $\mathscr{E}_q \subseteq \mathrm{Rep}(G_q)$ denote the full subcategory of quasi-classical representations.

Lemma 4.5. *The subcategory $\mathscr{E}_q \subseteq \mathrm{Rep}\, G_q$ is a tensor subcategory in $\mathrm{Rep}\, G_q$, and all objects in $\mathscr{E}_q$ are graded by the sublattice X^*.*

Proof. The fact that $\mathscr{E}_q$ is a tensor subcategory in $\mathrm{Rep}\, G_q$ follows from the fact that all simple root vectors E_α and F_α are skew primitive. As for the claim about the grading, if a representation V is annihilated by all E_α and F_α with $l_\alpha > 1$ then all of its composition factors are of the form $L(\lambda)$ with $\lambda \in X^*$, by Proposition 4.3. It follows that V is graded by X^*, since all of its composition factors are graded by X^*. $\qquad\square$

Remark 4.6. The category $\mathscr{E}_q$ is also closed under the formation of extensions in $\mathrm{Rep}(G_q)$, provided all $l_i > 2$, or provided G has no factors of types A_1 or B_n at $l_i = 2$. Indeed, in this case $\mathscr{E}_q$ is precisely the subcategory of X^*-graded objects in $\mathrm{Rep}(G_q)$.

4.4. **The dual group G^*.** In [42, 43, 44] Lusztig considers a dual group G^* to G which is specified by the following data: The character lattice for G^* is X^*, and the simple roots are

$$\Delta^* := \{\alpha^* : \alpha \in \Delta\} \subseteq lQ$$

where $\alpha^* := l_\alpha \alpha$. The Cartan integers are given by

$$\langle \alpha^*, \beta^* \rangle_l := \frac{2l_\beta}{l_\alpha} \frac{(\alpha, \beta)}{(\alpha, \alpha)}$$

[44, § 2.2.5]. The group G^* decomposes into almost-simple factors which correspond bijectively to the almost-simple factors for G,

$$G = G_1 \times \cdots \times G_t \ \Rightarrow \ G^* = G_1^* \times \cdots \times G_t^*.$$

One sees from the definition of the Cartan integers that the Killing form on the root lattice lQ for G^* is some rescaling of the form $(-,-)|_{lQ \times lQ}$. We have $(\alpha^*, \alpha^*) = l_\alpha^2(\alpha, \alpha)$ and, for

$$l_i' = \min\{l_\beta : \beta \in \Delta_i\},$$

we rescale by $1/l_i l_i'$ on each simple factor $\mathfrak{g}_i^*$ to obtain the properly normalized form $(-,-)_l$. At each simple root $\alpha \in \Delta_i$ we now have

$$\frac{1}{2}(\alpha^*, \alpha^*)_l = \frac{l_\alpha^2}{l_i l_i'} d_\alpha = \begin{cases} d_\alpha & \text{if } l_i/l_i' = 1 \\ 2 & \text{if } \alpha \text{ short and } l_i/l_i' = 2 \\ 3 & \text{if } \alpha \text{ short and } l_i/l_i' = 3 \\ 1 & \text{if } \alpha \text{ long and } l_i/l_i' = 2 \text{ or } 3 \end{cases}.$$

We therefore observe the fundamental weights

$$\omega_{\alpha^*} = l_\alpha \cdot \omega_\alpha$$

and weight lattice $P^* = \mathbb{Z} \cdot \{l_\alpha \omega_\alpha : \alpha \text{ simple}\}$ for G^*. From the above expressions we find the following.

Lemma 4.7. *When the lacing number for G_i is coprime to l_i, the dual group G_i^* is of the same Dynkin type as G_i. When the lacing number for G_i divides l_i, the dual group G_i^* is of Langlands dual Dynkin type to G_i.*

4.5. **Lusztig's quantum Frobenius.** Take G^* and P^* as above. Consider the form ε on P^* defined simply by restricting q,

$$\varepsilon(\lambda, \mu) := q(\lambda, \mu) \ \text{ for } \ \lambda, \mu \in P^*.$$

Since the Killing form on P^* is just a rescaling of the Killing form on the ambient lattice P, the form ε exponentiates the Killing form for G^* and so provides a quantum parameter for the dual group. We have

$$\varepsilon(\alpha^*, \alpha^*) = q(\alpha, \omega_\alpha)^{2l_\alpha^2} = 1$$

at each simple root, which forces $\varepsilon_i = \pm 1$ on each almost-simple factor G_i^*.

Remark 4.8. Scalar parameters of non-trivial value $\varepsilon_i = -1$ can occur, for example, when the corresponding parameter q_i for G_i is a $2p$-th root of 1 with p odd. This can also occurs in types B_n, C_n, and F_4 when q_i is a $4p$-th root of 1 with p odd, and in type G_2 when q_i is a $6p$-th root of 1 with p odd.

We now let $\dot{U}_\varepsilon^*$ denote the modified quantum enveloping algebra for G^* at ε [44, §31.2], and let

$$e_\alpha^{(m)} 1_\mu, \ f_\alpha^{(m)} 1_\mu \ \in \dot{U}_\varepsilon^*$$

denote the standard generators, for simple α and $\mu \in X^*$. By [44, Theorem 35.1.9, § 35.5.2] (see also [39]) there is a surjective map of non-unital algebras

$$fr : \dot{U}_q \to \dot{U}_\varepsilon^*$$

which maps the powers of the simple root vectors as

$$E_\alpha^{(n)} 1_\lambda \mapsto \begin{cases} e_\alpha^{(n/l_\alpha)} 1_\lambda & \text{if } l_\alpha | n \text{ and } \lambda \in X^* \\ 0 & \text{otherwise} \end{cases}$$

and

$$F_\alpha^{(n)} 1_\lambda \mapsto \begin{cases} f_\alpha^{(n/l_\alpha)} 1_\lambda & \text{if } l_\alpha | n \text{ and } \lambda \in X^* \\ 0 & \text{otherwise.} \end{cases}$$

The map fr is compatible with the Hopf structures on $\dot{U}_q$ and $\dot{U}_\varepsilon^*$, and restricting along fr provides an embedding of tensor categories

$$\mathrm{Fr} := \mathrm{res}_{fr} : \mathrm{Rep}(G_\varepsilon^*) \to \mathrm{Rep}(G_q). \tag{6}$$

The tensor compatibility $\mathrm{Fr}(V) \otimes \mathrm{Fr}(W) \to \mathrm{Fr}(V \otimes W)$ is given by the identity.

4.6. Quasi-classical representations via quantum Frobenius.

Theorem 4.9 ([44]). *The quantum Frobenius functor* $\mathrm{Fr} : \mathrm{Rep}(G_\varepsilon^*) \to \mathrm{Rep}(G_q)$ *restricts to an equivalence onto the subcategory* $\mathscr{E}_q$ *of quasi-classical representations in* $\mathrm{Rep}(G_q)$.

The proof is essentially the same as that of [44, Corollary 35.3.3]. We repeat it for the sake of completeness.

Proof. We first note that the image of Fr is contained in $\mathscr{E}_q$, directly by the definitions. Now, it is shown in [44, § 35.2.3] that the generators $E_\alpha^{(l_\alpha)}$ in U_q^+ satisfy the ε-Serre relations of the e_α in U_ε^+. The $F_\alpha^{(l_\alpha)}$ similarly satisfy the relations of the f_α. So any V in $\mathscr{E}_q$ comes equipped with actions of U_ε^+ and U_ε^- and is appropriately graded by X^*. The commutativity relations [43, 6.5 (a2)], in conjunction with the fact that V is annihilated by all E_α with $l_\alpha > 1$, implies that

$$[E_\alpha^{(l_\alpha)}, F_\beta^{(l_\beta)}] \cdot v = \delta_{\alpha\beta} \left[\begin{matrix} K_\alpha; 0 \\ l_\alpha \end{matrix} \right] \cdot v = \delta_{\alpha,\beta} \left[\begin{matrix} \langle \alpha, \lambda \rangle \\ l_\alpha \end{matrix} \right]_{q_\alpha} \cdot v$$

$$= \delta_{\alpha\beta} \left(q_\alpha^{(\frac{\langle \alpha, \lambda \rangle}{l_\alpha} + 1) l_\alpha^2} \frac{\langle \alpha, \lambda \rangle}{l_\alpha} \cdot v \right) = \delta_{\alpha\beta} \left(q_\alpha^{l_\alpha^2 (\langle \alpha^*, \lambda \rangle_l + 1)} \langle \alpha^*, \lambda \rangle_l \cdot v \right)$$

for arbitrary $\alpha, \beta \in \Delta$ [44, Lemma 34.1.2]. We have $q_\alpha^{l_\alpha^2} = \varepsilon_{\alpha^*}$ so that the final expression reduces to

$$[E_\alpha^{(l_\alpha)}, F_\beta^{(l_\beta)}] \cdot v = \delta_{\alpha\beta} \left(\varepsilon_{\alpha^*}^{(\langle \alpha^*, \lambda \rangle_l + 1)} \langle \alpha^*, \lambda \rangle_l \cdot v \right) = \delta_{\alpha,\beta} \left[\begin{matrix} \langle \alpha^*, \lambda \rangle_l \\ 1 \end{matrix} \right]_{\varepsilon_{\alpha^*}} \cdot v.$$

This information tells us that the $E_\alpha^{(l_\alpha)}$ and $F_\alpha^{(l_\alpha)}$ actions on V satisfy the necessarily relations to provide a $\dot{U}_\varepsilon^*$-action on V. This defines a functor

$$-|_{U_\varepsilon^*} : \mathscr{E}_q \to \mathrm{Rep}(G_\varepsilon^*).$$

Now, for V in $\mathscr{E}_q$, the identity map provides a natural isomorphism of G_q-representations $\mathrm{Fr}(V|_{\dot{U}_\varepsilon^*}) \xrightarrow{\sim} V$, and for any W in $\mathrm{Rep}(G_\varepsilon^*)$ the identity again provides a natural isomorphism of G_ε^*-representations $W \xrightarrow{\sim} \mathrm{Fr}(W)|_{\dot{U}_\varepsilon^*}$. So $\mathrm{Fr} : \mathrm{Rep}(G_\varepsilon^*) \to \mathrm{Rep}(G_q)$ restricts to an equivalence onto $\mathscr{E}_q$. $\square$

The following is an immediate corollary to Theorem 4.9 and Proposition 4.1.

Corollary 4.10. *The category* $\mathscr{E}_q$ *of quasi-classical representations in* $\mathrm{Rep}(G_q)$ *is semisimple.*

5. Small quantum algebras above the lacing number

We cover the construction of the smallest quantum algebra $\bar{u}_q = \bar{u}(G_q)$ at almost-every q. We address all pairings of G and q save for types B_n, C_n, and F_4 at a 4-th root of 1, and type G_2 at a 3-rd, 4-th, or 6-th root of 1. These low order cases are discussed independently in Section 6 below. We also explain how the category of representations $\mathrm{Rep}(\bar{u}_q)$ inherits the natural structure of a pointed module category over $\mathrm{Rep}(G_q)$.

We begin by recalling Lusztig's finite-dimensional Hopf subalgebra in U_q, which we denote by $\mathfrak{v}_q$. We view $\mathfrak{v}_q$ as associated to the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$, rather then the group G itself. We then construct an intermediate Hopf algebra v_q which admits highly structured maps from both $\mathfrak{v}_q$ and our "smallest quantum algebra"

$\bar{u}_q$. We use v_q, along with the aforementioned maps, to translate results for $\mathfrak{v}_q$ to $\bar{u}_q$;

$$
\xymatrix{ & v_q & \\ \mathfrak{v}_q \ar[ur] & & \bar{u}_q \ar[ul] \\ \mathfrak{v}_q \ar@{-->}[rr] & & \bar{u}_q. }
$$

5.1. **Lusztig's finite-dimensional subalgebra.** Consider a semisimple algebraic group G, and suppose that each l_i is greater than the lacing number for its associated almost-simple factors in G. We consider Lusztig's divided power algebra U_q, which we think of as a "copy" of the quantum enveloping algebra used in the construction of $\mathrm{Rep}(G_q)$. To distinguish between these two algebras we let K_α denote the standard grouplikes in our new copy of U_q.

We begin with Lusztig's finite-dimensional subalgebra in U_q [42, 43]. We adopt the unorthodox notation $\mathfrak{v}_q$ for this algebra, and recall the definition

$$
\mathfrak{v}_q = \mathfrak{v}_q(\mathfrak{g}) = \left\{ \begin{array}{c} \text{The subalgebra in } U_q \text{ generated} \\ \text{by the elements } E_\alpha,\ F_\alpha, \text{ and } \mathsf{K}_\alpha \\ \text{for simple roots } \alpha \in \Delta \end{array} \right\} \subseteq U_q.
$$

The algebra $\mathfrak{v}_q$ is a finite-dimensional Hopf subalgebra in U_q which has basis

$$
\mathbf{B} = \{ E_{\gamma_1}^{m_1} \dots E_{\gamma_r}^{m_r} \cdot \mathsf{K}_\nu \cdot F_{\gamma_1}^{m'_1} \dots F_{\gamma_r}^{m'_r} : \nu \in Q,\ \gamma_i \in \Phi^+, \text{ and } m_i, m'_i < l_{\gamma_i} \},
$$

under some ordering of the positive roots.

From this basis one determines the dimension of $\mathfrak{v}_q$ as

$$
\dim(\mathfrak{v}_q) = \Big(\prod_{\alpha \in \Delta} 2l_\alpha \Big) \Big(\prod_{\gamma \in \Phi^+} l_\gamma \Big)^2.
$$

The first factor accounts for the grouplikes in $\mathfrak{v}_q$ and the second factors accounts for the positive and negative subalgebras $\mathfrak{v}_q^\pm$ in $\mathfrak{v}_q$ [43, Theorem 8.3].

5.2. **Small quantum algebras.** Recall our root lattice $lQ = \mathbb{Z} \cdot \{l_\alpha \alpha : \alpha \in \Delta\}$ and character lattice X^* for the dual group G^*, from Section 4.5. We first consider the character groups

$$
A := (X/2 \cdot lQ)^\vee \quad \text{and} \quad \bar{A} := (X/X^*)^\vee,
$$

and note that the inclusion $2 \cdot lQ \subseteq X^*$ dualizes to an inclusion $\bar{A} \subseteq A$.

For any $\nu \in Q$ we have the associated character $K_\nu : X \to k^*$, $K_\nu(\lambda) = q(\lambda, \nu)$, which vanishes on $2 \cdot lQ$ and hence defines an element in A. We also have a natural action of A on $\mathfrak{v}_q$ by the Hopf algebra automorphisms

$$
\xi \cdot E_\gamma = \xi(\gamma) E_\gamma, \quad \xi \cdot F_\gamma = \xi(-\gamma) F_\gamma, \quad \xi \cdot \mathsf{K}_\alpha = \mathsf{K}_\alpha,
$$

and can consider the new Hopf algebra $\mathfrak{v}_q \rtimes A$.

We note that the differences $\mathsf{K}_\nu \cdot K_\nu^{-1}$ are central grouplike elements in $\mathfrak{v}_q \rtimes A$, and take the quotient

$$
v_q := \frac{\mathfrak{v}_q \rtimes A}{(\mathsf{K}_\nu \cdot K_\nu^{-1} : \nu \in Q)}.
$$

From the basis for $\mathfrak{v}_q$ we see that $\mathfrak{v}_q \rtimes A$ is a symmetric bimodule over the subgroup algebra generated by the differences $\mathsf{K}_\nu \cdot K_\nu^{-1}$, which is furthermore free on the left and right independently. We then have the induced basis,

$$
\text{Basis for } v_q = \{ E_{\gamma_1}^{m_1} \dots E_{\gamma_r}^{m_r} \cdot \xi \cdot F_{\gamma_1}^{m'_1} \dots F_{\gamma_r}^{m'_r} : \xi \in A,\ m_i, m'_i < l_{\gamma_i} \text{ for all } i \},
$$

and the triangular decomposition for $\mathfrak{v}_q$ induces a triangular decomposition for v_q,

$$\mathfrak{v}_q^+ \otimes kA \otimes \mathfrak{v}_q^- \xrightarrow{\cong} v_q.$$

In our new algebra v_q, we scale by the grouplikes to produce normalized root vectors $\mathbf{E}_\gamma = K_\gamma E_\gamma$ for each $\gamma \in \Phi^+$.

Definition 5.1. Let G be a semisimple algebraic group, and q be a quantum parameter at which each l_i is greater than the lacing number for its associated almost-simple factor in G. We define $\bar{u}_q$ as follows:

$$\bar{u}_q = \bar{u}(G_q) := \left\{ \begin{array}{c} \text{The subalgebra in } v_q \text{ generated by the} \\ \text{root vectors } \mathbf{E}_\alpha \text{ and } F_\alpha, \text{ for } \alpha \in \Delta, \\ \text{and the subgroup of characters } \bar{A} \text{ in } A \end{array} \right\}.$$

We refer to $\bar{u}_q$, somewhat informally, as the *smallest quantum algebra* for G at q. We consider the subalgebras $\bar{u}_q^+$ and $\bar{u}_q^-$ in $\bar{u}_q$ generated by the positive and negative root vectors $\mathbf{E}_\alpha$ and F_α respectively.

Lemma 5.2. *Under some ordering of the positive roots* $\{\gamma_1, \ldots, \gamma_r\} = \Phi^+$, *the collections*

$$\{\mathbf{E}_{\gamma_1}^{m_1} \ldots \mathbf{E}_{\gamma_r}^{m_r} : m_i < l_{\gamma_i} \text{ for all } i\} \quad and \quad \{F_{\gamma_1}^{m_1'} \ldots F_{\gamma_r}^{m_r'} : m_i' < l_{\gamma_i} \text{ for all } i\}$$

provide bases for $\bar{u}_q^+$ *and* $\bar{u}_q^-$ *respectively.*

Proof. Take $v_q^\pm = \bar{u}_q^\pm$, interpreted as subalgebras in the ambient algebra v_q. For v_q^+ we note that both of the subalgebras $v_q^+, \mathfrak{v}_q^+ \subseteq v_q$ are naturally graded by the root lattice Q, and we have linear isomorphisms

$$K_\nu \cdot - : (\mathfrak{v}_q^+)_\nu \xrightarrow{\sim} (v_q^+)_\nu$$

at all $\nu \in Q$. So we deduce the claimed basis for v_q^+ from that of $\mathfrak{v}_q^+$. For $\bar{u}_q^-$, we simply have $\bar{u}_q^- = v_q^- = \mathfrak{v}_q^-$. $\qquad \square$

From the triangular decomposition for v_q we deduce a triangular decompositions for $\bar{u}_q$,

$$\bar{u}_q^+ \otimes k\bar{A} \otimes \bar{u}_q^- \xrightarrow{\cong} \bar{u}_q. \tag{7}$$

The above isomorphism is given explicitly by including the relevant algebras into $\bar{u}_q$, then composing with multiplication. From the decomposition (7), and the bases of Lemma 5.2, we calculate the dimension of $\bar{u}_q$.

Corollary 5.3.

$$\dim(\bar{u}_q) = [X : X^*] \cdot \left(\prod_{\gamma \in \Phi^+} l_\gamma \right)^2.$$

Remark 5.4. Our rescaling of the generators $E_\gamma \mapsto \mathbf{E}_\gamma$ is adapted from Arkhipov-Gaitsgory [7].

5.3. **Tensor structures.** The Hopf structure on v_q is given explicitly by the formulas $\Delta(\xi) = \xi \otimes \xi$, for $\xi \in A$, and

$$\Delta(E_\alpha) = E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha, \quad \Delta(F_\alpha) = F_\alpha \otimes K_{-\alpha} + 1 \otimes F_\alpha, \tag{8}$$

for each $\alpha \in \Delta$.

Lemma 5.5. *The algebra $\bar{u}_q$ is a right coideal subalgebra in v_q.*

Proof. We have $\Delta(F_\alpha) \in \bar{u}_q \otimes v_q$ and $\Delta(\xi) = \xi \otimes \xi \in \bar{u}_q \otimes v_q$ for all $\xi \in \bar{A}$, by direct inspection. For any $\lambda \in X^*$ we have $q^2(\alpha, \lambda) = 1$ at all $\alpha \in \Delta$, so that $K_\alpha^2 : X \to k^\times$ vanishes on X^*. Hence each character K_α^2 lies in $\bar{A} \subseteq \bar{u}_q$. So we have

$$\Delta(\mathbf{E}_\alpha) = \mathbf{E}_\alpha \otimes K_\alpha + K_\alpha^2 \otimes \mathbf{E}_\alpha \ \in \ \bar{u}_q \otimes v_q,$$

and see that $\bar{u}_q$ is a right coideal subalgebra in v_q. $\qquad\qquad\qquad\square$

Via the actions of $\mathbf{E}_\alpha$, F_α, and $\bar{A}$ on big quantum group representations, $\mathrm{Rep}(\bar{u}_q)$ inherits a natural right module category structure over $\mathrm{Rep}(G_q)$.

To elaborate, any G_q-representation carries a natural X/X^*-grading, via the X-grading and the projection $X \to X/X^*$. Hence any G_q-representation carries a natural action of the character group $\bar{A} = (X/X^*)^\vee$. Via this $\bar{A}$-action, the actions of $\mathbf{E}_\alpha = K_\alpha E_\alpha$ and F_α on G_q-representations, and the coproducts (8) we obtain a natural action of $\bar{u}_q$ on products

$$W \otimes V \ \text{ for } W \text{ in } \mathrm{Rep}(\bar{u}_q) \text{ and } V \text{ in } \mathrm{Rep}(G_q).$$

In this way $\mathrm{Rep}(\bar{u}_q)$ is realized as a right module category over $\mathrm{Rep}(G_q)$.

Furthermore, the restriction functor

$$\mathrm{res}_q^- : \mathrm{Rep}(G_q) \to \mathrm{Rep}(\bar{u}_q)$$

is $\mathrm{Rep}(G_q)$-linear, with trivial compatibility $\mathrm{res}_q^-(V' \otimes V) = \mathrm{res}_q^-(V') \otimes V$, so that $\mathrm{Rep}(\bar{u}_q)$ becomes a *pointed* module category over $\mathrm{Rep}(G_q)$. The unit object in $\mathrm{Rep}(\bar{u}_q)$ is given by the trivial representation $\mathbf{1}_{\mathrm{Rep}(\bar{u}_q)} = k$.

We find in Section 7 below that the restriction functor $\mathrm{res}_q^- : \mathrm{Rep}(G_q) \to \mathrm{Rep}(\bar{u}_q)$ is in fact normal, and has kernel equal to the subcategory of quasi-classical representations in $\mathrm{Rep}(G_q)$.

6. Small quantum algebras below the lacing number

We extend our construction of $\bar{u}_q$ at not-too-small order q to arbitrary q. Here we replace Lusztig's usual small quantum algebra, from Section 5.1, with a slight modification which already appears in [44, §8.2] and was studies extensively by Lentner [39].

Formally speaking, the materials of this section recover (and extend) all of our findings from Section 5. However, our presentation becomes much more delicate as we must account for degenerations of certain "linkings" between root vectors in U_q at low order q.

Remark 6.1. The reader who is happy to work away from extraordinarily small order parameters can safely skip this section, though they should heed the change in notation from Δ to Δ_l (see Section 6.2).

6.1. **Finite-dimensional subalgebras at small order q.** We consider the quantum enveloping algebra U_q, now at arbitrary q. We again think of U_q as a "copy" of our original enveloping algebra from Section 3.3, and let K_ν denote the grouplike element in U_q associated to $\nu \in Q$. In [43, §8.1, 8.2] Lusztig introduces the following

subalgebra in U_q,[3]

$$\mathfrak{v}_q = \mathfrak{v}_q(\mathfrak{g}) = \left\{ \begin{array}{c} \text{The smallest subalgebra in } U_q \text{ which contains all} \\ \text{of the simple generators } E_\alpha \text{ and } F_\alpha \text{ with } l_\alpha > 1, \\ \text{contains all the grouplikes } \mathsf{K}_\nu \text{ for } \nu \in Q, \text{ and} \\ \text{which is stable under the braid group action on } U_q \end{array} \right\}. \quad (9)$$

We note that $\mathfrak{v}_q$ is *not* generated by the simple root vectors E_α and F_α in general. For example, in type B_2 at a 4-th root of 1 we have $l_\beta = 1$ at the long simple root β, so that neither E_β nor F_β appears in $\mathfrak{v}_q$. However, the root vectors $E_{\beta+\alpha} = E_\beta E_\alpha + E_\alpha E_\beta$ and $F_{\beta+\alpha}$ do appear in $\mathfrak{v}_q$ in this case.

Lentner establishes the following structural result.

Proposition 6.2 ([39, Theorems 5.2 & 5.4]). *Suppose that G is almost-simple, and take $l = l_i$ for the unique component $\Delta_i = \Delta$. At an arbitrary quantum parameter q we have a triangular decomposition*

$$\mathfrak{v}_q^+ \otimes k[\mathsf{K}_\nu : \nu \in Q] \otimes \mathfrak{v}_q^- \xrightarrow{\sim} \mathfrak{v}_q$$

and corresponding linear basis

$$\mathbf{B} = \{E_{\gamma_1}^{m_1} \dots E_{\gamma_r}^{m_r} \cdot \mathsf{K}_\nu \cdot F_{\gamma_1}^{m_1'} \dots F_{\gamma_r}^{m_r'} : \nu \in Q \text{ and } m_i, \ m_i' < l_{\gamma_i}\} \quad (10)$$

for $\mathfrak{v}_q$, where $\{\gamma_1, \dots, \gamma_r\} = \Phi^+$ is some ordering of the positive roots.

In the specific cases where l is equal to the lacing number for G, the algebra $\mathfrak{v}_q$ admits the following description:

- *When $l = 1$, $\mathfrak{v}_q = k[\mathsf{K}_\nu : \nu \in Q]$.*
- *In type C_n at $l = 2$, $\mathfrak{v}_q$ is generated by the simple root vectors E_α and F_α with α short, the grouplikes K_ν, and*

$$E_{\beta+\alpha'} := (E_\beta E_{\alpha'} + E_{\alpha'} E_\beta), \quad F_{\beta+\alpha'} := -(F_\beta F_{\alpha'} + F_{\alpha'} F_\beta), \quad (11)$$

  where β is the long simple root and α' is its unique short neighbor.
- *In type F_4 at $l = 2$, list the roots as $\{\beta_4, \beta_3, \alpha_2, \alpha_1\}$ where the α_i are short and $(\beta_3, \alpha_2) = -2$. Then $\mathfrak{v}_q$ is generated by the short vectors E_{α_i} and F_{α_i}, the grouplikes K_ν, $E_{\beta_3+\alpha_2}$ and $F_{\beta_3+\alpha_2}$ defined as in (11), and*

$$E_{\beta_4+\beta_3+\alpha_2} = (E_{\beta_4} E_{\beta_3+\alpha_2} + E_{\beta_3+\alpha_2} E_{\beta_4}), \quad F_{\beta_4+\beta_3+\alpha_2} = -(F_{\beta_4} F_{\beta_3+\alpha_2} + F_{\beta_3+\alpha_2} F_{\beta_4}).$$

- *In type B_n at $l = 2$, list the roots as $\{\beta_n, \dots, \beta_2, \alpha_1\}$, where consecutive roots are neighbors and α_1 is short. Then $\mathfrak{v}_q$ is generated by E_{α_1} and F_{α_1}, the grouplikes K_ν, and the root vectors $E_{\beta_r+\dots\beta_2+\alpha_1}$ and $F_{\beta_r+\dots\beta_2+\alpha_1}$ for all $r \geq 2$, which we define recursively as*

$$E_{\beta_{r+1}+\dots\alpha_1} = (E_{\beta_{r+1}} E_{\beta_r+\dots\alpha_1} + E_{\beta_r+\dots\alpha_1} E_{\beta_{r+1}}),$$

$$F_{\beta_{r+1}+\dots\alpha_1} = -(F_{\beta_{r+1}} F_{\beta_r+\dots\alpha_1} + F_{\beta_r+\dots\alpha_1} F_{\beta_{r+1}}).$$

- *In type G_2 at $l = 3$, $\mathfrak{v}_q$ is generated by E_α and F_α for short α, all the grouplikes K_ν, and the vectors*

$$E_{\beta+\alpha} = (E_\beta E_\alpha - q_\alpha^3 E_\alpha E_\beta), \quad F_{\beta+\alpha} = -(F_\beta F_\alpha - q_\alpha^3 F_\beta F_\alpha).$$

Furthermore, each of the generating E_γ and F_γ above are skew primitive in $\mathfrak{v}_q$.

---

[3]It is not obvious that our algebra $\mathfrak{v}_q$ agrees with Lusztig's algebra from [43, §8.1, 8.2]. However, such an identification follows from Propositions 6.2 and 6.4 below in conjunction with stability under the action of the braid group.

Proof. Each anomalous generator $E_\gamma = E_{\beta+\gamma'}$ (resp. $F_\gamma = F_{\beta+\gamma'}$) is obtained from the preceding generator $E_{\gamma'}$ (resp. $F_{\gamma'}$) via an application of the braid group operator T_β. So it is clear that these elements live in $\mathfrak{v}_q$. In [39] it is shown that these generators E_γ and F_γ are also skew primitive in $\mathfrak{v}_q$, and that they generate a finite-dimensional subalgebra in U_q with the prescribed basis. So, all that is left to show is that the subalgebra generated by the prescribed elements is all of $\mathfrak{v}_q$. Equivalently, we must show that this subalgebra is stable under the braid group action on U_q. We verify such stability via direct computation in Appendix A.   □

Remark 6.3. Though the algebra $\mathfrak{v}_q$ already appears in [43, §8.2], Lusztig immediately relocates to the case of an odd order scalar parameter. Such low order cases are also avoided in the text [44]. So we rely on, and emphasize, the more extensive analysis of Lentner [39] in this section.

One should note that, in the above basis, we have $l_\gamma = 1$ whenever $\gamma \in \Phi^+$ is long and l is equal to the lacing number for G. Hence many root vectors do not contribute to the basis **B** at low order q. We also highlight the validity of the expression (10) at *all* quantum parameters (cf. Section 5.1).

The final case of G_2 at a 4-th root of unity is also interesting. Here not all root vectors lie in the subalgebra of U_q generated by the simple root vectors, though all root vectors E_γ and F_γ do appear in $\mathfrak{v}_q$.

Proposition 6.4 ([39, Theorem 5.4]). *Consider G of type G_2 at $l = 2$, and let β and α be the long and short simple roots respectively. The algebra $\mathfrak{v}_q$ is generated by the grouplikes K_ν and the elements*

$$E_\alpha,\ E_\beta,\ E_{2\alpha+\beta},\ \ F_\alpha,\ F_\beta,\ F_{2\alpha+\beta},$$

where

$$E_{2\alpha+\beta} = E_\alpha^{(2)}E_\beta + E_\beta E_\alpha^{(2)} + E_\alpha E_\beta E_\alpha \ \ and \ \ F_{2\alpha+\beta} = F_\alpha^{(2)}F_\beta + F_\beta F_\alpha^{(2)} + F_\alpha F_\beta F_\alpha.$$

Furthermore, these generating E's and F's are skew primitive.

Proof. We have

$$E_{2\alpha+\beta} = T_\beta T_\alpha T_\beta(E_\alpha) \ \ and \ \ F_{2\alpha+\beta} = T_\beta T_\alpha T_\beta(F_\alpha)$$

[44, §39.2.2]. Lenter proves skew primitivity of these elements in $\mathfrak{v}_q$, and that the subalgebra in U_q which is generated by these elements has the prescribed basis [39, Theorem 5.4]. So, again, we need only verify that the subalgebra generated by the specified elements E_γ, F_γ, and K_ν is stable under the braid group action. We verify such stability in Appendix A.   □

For general G, $\mathfrak{v}_q$ decomposes into a product

$$\mathfrak{v}_q = \mathfrak{v}_q(\mathfrak{g}_1) \otimes \ldots \otimes \mathfrak{v}_q(\mathfrak{g}_t)$$

in accordance with the decomposition of G into almost-simple factors. So Propositions 6.2 and 6.4 provide an explicit description of $\mathfrak{v}_q$ in all cases.

Though there are many more interesting things to say about this algebra, and in particular about its positive and negative subalgebras $\mathfrak{v}_q^\pm$ at low order q, we close our introduction to $\mathfrak{v}_q$ here and invite the reader to see [39] for more details.

6.2. **Distinguished root vectors at low order** q. It will be convenient to define a subset of positive roots $\Delta_l \subseteq \Phi^+$ which labels the skew-primitive generators in $\mathfrak{v}_q$ at arbitrary q. Though the definition of Δ_l is implicit in the statements of Propositions 6.2 and 6.4 above, let us take a moment to identify this subset clearly.

First, we decompose G into its almost-simple factors $G = G_1 \times \cdots \times G_t$, each of which has its associated simple roots $\Delta_i \subseteq \Delta$. For each index $i \in \pi_0(\Delta)$ we consider a new set of "simple roots" Δ_{l_i}, and then define $\Delta_l = \coprod_i \Delta_{l_i} \subseteq \Phi^+$.

In the case where l_i is greater than the lacing number for G_i we just take $\Delta_{l_i} = \Delta$. In the case where l_i is equal to the lacing number for G_i, we first enumerate the simple roots as

$$\Delta_i := \{\beta_n, \ldots, \beta_{r+1}, \alpha_r, \ldots, \alpha_1\}$$

with the α_i short, the β_j long, and all consecutive roots neighbors. We take

- $\Delta_{l_i} = \{\beta_n + \cdots + \beta_2 + \alpha_1, \ldots, \beta_2 + \alpha_1, \alpha_1\}$ when G_i is of type B_n at $l_i = 2$.

- $\Delta_{l_i} = \{\beta_n + \alpha_{n-1}, \alpha_{n-1}, \ldots, \alpha_1\}$ in type C_n at $l_i = 2$.

- $\Delta_{l_i} = \{\beta_4 + \beta_3 + \alpha_2, \beta_3 + \alpha_2, \alpha_2, \alpha_1\}$ in type F_4 at $l_i = 4$.

- $\Delta_{l_i} = \{\beta_2 + \alpha_1, \alpha_1\}$ in type G_2 at $l_i = 3$.

Finally, for a factor of type G_2 at $l_i = 2$ we take $\Delta_{l_i} = \{2\alpha + \beta, \beta, \alpha\}$. This covers all cases, and we now have the explicit subset of positive roots

$$\Delta_l := \coprod_{i \in \pi_0(\Delta)} \Delta_{l_i}.$$

According to Propositions 6.2 and 6.4, the roots $\alpha \in \Delta_l$ label the skew-primitive generators E_α and F_α in $\mathfrak{v}_q$, for arbitrary G at an arbitrary torsion parameter q. One can check that Δ_l also provides a basis for the subsystem

$$\{\gamma \in \Phi : l_\gamma > 1\} \subseteq \Phi, \tag{12}$$

whenever G has no factor of type G_2 at $l_i = 2$. In all cases, the subsystem (12) labels the root vectors E_γ and F_γ which appear in $\mathfrak{v}_q$.

6.3. **Small quantum algebras below the lacing number.** Having taken proper account of the above details, we can now proceed exactly as in Section 5.

Take

$$A = (X/2 \cdot lQ)^\vee, \quad \bar{A} = (X/X^*)^\vee,$$

and consider the characters $K_\nu : X \to k^*$, $K_\nu(\lambda) = q(\lambda, \nu)$. These characters vanish on $2 \cdot lQ$, and so define elements in A.

We note that $\mathfrak{v}_q$ is a Hopf subalgebra in U_q and we have the modification of the grouplikes realized by the algebra

$$v_q := \frac{\mathfrak{v}_q \rtimes A}{(\mathsf{K}_\nu \cdot \mathsf{K}_\nu^{-1} : \nu \in Q)}.$$

The algebra v_q inherits a unique Hopf structure so that the characters A are grouplike and the map $\mathfrak{v}_q \to v_q$ is a map of Hopf algebras. We additionally have a triangular decomposition for v_q which is inherited from that of $\mathfrak{v}_q$.

Definition 6.5. Define $\bar{u}_q = \bar{u}(G_q)$ to be the subalgebra in v_q which is generated by the $\mathbf{E}_\alpha = K_\alpha E_\alpha$ and F_α, for $\alpha \in \Delta_l$, and the subgroup of characters $\bar{A} \subseteq A$.

We again refer to $\bar{u}_q$ as the *smallest quantum algebra* for G at q. We have the triangular decomposition, distinguished basis, and dimension calculation

$$\dim(\bar{u}_q) = [X : X^*] \cdot (\prod_{\gamma \in \Phi^+} l_\gamma)^2,$$

just as in Section 5.

6.4. **Realization via coinvariants.** We describe an alternate means of locating the subalgebra $\bar{u}_q$ in v_q. Consider the group of characters $\Psi = (X^*/2 \cdot lQ)^\vee$ along with the natural quotient map $A \to \Psi$, which is given by restriction.

The composition

$$\mathrm{Rep}(G^*_\varepsilon) \to \mathrm{Rep}(G_q) \xrightarrow{\mathrm{res}} \mathrm{Rep}(v_q)$$

provides a tensor functor from $\mathrm{Rep}(G^*_\varepsilon)$ to $\mathrm{Rep}(v_q)$, and each G^*_ε-representation decomposes into a sum of 1-dimensional representations over v_q. These representations are labeled by elements in the quotient $X^*/2 \cdot lQ$ and define a map of Hopf algebras $\pi : v_q \to k\Psi$. The map π annihilates all of the generators E_α and F_α, and lifts the group map $A \to \Psi$.

Via Nichols-Zoeller freeness [48] the subalgebra of left $k\Psi$-coinvariants in v_q is seen to be of dimension

$$\dim(^{k\Psi}v_q) = \frac{\dim(v_q)}{|\Psi|} = [X : X^*] \cdot (\prod_{\gamma \in \Phi^+} l_\gamma)^2.$$

Furthermore, by checking on the generators, one observes an algebra inclusion $\bar{u}_q \subseteq {}^{k\Psi}v_q$. Via the triangular decomposition for v_q one sees that $\bar{u}_q$ has dimension greater than or equal to $[X : X^*] \cdot (\prod_{\gamma \in \Phi^+} l_\gamma)^2$, which forces a calculation

$$\dim(\bar{u}_q) = [X : X^*] \cdot (\prod_{\gamma \in \Phi^+} l_\gamma)^2, \quad \text{an equality of algebras} \quad \bar{u}_q = {}^{k\Psi}v_q,$$

and verifies the triangular decomposition for $\bar{u}_q$ as well.

6.5. **Tensor structures.** As before, the algebra $\bar{u}_q$ sits in v_q as a right coideal subalgebra in v_q. This follows from the computations

$$\Delta(\mathbf{E}_\alpha) = \mathbf{E}_\alpha \otimes K_\alpha + K_\alpha^2 \otimes \mathbf{E}_\alpha \quad \text{and} \quad \Delta(F_\alpha) = F_\alpha \otimes K_\alpha^{-1} + 1 \otimes F_\alpha, \qquad (13)$$

for $\alpha \in \Delta_l$, and the fact that all $K_\alpha^2|_{X^*} = 1$.

Via the coaction (13) we obtain a natural right action of $\mathrm{Rep}(G_q)$ on $\mathrm{Rep}(\bar{u}_q)$, and the restriction functor

$$\mathrm{res}_q^- : \mathrm{Rep}(G_q) \to \mathrm{Rep}(\bar{u}_q)$$

becomes a map of $\mathrm{Rep}(G_q)$-module categories. So, $\mathrm{Rep}(\bar{u}_q)$ again carries a natural pointed module category structure over $\mathrm{Rep}(G_q)$. The details here are exactly as in Section 5.3.

6.6. **Braid group actions.** By construction, the action of the braid group $B_\mathfrak{g}$ on U_q restricts to an action of $B_\mathfrak{g}$ on the Hopf subalgebra $\mathfrak{v}_q$. This action extends uniquely to an action on v_q under which the map $\mathfrak{v}_q \to v_q$ is $B_\mathfrak{g}$-equivariant, and under which $B_\mathfrak{g}$ acts on the grouplikes $\xi \in A$ as $T_\alpha \cdot \xi = \xi(\sigma_\alpha^{-1}-)$. We claim that the subalgebra $\bar{u}_q \subseteq v_q$ is stable under this braid group action.

Proposition 6.6. *The algebra $\bar{u}_q$ admits a natural action of the braid group $B_\mathfrak{g}$ under which the restriction functor $\mathrm{res}_q^- : \mathrm{Rep}(G_q) \to \mathrm{Rep}(\bar{u}_q)$ is $B_\mathfrak{g}$-equivariant.*

For the proof one employs the identification of $\bar{u}_q$ with the $k\Psi$-coinvariants in v_q, and leverages $B_{\mathfrak{g}}$-equivariance of the map $\pi : v_q \to k\Psi$. We leave the details to the interested reader.

7. Normality of restriction functors

In this section we show that the restriction functor $\mathrm{res}_q^- : \mathrm{Rep}(G_q) \to \mathrm{Rep}(\bar{u}_q)$ is normal. Here $\mathrm{Rep}(\bar{u}_q)$ is considered along with its natural structure as a pointed module category over $\mathrm{Rep}(G_q)$ (see Sections 5.3 and 6.5). We show additionally that the kernel of this functor is precisely the image of quantum Frobenius in $\mathrm{Rep}(G_q)$.

At certain parameters such normality is well-known. See for example [17] [15, § 2.3] for the case of simple G at an odd order scalar parameter which is coprime to the determinant of the Cartan matrix. The general odd order case also follows from standard relations which can be found in [43, 44]. One can see [7, Proposition 1.5] for the case of a semisimple simply-connected group G at a quantum parameter which is of "sufficiently large" even order.[4]

7.1. Normality of restriction.

Proposition 7.1 (cf. [39, Problem 7.1]). *The restriction functor* $\mathrm{res}_q^- : \mathrm{Rep}(G_q) \to \mathrm{Rep}(\bar{u}_q)$ *is normal. Furthermore, the kernel of* res_q^- *is equal to the image of quantum Frobenius*

$$\ker(\mathrm{res}_q^-) = \mathrm{im}\left(\mathrm{Fr} : \mathrm{Rep}(G_\varepsilon^*) \to \mathrm{Rep}(G_q) \right).$$

The proposition is obtained as an application of Lemma 7.3 below. We provide a proof which is contingent on Lemma 7.3, then return to cover the necessary details.

Proof. All objects in $\ker(\mathrm{res}_q^-)$ necessarily have grading in the kernel of the projection $X \to (A)^\vee = X/X^*$, and are annihilated by all E_α with $l_\alpha > 1$. So $\ker(\mathrm{res}_q^-) \subseteq \mathrm{im}(\mathrm{Fr})$ by Lemma 4.5 and Theorem 4.9. We refer again to Lemma 4.5 to see that $\mathrm{im}(\mathrm{Fr}) \subseteq \ker(\mathrm{res}_q^-)$, and hence that the kernel is precisely the image of quantum Frobenius.

We need to show now that the $\bar{u}_q$-invariants $V^{\bar{u}_q}$ in any G_q-representation V form a G_q-subrepresentation. We have explicitly

$$V^{\bar{u}_q} = \left\{ \begin{array}{c} \text{The sum of all homogeneous } v \in V \text{ with} \\ \deg(v) \in X^* \text{ and } \mathbf{E}_\alpha \cdot v = F_\alpha \cdot v = 0 \text{ at all } \alpha \in \Delta_l \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \text{The sum of all homogeneous } v \in V \text{ with} \\ \deg(v) \in X^* \text{ and } E_\alpha \cdot v = F_\alpha \cdot v = 0 \text{ at all } \alpha \in \Delta_l \end{array} \right\}.$$

By definition, this subspace $V^{\bar{u}_q}$ is stable under the actions of E_α and F_α, for $\alpha \in \Delta_l$, and we need only show it is stable under the actions of the divided powers $E_\beta^{(l_\beta)}$ and $F_\beta^{(l_\beta)}$ for simple β. Rather, we need to show that, for any $v \in V^{\bar{u}_q}$, the vectors

$$E_\beta^{(l_\beta)} \cdot v \quad \text{and} \quad F_\beta^{(l_\beta)} \cdot v$$

are still annihilated the actions of the elements E_α, F_α, and $(\xi - 1)$ with $\xi \in \bar{A}$. However, this is precisely what Lemma 7.3 tells us. $\qquad\square$

---

[4]As far as we can tell, one also assumes that a sufficiently large power of 2 divides all l_i in [7].

We now cover the requisite details. In our arguments below we consider the topological Hopf algebra

$$\widehat{U}_q := \varprojlim_V \dot{U}_q / \operatorname{Ann}(V),$$

where the limit is over all finite-dimensional G_q-representations V. Here we employ the partial ordering where $V \leq W$ whenever W admits an injective map of representations $V \to W$. Alternatively, $\widehat{U}_q$ is constructed as the endomorphism algebra of the forgetful functor

$$\widehat{U}_q = \operatorname{End}\left(forget : \operatorname{Rep} G_q \to \operatorname{Vect}\right)$$

[47, §2.6]. This algebra is generated topolgically by the divided powers of the simple root vectors $E_\beta^{(n)}$ and $F_\beta^{(n)}$, and the characters $X^\vee = \operatorname{Hom}_{\operatorname{Grp}}(X, k^\times)$.

By Tannakian reconstruction, the restriction functor $\operatorname{Rep}(G_q) \to \operatorname{Rep}(\bar{u}_q)$ defines an algebra map $\bar{u}_q \to \widehat{U}_q$, which explicitly sends the $\mathbf{E}_\gamma$ and F_γ in $\bar{u}_q$ to the corresponding vectors in $\widehat{U}_q$ and the characters $\xi \in \bar{A}$ to their corresponding functions on X.

Remark 7.2. From a simple-minded perspective, $\widehat{U}_q$ is just a version of the quantum enveloping algebra which is unital–unlike the modified algebra $\dot{U}_q$–and which contains all naturally occurring toral characters–unlike the usual quantum enveloping algebra U_q. In $\widehat{U}_q$ we can directly compare elements from all of our favorite algebras; U_q, $\mathfrak{v}_q$, and $\bar{u}_q$.

Lemma 7.3. *Let $\mathfrak{m} \subseteq \bar{u}_q$ be the kernel of augmentation $\bar{u}_q \to k$. For all $\xi \in \bar{A}$, and all positive roots γ, we have*

$$(\xi - 1)E_\gamma^{(l_\gamma)} = E_\gamma^{(l_\gamma)}(\xi - 1) \quad and \quad (\xi - 1)F_\gamma^{(l_\gamma)} = F_\gamma^{(l_\gamma)}(\xi - 1). \tag{14}$$

Additionally, for all $\alpha \in \Delta_l$ and all $\beta \in \Delta$ we have

$$E_\alpha E_\beta^{(l_\beta)}, \ F_\alpha F_\beta^{(l_\beta)}, \ E_\alpha F_\beta^{(l_\beta)}, \ F_\alpha E_\beta^{(l_\beta)} \ \in \ \widehat{U}_q\mathfrak{m}. \tag{15}$$

The result essentially follows by computations of Lusztig [43, 44] and Lentner [39].

Proof. It suffices to prove the result when G is almost-simple. Hence we take $l = l_i$ at the unique index $i \in \pi_0(\Delta)$. When $l = 1$ the group $\bar{A}$ is trivial and Δ_l is empty, so that $\bar{u}_q = k$ and there is nothing to check. We therefore suppose that $l > 1$.

Before beginning, let us note that we have containments

$$\widehat{U}_q E_\gamma \ \subseteq \ \widehat{U}_q\mathfrak{m} \quad \text{and} \quad \widehat{U}_q F_\gamma \subseteq \widehat{U}_q\mathfrak{m}$$

whenever γ is a positive root with $l_\gamma > 1$. We take these inclusions for granted, and use them throughout the proof.

Each characters ξ in $(X/X^*)^\vee = \bar{A}$ has $\xi|_{lQ} = 1$, by Lemma 4.2. Hence $\xi(l_\gamma\gamma) = 1$ at each root γ, and the commutativity relations (14) follow.

As for the relations (15), it suffices to establish inclusions

$$\left.\begin{array}{c} [E_\beta^{(l_\beta)}, E_\alpha], \ [F_\beta^{(l_\beta)}, F_\alpha] \\ [E_\beta^{(l_\beta)}, F_\alpha], \ [F_\beta^{(l_\beta)}, E_\alpha] \end{array}\right\} \ \in \ \widehat{U}_q\mathfrak{m} \tag{16}$$

at arbitrary $\alpha \in \Delta_l$ and $\beta \in \Delta$. When both $\alpha, \beta \in \Delta$, and $l_\alpha, l_\beta > 1$, one employs the higher Serre relations [44, Proposition 35.3.1] to see that the first two commutators lie in $\widehat{U}_q\mathfrak{m}$. Similarly, in this case the relations of [43, §6.5 (a2), (a6)]

tell us that the latter two commutators lie in $\widehat{U}_q\mathfrak{m}$ as well. We therefore have both of the desired containments (14) and (15), provided l is greater than the lacing number for G.

The cases where l is less than or equal to the lacing number for G are somewhat involved, and are delayed until Appendix B. $\qquad\square$

7.2. **Normality and Hopf structures.** We note that the fiber functor for $\text{Rep}(G_q)$ endows Vect with the structure of a pointed $\text{Rep}(G_q)$-module category, and that the forgetful functor $\text{Rep}(\bar{u}_q) \to \text{Vect}$ is naturally a map of pointed module categories over $\text{Rep}(G_q)$. It follows that the restriction functor $\text{res}_q^- : \text{Rep}(G_q) \to \text{Rep}(\bar{u}_q)$ determines, and is determined by, a map of $\mathscr{O}(G_q)$-module coalgebras $\pi : \mathscr{O}(G_q) \to \bar{u}_q^*$ [19, Definition 1.1], via Tannakian reconstruction [51, Theorem 2.2.8].

Similarly, the quantum Frobenius functor determines an embedding of Hopf algebras $fr^* : \mathscr{O}(G_\varepsilon^*) \to \mathscr{O}(G_q)$ [51, Lemma 2.2.13]. The following codifies normality of the restriction functor in algebraic terms.

Lemma 7.4. *Consider a Hopf algebra A, and let $\pi : A \to C$ be a map of A-module coalgebras for which the restriction functor $\text{res}_\pi : \text{Corep}(A) \to \text{Corep}(C)$ is normal. Let B be the (unique) Hopf subalgebra in A whose corepresentations are identified with the kernel of res_π. Then the left and right C-coinvariants in A agree, and $^C A = A^C = B$.*

Proof. The fact that B is in the kernel of res_π implies $B \subseteq A^C$, by definition. The left counit axiom for B then gives $\pi|_B = \epsilon_B$. Explicitly, for $b \in B$ we have

$$\epsilon(b) = \epsilon(b_1)\pi(b_2) = \pi(\epsilon(b_1)b_1) = \pi(b).$$

Since B is a subcoalgebra in A, this implies $B \subseteq {}^C A$ as well.

For the opposite inclusion $A^C \subseteq B$, normality of restriction tells us that A^C lies in $\text{Corep}(B)$, so that $\Delta(A^C) \subseteq A \otimes B$. We apply the counit on the left to find $A^C \subseteq B$, and hence $A^C = B$.

The final inclusion $^C A \subseteq B$ follows from the A-module coalgebra structure on C. Indeed, for any $a \in {}^C A$ we have $\pi(a_1) \otimes a_2 = 1 \otimes a \in C \otimes A$, and hence

$$\pi(Sa_1) \otimes a_2 = \big(1 \cdot Sa_1\big) \otimes a_2 = \big(\pi(a_1) \cdot Sa_2\big) \otimes a_3 = 1 \otimes a.$$

Apply $id_C \otimes S$ to the above expression to see that $S(a) \in A^C = B$ whenever $a \in {}^C A$. This gives $a = S^{-1}S(a) \in S^{-1}(B) = B$, and verifies the equality $^C A = B$. $\qquad\square$

We apply Lemma 7.4 to the quantum group setting, as described above.

Corollary 7.5. *For arbitrary G and q,*

$$^{\bar{u}_q^*}\mathscr{O}(G_q) = \mathscr{O}(G_q)^{\bar{u}_q^*} = \mathscr{O}(G_\varepsilon^*).$$

7.3. **Normality of restriction for the quantum Borel.** We consider the category $\text{Rep}(B_q)$ of representations for the positive quantum Borel. These are integrable U_q^+-representations which are equipped with a compatible grading by the character lattice X for G. As with the full quantum group, we have the tensor embedding

$$\text{Fr}^{\geq 0} : \text{Rep}(B_\varepsilon^*) \to \text{Rep}(B_q)$$

which is obtained by restricting along the algebra map $fr|_{U_q^+} : U_q^+ \to (U_\varepsilon^*)^+$ [44, 39].

We also have the non-negative subalgebra $\bar{u}_q^{\geq 0}$ in $\bar{u}_q$, which sits as a subcomodule subcoalgebra in $\bar{u}_q$. Hence we obtain a natural action of $\mathrm{Rep}(B_q)$ on the category of representations $\mathrm{Rep}(\bar{u}_q^{\geq 0})$, and the restriction functor

$$\mathrm{Rep}(B_q) \to \mathrm{Rep}(\bar{u}_q^{\geq 0})$$

endows $\mathrm{Rep}(\bar{u}_q^{\geq 0})$ with a pointed module category structure over $\mathrm{Rep}(B_q)$.

One can use Lentner's normality result [39, Lemma 6.6], or [44, Proposition 35.3.1], to obtain the following non-negative analog of Proposition 7.1.

Proposition 7.6 ([44, 39]). *The restriction functor $\mathrm{Rep}(B_q) \to \mathrm{Rep}(\bar{u}_q^{\geq 0})$ is normal, and has kernel equal to the image of $\mathrm{Rep}(B_\varepsilon^*)$ in $\mathrm{Rep}(B_q)$ under quantum Frobenius.*

8. Restriction and simple G_q-representations

Before turning to a discussion of the Steinberg representation, we record some helpful results which describe simple G_q-representations and their behaviors under restriction. For convenience of presentation,

We suppose that G is simply-connected throughout this section.

Remark 8.1. One recovers an analysis of the simples for non-simply-connected G by noting that $\mathrm{Rep}(G_q)$ sits in its simply-connected form $\mathrm{Rep}(G_q^{sc})$ as the full subcategory of representations whose P-grading is supported on the character lattice X for G.

8.1. **Generic information.** The following is well-known at most parameters q.

Lemma 8.2 ([4, Lemma 1.13]). *Consider a G_q-representation V, and let $\Pi_l(V)$ denote the subset of weights $\mu \in P$ with V_μ non-vanishing. Then $\Pi_l(V)$ is stable under the action of the Weyl group on P.*

Proof. It suffices to consider the case where V is finite-dimensional. Let $v \in V$ be nonzero and of weight μ. Fix simple α and take $m = -(\alpha, \mu)/d_\alpha$. We claim that if $m \geq 0$ then $E_\alpha^{(m)} \cdot v \neq 0$, and if $m \leq 0$ then $F_\alpha^{(m)} \cdot v \neq 0$. Supposing this claim is valid, we find that $V_{\sigma_\alpha(\mu)} = V_{\mu+m\alpha}$ is non-zero, that $\Pi_l(V)$ is stable under the actions of the simple reflections in $\mathscr{W}$, and hence that $\Pi_l(V)$ is stable under the action of the Weyl group.

To establish the claim, we reduce to the case of $G = \mathrm{SL}(2)$ at a parameter ζ by restricting along the root subgroup for α. Now, by induction on the length, we reduce further to the case where V is a simple $\mathrm{SL}(2)_\zeta$-representation. Here all the simples are of the form $\mathrm{Fr}(L_1) \otimes L_2$ where L_1 is a representation for classical $\mathrm{SL}(2)$ and L_2 has highest weight $< \mathrm{ord}\,\zeta(\alpha, \alpha)$, and the claim follows by a direct inspection. $\square$

For a given simple representation $L(\lambda)$ we take $\Pi_l(\lambda) = \Pi_l(L(\lambda))$. A classical fact tells us that the minimal element in this set is $w_0\lambda$ [40, Theorem 4.12] [33, Proposition 21.3, Exercise 21.4(6)]. We therefore obtain the following.

Lemma 8.3. *For $\lambda \in P^+$, $L(\lambda)$ has lowest weight $w_0\lambda$, and $L(\lambda)^* = L(-w_0\lambda)$.*

8.2. Restrictions of simple G_q-representations. Below we take

$$P_l = \left\{ \lambda \in P : 0 \leq \frac{(\alpha, \lambda)}{d_\alpha} < l_\alpha \ \forall \ \alpha \in \Delta \right\}.$$

In [49] such weights are called *(l-)restricted*. These are weights of the form $\lambda = \sum_\alpha c_\alpha \omega_\alpha$, where the ω_α are the fundamental weights associated to simple roots α, and the coefficients satisfy $0 \leq c_\alpha < l_\alpha$. We note that the class of simples $L(\lambda)$ of restricted highest weight is closed under duality. This just follows from the fact that $w_0 \Delta = -\Delta$.

Lemma 8.4 ([49, Lemma 9.3.2]). *For any $\lambda \in P_l$, the simple representation $L(\lambda)$ satisfies*

$$L(\lambda)^{\bar{u}_q^+} = L(\lambda)^{\mathfrak{v}_q^+} = L(\lambda)_\lambda.$$

To be clear, by the $\bar{u}_q^+$ or $\mathfrak{v}_q^+$-invariants here we mean the collections of vectors which are annihilated by all $\mathbf{E}_\alpha$ or E_α, for $\alpha \in \Delta_l$, respectively.

Proof. The equality $L(\lambda)^{\bar{u}_q^+} = L(\lambda)^{\mathfrak{v}_q^+}$ is immediate, since all K_γ act as units on $L(\lambda)$. We restrict to B_q and, for any $\mu \in P$, consider the product $L(\lambda) \otimes k_\mu$. It suffices to show that the invariants $(L(\lambda) \otimes k_\mu)^{\bar{u}_q^{\geq 0}}$ lie in the span of the highest weight vector, at arbitrary $\mu \in P$. Let $v(\mu)$ denote the (unique-up-to-scaling) highest weight vector in $L(\lambda) \otimes k_\mu$, which is just the highest weight vector in $L(\lambda)$ times any nonzero vector in k_μ.

By the normality result of Proposition 7.6, the invariants $(L(\lambda) \otimes k_\mu)^{\bar{u}_q^{\geq 0}}$ are identified with a B_ε^*-representation via quantum Frobenius, and the socle of this representation over B_ε^* lies in the socle of $L(\lambda) \otimes k_\mu$ over B_q, i.e. in the highest weight space. It follows that $v(\mu)$ is contained in the B_ε^*-subrepresentation generated by any nonzero vector w in $(L(\lambda) \otimes k_\mu)^{\bar{u}_q^{\geq 0}}$.

From the expression $\lambda = \sum_\alpha c_\alpha \omega_\alpha$, with each $c_\alpha < l_\alpha$ by hypothesis, we find at each simple root α that

$$\sigma_\alpha(\lambda - l_\alpha \alpha) = \lambda - c_\alpha \alpha + l_\alpha \alpha = \lambda + (l_\alpha - c_\alpha)\alpha > \lambda.$$

In particular $\lambda - l_\alpha \alpha \notin \Pi_l(\lambda)$. It follows that any nonzero homogeneous w in $(L(\lambda) \otimes k_\mu)^{\bar{u}_q^{\geq 0}}$ which is of degree $< \lambda + \mu$ generates a subrepresentation which does *not* contain $v(\mu)$. By our previous observations about B_ε^*-subrepresentations in $(L(\lambda) \otimes k_\mu)^{\bar{u}_q^{\geq 0}}$, no such nonzero vector exists. This forces

$$(L(\lambda) \otimes k_\mu)^{\bar{u}_q^{\geq 0}} \subseteq k \cdot v(\mu).$$

$\square$

Proposition 8.5 ([49, Proposition 9.3.4]). *For each $\lambda \in P_l$, the associated simple $L(\lambda)$ in $\mathrm{Rep}(G_q)$ restricts to a simple object in $\mathrm{Rep}(\bar{u}_q)$.*

Proof. One employs Lemma 8.4 and proceeds exactly as in [49]. $\square$

By the usual analysis with baby Verma modules, one sees that simple representations in $\mathrm{Rep}(\bar{u}_q)$ are characterized by their highest weights. These highest weights are, by definition, characters for the group $\bar{A} = (P/P^*)^\vee$, i.e. elements in the quotient P/P^*. Now, the projection $P \to P/P^*$ restricts to provide a bijection $P_l \xrightarrow{\sim} P/P^*$, so that

$$|\mathrm{Irrep}(\bar{u}_q)| = |P/P^*| = |P_l|.$$

Hence Proposition 8.5 implies the following.

Corollary 8.6. *For any simple representation $L(\bar{\lambda})$ in $\mathrm{Rep}(\bar{u}_q)$ there is a simple representation $L(\lambda)$ in $\mathrm{Rep}(G_q)$ for which*

$$\mathrm{res}_q^-(L(\lambda)) = L(\bar{\lambda}).$$

One sees from Corollary 8.8 below that this simple lift $L(\lambda)$ of $L(\bar{\lambda})$ is unique.

8.3. **Steinberg decomposition.** We use Lemma 8.4 and Proposition 8.5 to recover Steinberg's tensor product theorem at arbitrary q.

Theorem 8.7 ([41, Theorem 7.4]). *Suppose that G is simply-connected. For any $\lambda \in P^+$ we have a unique decomposition $\lambda = \lambda_0 + \lambda_1$ with $\lambda_1 \in P^*$ and $\lambda_0 \in P_l$, and a corresponding isomorphism of G_q-representations*

$$\mathrm{Fr}(L(\lambda_1)) \otimes L(\lambda_0) = L(\lambda_1) \otimes L(\lambda_0) \xrightarrow{\sim} L(\lambda).$$

Proof. We check highest weight vectors in the product $L(\lambda_1) \otimes L(\lambda_0)$. Since $L(\lambda_1)$ is in the kernel of the restriction functor $\mathrm{Rep}(B_q) \to \mathrm{Rep}(\mathfrak{v}_q^+)$ we have

$$(L(\lambda_1) \otimes L(\lambda_0))^{\mathfrak{v}_q^+} = L(\lambda_1) \otimes L(\lambda_0)^{\mathfrak{v}_q^+} = L(\lambda_1) \otimes k_{\lambda_0}.$$

The final equality here follows by Proposition 8.5. We note that $L(\lambda_1) \otimes k_{\lambda_0}$ has 1-dimensional socle over B_q to see now that $L(\lambda_1) \otimes L(\lambda_0)$ has a 1-dimensional socle over B_q. This B_q-socle is spanned by the unique-up-to-scaling element of weight λ. This calculates the socle over G_q as $\mathrm{soc}(L(\lambda_1) \otimes L(\lambda_0)) = L(\lambda)$.

We note at this point that $L(\lambda_1) \otimes L(\lambda_0)$ and $L(\lambda)$ both have lowest weights $w_0\lambda$. Take duals and apply a similar analysis to see that $L(\lambda_0)^* \otimes L(\lambda_1)^*$ has socle $L(-w_0\lambda)$. We therefore observe a projection

$$p : L(\lambda_1) \otimes L(\lambda_0) \to L(-w_0\lambda)^* = L(\lambda).$$

The kernel of this map must be trivial, since $L(\lambda_1) \otimes L(\lambda_0)$ has no additional highest weight vectors, outside of the one of weight λ. Hence p is injective. Surjectivity follows by simplicity of $L(\lambda)$, so that we observe the claimed isomorphism. $\qquad\square$

From Proposition 8.5 we know that $\mathrm{res}_q^-(L(\lambda))$ is simple in $\mathrm{Rep}(\bar{u}_q)$ whenever $\lambda \in P_l$. So the Steinberg decomposition can alternately be expressed in terms of the restriction functor res_q^-.

Corollary 8.8. *Suppose that G is simply-connected. Then for any $\lambda \in P^+$ the simple representation $L(\lambda)$ admits a unique decomposition*

$$L(\lambda) = L(\lambda'') \otimes L(\lambda')$$

where $L(\lambda')$ restricts to a simple representation in $\mathrm{Rep}(\bar{u}_q)$, and $L(\lambda'')$ is in the kernel of res_q^-.

9. THE STEINBERG REPRESENTATION

We consider the dominant weight $\rho_l := \frac{1}{2} \sum_{\gamma \in \Phi^+} (l_\gamma - 1)\gamma$ and its corresponding simple representation

$$\mathrm{St} := L(\rho_l)$$

in $\mathrm{Rep}(G_q)$, for simply-connected G. The representation St is known as the *Steinberg representation*, and it has proved extraordinarily useful in studies of both modular and quantum representation theory. See for example [34, 49, 4, 5, 3].

Our next goal is to show that the Steinberg representation is both projective and injective in $\mathrm{Rep}(G_q)$, and that it restricts to a projective and injective object in $\mathrm{Rep}(\bar{u}_q)$ (cf. [3, 49, 4, 5, 2]). However, these points are more easily argued in a locale which lies between these two categories.

In this section we analyze the behaviors of the Steinberg representation in the category of *torally extended* small quantum group representations $\mathrm{Rep}(\dot{u}_q)$. We then return to a discussion of G_q-representations and $\bar{u}_q$-representations in Sections 10 and 11. Our approach follows those of Parshall-Wang [49] and Andersen-Polo-Wen [4], to an extent, though we employ basic results for Hopf algebras [38] in order to avoid intricate arguments with induction.

We note that ρ_l does not lie in the character lattice for G, at a general non-simply-connected group. So we restrict our attention to the simply-connected setting.

Throughout the section we assume G is simply-connected!

9.1. **Torally extended small quantum group representations.** We consider the tensor category $\mathrm{Rep}(\dot{u}_q)$ of "torally extended" small quantum group representations. Specifically we consider

$$\mathrm{Rep}(\dot{u}_q) = \left\{ \begin{array}{c} \text{The category of } \mathfrak{v}_q\text{-modules with a} \\ \text{compatible grading by the character lattice } X \end{array} \right\}. \qquad (17)$$

Since we've restricted ourselves to the simply-connected setting, this character lattice is just the weight lattice $X = P$. By a compatible P-grading we mean that E_γ and F_γ shift degrees on a $\dot{u}_q$-representation V by γ and $-\gamma$ respectively, and that the K_ν act on each homogeneous subspace $V_\lambda \subseteq V$ as the corresponding scalar $q(\nu, \lambda)$.

Our notation (17) is taken from [44, § 31.1 & 36.2], where $\mathrm{Rep}(\dot{u}_q)$ is constructed as the category of "unital" representations for a non-unital algebra $\dot{u}_q$ with generators $E_\alpha 1_\lambda$ and $F_\alpha 1_\lambda$, for $\alpha \in \Delta_l$ and $\lambda \in P$, which satisfy

$$E_\alpha 1_\lambda \cdot F_\beta 1_\mu = \delta_{\mu, \lambda+\beta} E_\alpha F_\beta 1_\mu \quad \text{and} \quad F_\beta 1_\mu \cdot E_\alpha 1_\lambda = \delta_{\lambda, \mu-\alpha} F_\beta E_\alpha 1_\lambda,$$

as well as relations deduced from the usual quantum Serre relations.

9.2. **The smallest quantum algebra and $\dot{u}_q$-representations.** Each of the generators for $\bar{u}_q$ act in the obvious way on $\dot{u}_q$-representations. In particular, the elements $\mathbf{E}_\gamma$ and F_γ act on a $\dot{u}_q$-representation via the corresponding elements in $\mathfrak{v}_q$, and the characters $\xi \in \bar{A}$ act as the semisimple operators

$$\xi \cdot v = \xi(\lambda) \cdot v \quad \text{for homogeneous } v \in V, \ \lambda = \deg(v).$$

In this way we obtain a natural restriction functor $\mathrm{Rep}(\dot{u}_q) \to \mathrm{Rep}(\bar{u}_q)$ and, as in Section 5.3, $\mathrm{Rep}(\bar{u}_q)$ has the natural structure of a pointed module category over $\mathrm{Rep}(\dot{u}_q)$.

For the maximal torus T^* in the dual group G^*, we consider the tensor subcategory $\mathrm{Rep}(T_\varepsilon^*)$ of P^*-graded vector spaces in $\mathrm{Rep}(\dot{u}_q)$. One checks the following directly.

Lemma 9.1. *The restriction functor $\mathrm{Rep}(\dot{u}_q) \to \mathrm{Rep}(\bar{u}_q)$ is normal, and has kernel equal to the subcategory of P^*-graded vector spaces $\mathrm{Rep}(T_\varepsilon^*) \subseteq \mathrm{Rep}(\dot{u}_q)$.*

We also have a natural restriction functor from $\mathrm{Rep}(G_q)$ to $\mathrm{Rep}(\dot{\mathfrak{u}}_q)$, which is a tensor functor. This tensor functor factors res_q^- as

$$
\begin{array}{ccc}
 & \mathrm{Rep}(\dot{\mathfrak{u}}_q) & \\
 \nearrow & & \searrow \\
\mathrm{Rep}(G_q) \xrightarrow[\mathrm{res}_q^-]{} & & \mathrm{Rep}(\bar{u}_q).
\end{array}
$$

The above diagram and Proposition 8.5 imply the following.

Lemma 9.2. *For any restricted weight* $\lambda \in P_l$, *the associated simple* $L(\lambda)$ *in* $\mathrm{Rep}(G_q)$ *restricts to a simple object in* $\mathrm{Rep}(\dot{\mathfrak{u}}_q)$.

9.3. **Induction for** $\dot{\mathfrak{u}}_q$**-representations.** We consider the category $\mathrm{Rep}(\dot{\mathfrak{u}}_q^{\leq 0})$ of character graded representations for the non-positive subalgebra $\mathfrak{v}_q^{\leq 0}$ in $\mathfrak{v}_q$, and the restriction functor $\mathrm{Rep}(\dot{\mathfrak{u}}_q) \to \mathrm{Rep}(\dot{\mathfrak{u}}_q^{\leq 0})$. We have the right adjoint to restriction

$$
\mathrm{R.ind} := \mathrm{R.ind}_{\dot{\mathfrak{u}}_q^{\leq 0}}^{\dot{\mathfrak{u}}_q} : \mathrm{Rep}(\dot{\mathfrak{u}}_q^{\leq 0}) \to \mathrm{Rep}(\dot{\mathfrak{u}}_q),
$$

as well as the left adjoint

$$
\mathrm{L.ind} := \mathrm{L.ind}_{\dot{\mathfrak{u}}_q^{\leq 0}}^{\dot{\mathfrak{u}}_q} : \mathrm{Rep}(\dot{\mathfrak{u}}_q^{\leq 0}) \to \mathrm{Rep}(\dot{\mathfrak{u}}_q).
$$

One can write $\mathrm{L.ind}(V)$ explicitly as $\mathrm{L.ind}(V) = \mathfrak{v}_q \otimes_{\mathfrak{v}_q^{\leq 0}} V \cong \mathfrak{v}_q^+ \otimes V$, where the E_γ and F_γ act via the left factor and the P-grading is given by

$$
\deg(E_{\gamma_1}^{m_1} \cdots E_{\gamma_r}^{m_r} \otimes v) = \deg(v) + \sum_i m_i \gamma_i.
$$

Lemma 9.3. *For any finite-dimensional* $\dot{\mathfrak{u}}_q^{\leq 0}$*-representation* V,

$$
\mathrm{R.ind}(V) = \mathrm{L.ind}(V^*)^* \quad and \quad \mathrm{L.ind}(V) = \mathrm{R.ind}(V^*)^*.
$$

In particular, $\mathrm{R.ind}$ *sends finite-dimensional* $\dot{\mathfrak{u}}_q^{\leq 0}$*-representations to finite-dimensional* $\dot{\mathfrak{u}}_q$*-representations.*

In the proof we employ the pivotal structure $id \xrightarrow{\sim} (-)^{**}$ on $\mathrm{Rep}(\dot{\mathfrak{u}}_q)$ which is provided by the action of the character K_ρ.

Proof. We consider the first identification. Since $\mathrm{Rep}(\dot{\mathfrak{u}}_q)$ is generated by its finite-dimensional representations, under filtered colimits, it suffices to provide natural isomorphisms

$$
\mathrm{Hom}_{\dot{\mathfrak{u}}_q}(W, \mathrm{L.ind}(V^*)^*) \cong \mathrm{Hom}_{\dot{\mathfrak{u}}_q^{\leq 0}}(W, V)
$$

at all finite-dimensional W in $\mathrm{Rep}(\dot{\mathfrak{u}}_q)$. Now, for finite-dimensional V and W we have the sequence of natural isomorphisms

$$
\begin{aligned}
\mathrm{Hom}_{\dot{\mathfrak{u}}_q}(W, \mathrm{L.ind}(V^*)^*) \ &\cong \mathrm{Hom}_{\dot{\mathfrak{u}}_q}(W^{**}, \mathrm{L.ind}(V^*)^*) \\
&\cong \mathrm{Hom}_{\dot{\mathfrak{u}}_q}(\mathrm{L.ind}(V^*), W^*) \\
&\cong \mathrm{Hom}_{\dot{\mathfrak{u}}_q^{\leq 0}}(V^*, W^*) \cong \mathrm{Hom}_{\dot{\mathfrak{u}}_q^{\leq 0}}(W, V)
\end{aligned}
$$

which realize $\mathrm{L.ind}(V^*)^*$ as a, and hence the, right adjoint to restriction. As a consequence we see that $\mathrm{R.ind}$ preserves finite-dimensional representations, since $\mathrm{L.ind}$ preserves finite-dimensional representations. The identification $\mathrm{L.ind}(V) = \mathrm{R.ind}(V^*)^*$ is established similarly. $\square$

For any $\lambda \in P$, with associated 1-dimensional simple $\dot{\mathfrak{u}}_q^{\leq 0}$-representation k_λ, we take

$$\text{R.ind}(\lambda) = \text{R.ind}(k_\lambda) \quad \text{and} \quad \text{L.ind}(\lambda) = \text{L.ind}(k_\lambda).$$

We have

$$\dim(\text{R.ind}(\lambda)) = \dim(\text{L.ind}(\lambda)) = \dim(\mathfrak{v}_q^+) = \prod_{\gamma \in \Phi^+} l_\gamma$$

at each λ. We note that $\text{L.ind}(\lambda)$ has a 1-dimensional lowest weight space of weight λ, and 1-dimensional highest weight space of weight $\lambda + 2\rho_l$, where $\rho_l = \frac{1}{2}\sum_{\gamma \in \Phi^+}(l_\gamma - 1)\gamma$. Lemma 9.3 now tells us that $\text{R.ind}(\lambda)$ has 1-dimensional highest and lowest weights spaces as well, of weights λ and $\lambda - 2\rho_l$ respectively.

Lemma 9.4. *For any $\lambda \in P$, $\text{R.ind}(\lambda)$ has a unique highest weight vector which is of weight λ. This highest weight vector generates a simple subrepresentation in $\text{R.ind}(\lambda)$, and identifies the socle as*

$$\text{soc}(\text{R.ind}(\lambda)) = \text{the unique simple } \dot{\mathfrak{u}}_q\text{-rep of highest weight } \lambda.$$

Proof. Let $L'(\mu)$ denote the simple $\dot{\mathfrak{u}}_q$-representation of highest weight μ. Highest weight vectors in $\text{R.ind}(\lambda)$, of weight $\mu \in P$ say, are identified with maps $M(\mu) \to \text{R.ind}(\lambda)$ from the corresponding baby Verma module $M(\mu) = \text{L.ind}_{\dot{\mathfrak{u}}_q^{\geq 0}}^{\dot{\mathfrak{u}}_q}(\mu)$. We have by adjunction

$$\text{Hom}_{\dot{\mathfrak{u}}_q}(M(\mu), \text{R.ind}(\lambda)) \cong \text{Hom}_{\dot{\mathfrak{u}}_q^{\leq 0}}(M(\mu), k_\lambda).$$

Since $M(\mu)$ is cyclically generated over $\dot{\mathfrak{u}}_q^{\leq 0}$ by its highest weight space $M(\mu)_\mu \cong k_\mu$ we restrict along the projection $M(\mu) \to k_\mu$ to get

$$\text{Hom}_{\dot{\mathfrak{u}}_q^{\leq 0}}(M(\mu), k_\lambda) \xleftarrow{\sim} \text{Hom}_{\dot{\mathfrak{u}}_q^{\leq 0}}(k_\mu, k_\lambda) = \begin{cases} k & \text{if } \mu = \lambda \\ 0 & \text{otherwise,} \end{cases}$$

and also

$$\text{Hom}_{\dot{\mathfrak{u}}_q^{\leq 0}}(k_\mu, k_\lambda) \xrightarrow{\sim} \text{Hom}_{\dot{\mathfrak{u}}_q^{\leq 0}}(L'(\mu), k_\lambda) \cong \text{Hom}_{\dot{\mathfrak{u}}_q}(L'(\mu), \text{R.ind}(\lambda)).$$

The first expression tells us that $\text{R.ind}(\lambda)$ admits a highest weight vector of weight μ if and only if $\lambda = \mu$. The second expression says that this vector defines a map $L'(\lambda) \to \text{R.ind}(\lambda)$ which is unique up to scaling. $\qquad\square$

We similarly calculate the cosocle for $\text{R.ind}(\lambda)$.

Lemma 9.5. *At any weight λ we have*

$$\text{cosoc}(\text{R.ind}(\lambda)) = \text{the unique simple } \dot{\mathfrak{u}}_q\text{-rep of lowest weight } \lambda - 2\rho_l.$$

Proof. Via duality and Lemma 9.3 it suffices to show that the socle in $\text{L.ind}(\mu)$ is identified with the unique simple $\dot{\mathfrak{u}}_q$-representation of highest weight $\mu + 2\rho_l$, at arbitrary μ. For this we note that $\text{L.ind}(\mu) = \mathfrak{v}_q^+$ as a $\mathfrak{v}_q^+$-representation. So this representation has a unique highest weight vector of weight $\mu + 2\rho_l$ if and only if $\mathfrak{v}_q^+$ has 1-dimensional socle as a module over itself, i.e. 1 dimensional space of $\mathfrak{v}_q^+$-invariants.

Now, any invariant vector in $\mathfrak{v}_q^+$ decomposes into a sum of homogeneous invariant vectors, under the natural Q-grading on $\mathfrak{v}_q^+$, and any homogeneous invariant vector contributes a unique non-vanishing $\mathfrak{v}_q^{\geq 0}$-invariant vector in the free module

$$\mathfrak{v}_q^{\geq 0} = \bigoplus_\chi \mathfrak{v}_q^{\geq 0} \otimes_{k[K_\alpha : \alpha \in \Delta]} k_\chi.$$

Here the χ run over all characters for the (finite abelian) group generated by the units K_α in $\mathfrak{v}_q$. However, we know that the invariants in any finite-dimensional Hopf algebra are 1-dimensional [45, Theorem 2.1.3] [38]. So we have

$$\dim \mathrm{L.ind}(\mu)^{\mathfrak{v}_q^+} = \dim(\mathfrak{v}_q^+)^{\mathfrak{v}_q^+} \leq 1.$$

Since the unique vector v of weight $\mu + 2\rho_l$ in $\mathrm{L.ind}(\mu)$ provides such an invariant vector, we conclude

$$\mathrm{L.ind}(\mu)^{\mathfrak{v}_q^+} = \mathrm{L.ind}(\mu)_{\mu+2\rho} = k \cdot v.$$

Uniqueness of the highest weight vector calculates the socle in $\mathrm{L.ind}(\mu)$ as the simple $\dot{\mathfrak{u}}_q$-rep of highest weight $\mu + 2\rho_l$. $\qquad\square$

Lemma 9.6. *For any weight $\lambda \in P$, the identity map is the unique $\dot{\mathfrak{u}}_q$-linear endomorphism of $\mathrm{R.ind}(\lambda)$, up to scaling.*

Proof. Since $\mathrm{R.ind}(\lambda)$ is of highest weight λ, and $\mathrm{R.ind}(\lambda)_\lambda$ is 1-dimensional, we have

$$\mathrm{End}_{\dot{\mathfrak{u}}_q}(\mathrm{R.ind}(\lambda)) \cong \mathrm{Hom}_{\dot{\mathfrak{u}}_{\tilde{q}}^{\leq 0}}(\mathrm{R.ind}(\lambda), k_\lambda) \cong \mathrm{End}_{\dot{\mathfrak{u}}_{\tilde{q}}^{\leq 0}}(k_\lambda) = k \cdot id.$$

$\qquad\square$

9.4. **The Steinberg representation and $\dot{\mathfrak{u}}_q$-extensions.** We consider the Steinberg representation St in $\mathrm{Rep}(G_q)$, which we recall is the simple representation $\mathrm{St} = L(\rho_l)$ of highest weight $\rho_l = \frac{1}{2} \sum_{\gamma \in \Phi^+} (l_\gamma - 1)\gamma$. By Lemma 9.2, the restriction $\mathrm{St}\,|_{\dot{\mathfrak{u}}_q}$ to $\mathrm{Rep}(\dot{\mathfrak{u}}_q)$ is still simple. It is, more specifically, the unique simple $\dot{\mathfrak{u}}_q$-representation of highest weight ρ_l.

Lemma 9.7 (cf. [49, Theorem 9.10.3]). *At ρ_l we have $\mathrm{R.ind}(\rho_l) \cong \mathrm{St}\,|_{\dot{\mathfrak{u}}_q}$. Furthermore, this object is self-dual.*

Proof. We have $w_0 \rho_l = -\rho_l$ so that St has lowest weight $-\rho_l$ and is self-dual, by Lemma 8.3. Hence its restriction $\mathrm{St}\,|_{\dot{\mathfrak{u}}_q}$ is self-dual as well. So we need only establish the claimed isomorphism $\mathrm{R.ind}(\rho_l) \cong \mathrm{St}\,|_{\dot{\mathfrak{u}}_q}$.

By Lemmas 9.4 and 9.5 there is a nontrivial inclusion $i : \mathrm{St}\,|_{\dot{\mathfrak{u}}_q} \to \mathrm{R.ind}(\rho_l)$ and projection $p : \mathrm{R.ind}(\rho_l) \to \mathrm{St}\,|_{\dot{\mathfrak{u}}_q}$. Since these two representations have 1-dimensional highest weight space of weight ρ_l, both i and p restrict to isomorphisms on the respective highest weight spaces. In particular, the composites ip and pi are both nonzero. Since

$$\dim \mathrm{End}_{\dot{\mathfrak{u}}_q}(\mathrm{St}\,|_{\dot{\mathfrak{u}}_q}) = \dim \mathrm{End}_{\dot{\mathfrak{u}}_q}(\mathrm{R.ind}(\rho_l)) = 1$$

we conclude that both of these composites are isomorphisms, and hence that both i and p are isomorphisms. $\qquad\square$

Lemma 9.8. *We have isomorphisms of $\dot{\mathfrak{u}}_q$-representations*

$$\mathrm{R.ind}(\rho_l) \cong \mathrm{L.ind}(-\rho_l) \cong \mathrm{R.ind}_{\dot{\mathfrak{u}}_{\tilde{q}}^{\geq 0}}^{\dot{\mathfrak{u}}_q}(-\rho_l) \cong \mathrm{L.ind}_{\dot{\mathfrak{u}}_{\tilde{q}}^{\geq 0}}^{\dot{\mathfrak{u}}_q}(\rho_l).$$

Proof. By Lemmas 9.3 and Lemma 9.7 we have

$$\mathrm{L.ind}(-\rho_l) = \mathrm{R.ind}(\rho_l)^* \cong \mathrm{R.ind}(\rho_l).$$

By exchanging q with q^{-1} and considering the algebra isomorphism $\psi : \dot{\mathfrak{u}}_q \xrightarrow{\sim} \dot{\mathfrak{u}}_{q^{-1}}$ which exchanges E_α with F_α, and takes 1_λ to $1_{-\lambda}$, we see that

$$\mathrm{res}_\psi \, \mathrm{R.ind}_{\dot{\mathfrak{u}}_{q^{-1}}^{\leq 0}}^{\dot{\mathfrak{u}}_{q^{-1}}}(\rho_l) = \mathrm{R.ind}_{\dot{\mathfrak{u}}_{\bar{q}}^{\geq 0}}^{\dot{\mathfrak{u}}_q}(-\rho_l).$$

Since $\mathrm{St}\,|_{\dot{\mathfrak{u}}_q}$ is simple with highest and lowest weights ρ_l and $-\rho_l$ respectively we also have

$$\mathrm{St}\,|_{\dot{\mathfrak{u}}_q} = \mathrm{res}_\psi(\mathrm{St}\,|_{\dot{\mathfrak{u}}_{q^{-1}}}) = \mathrm{res}_\psi \, \mathrm{R.ind}_{\dot{\mathfrak{u}}_{q^{-1}}^{\leq 0}}^{\dot{\mathfrak{u}}_{q^{-1}}}(\rho_l).$$

So $\mathrm{R.ind}(\rho_l) \cong \mathrm{St}\,|_{\dot{\mathfrak{u}}_q} \cong \mathrm{R.ind}_{\dot{\mathfrak{u}}_{\bar{q}}^{\geq 0}}^{\dot{\mathfrak{u}}_q}(-\rho_l)$. Finally by fiddling with (self-)dualities again we obtain $\mathrm{R.ind}_{\dot{\mathfrak{u}}_{\bar{q}}^{\geq 0}}^{\dot{\mathfrak{u}}_q}(-\rho_l) \cong \mathrm{L.ind}_{\dot{\mathfrak{u}}_{\bar{q}}^{\geq 0}}^{\dot{\mathfrak{u}}_q}(\rho_l)$. $\qquad\square$

We now observe vanishing of extensions for the Steinberg representation.

Proposition 9.9 (cf. [5, Lemma 4.2]). *For any finite-dimensional $\dot{\mathfrak{u}}_q$-representation V we have*

$$\mathrm{Ext}^1_{\dot{\mathfrak{u}}_q}(\mathrm{St}\,|_{\dot{\mathfrak{u}}_q}, V) = \mathrm{Ext}^1_{\dot{\mathfrak{u}}_q}(V, \mathrm{St}\,|_{\dot{\mathfrak{u}}_q}) = 0.$$

Proof. By duality it suffices to prove $\mathrm{Ext}^1_{\dot{\mathfrak{u}}_q}(V, \mathrm{St}\,|_{\dot{\mathfrak{u}}_q}) = 0$, and by considering exact sequences on cohomology it suffices to establish such vanishing at simple V. Consider an extension

$$0 \to \mathrm{St}\,|_{\dot{\mathfrak{u}}_q} \to W \to V \to 0, \tag{18}$$

with V simple and of highest weight μ. Then, by Lemma 9.4, V has lowest weight μ' with $\mu' \geq \mu - 2\rho_l$.

In the case where $\mu \not\geq \rho_l$, W has a unique nonzero highest weight vector of weight ρ_l, up to scaling, and the $\dot{\mathfrak{u}}_{\bar{q}}^{\leq 0}$-submodule W' in W generated by the all weight spaces W_τ with $\tau \neq \rho_l$ is of codimension 1 with quotient $W/W' \xrightarrow{\sim} W_{\rho_l} \cong k_{\rho_l}$. This projection lifts to a map $W \to \mathrm{R.ind}(\rho_l) = \mathrm{St}\,|_{\dot{\mathfrak{u}}_q}$ which splits the sequence (18). A similar argument splits the sequence when $\mu = \rho_l$.

In the final case where $\mu > \rho_l$, W has 1-dimensional lowest weight space of weight $-\rho_l$. We have the $\dot{\mathfrak{u}}_{\bar{q}}^{\geq 0}$-projection $W \to W_{-\rho_l} \cong k_{-\rho_l}$ and corresponding map $W \to \mathrm{R.ind}_{\dot{\mathfrak{u}}_{\bar{q}}^{\geq 0}}^{\dot{\mathfrak{u}}_q}(-\rho_l) = \mathrm{St}\,|_{\dot{\mathfrak{u}}_q}$, which again splits (18). $\qquad\square$

9.5. Projective and injective $\dot{\mathfrak{u}}_q$-representations.

Lemma 9.10. *The category* $\mathrm{rep}(\dot{\mathfrak{u}}_q)$ *of finite-dimensional* $\dot{\mathfrak{u}}_q$-*representations has enough injectives, and any injective object in* $\mathrm{rep}(\dot{\mathfrak{u}}_q)$ *is injective in* $\mathrm{Rep}(\dot{\mathfrak{u}}_q)$.

Proof. As in [49, pg 118] we consider the right adjoint $I : \mathrm{Vect}_P = \mathrm{Rep}(T_q) \to \mathrm{Rep}(\dot{\mathfrak{u}}_q)$ to the restriction functor $\mathrm{Rep}(\dot{\mathfrak{u}}_q) \to \mathrm{Rep}(T_q)$. We note that I sends injectives to injectives since restriction is exact. By an explicit expression

$$I(V) = \varinjlim_\alpha \mathrm{Hom}_{T_q}(\dot{\mathfrak{u}}_q, V_\alpha) \cong \varinjlim_\alpha \mathrm{Hom}_k(\mathfrak{v}_q^+ \otimes \mathfrak{v}_q^-, V_\alpha) \cong \mathrm{Hom}_k(\mathfrak{v}_q^+ \otimes \mathfrak{v}_q^-, V),$$

where the colimit here is indexed over finite-dimensional subrepresentations $V_\alpha \subseteq V$, we see that I preserves finite-dimensional objects.

Now, since all objects in $\mathrm{Rep}(T_q)$ are injective, the unit of the adjunction $V \to I(V)$ maps any finite-dimensional $\dot{\mathfrak{u}}_q$-representation to a finite-dimensional injective object. Since the restriction functor is faithful, this unit map $V \to I(V)$ has trivial kernel. So we see that $\mathrm{rep}(\dot{\mathfrak{u}}_q)$ has enough injectives, and injectives in $\mathrm{rep}(\dot{\mathfrak{u}}_q)$ are injective in $\mathrm{Rep}(\dot{\mathfrak{u}}_q)$. $\qquad\square$

Proposition 9.11. *The Steinberg representation* St *restricts to a simultaneous projective and injective object in* $\mathrm{Rep}(\dot{\mathfrak{u}}_q)$.

Proof. Proposition 9.9 tells us that $\mathrm{St}\,|_{\dot{\mathfrak{u}}_q}$ is injective in $\mathrm{rep}(\dot{\mathfrak{u}}_q)$. By Lemma 9.10, it follows that $\mathrm{St}\,|_{\dot{\mathfrak{u}}_q}$ is injective in $\mathrm{Rep}(\dot{\mathfrak{u}}_q)$. For projectivity, Proposition 9.9 says that $\mathrm{St}\,|_{\dot{\mathfrak{u}}_q}$ is also projective in $\mathrm{rep}(\dot{\mathfrak{u}}_q)$. For a possibly infinite-dimensional $\dot{\mathfrak{u}}_q$-representation M, we can write M as a filtered colimit $M = \varinjlim_\alpha M_\alpha$ of finite-dimensional subrepresentations. We then employ compactness of $\mathrm{St}\,|_{\dot{\mathfrak{u}}_q}$ in $\mathrm{Rep}(\dot{\mathfrak{u}}_q)$ and exactness of filtered colimits to find

$$\mathrm{Ext}^1_{\dot{\mathfrak{u}}_q}(\mathrm{St}\,|_{\dot{\mathfrak{u}}_q}, M) = \varinjlim_\alpha \mathrm{Ext}^1_{\dot{\mathfrak{u}}_q}(\mathrm{St}\,|_{\dot{\mathfrak{u}}_q}, M_\alpha) \underset{\mathrm{Prop}\ 9.9}{=} 0.$$

Such vanishing ensures projectivity of $\mathrm{St}\,|_{\dot{\mathfrak{u}}_q}$ in $\mathrm{Rep}(\dot{\mathfrak{u}}_q)$. $\qquad\square$

Corollary 9.12. *The categories* $\mathrm{rep}(\dot{\mathfrak{u}}_q)$ *and* $\mathrm{Rep}(\dot{\mathfrak{u}}_q)$ *have enough projectives, and projectives in* $\mathrm{rep}(\dot{\mathfrak{u}}_q)$ *remain projective in* $\mathrm{Rep}(\dot{\mathfrak{u}}_q)$.

Proof. Take $P = (\mathrm{St}\,|_{\dot{\mathfrak{u}}_q}) \otimes (\mathrm{St}\,|_{\dot{\mathfrak{u}}_q}) = (\mathrm{St}\,|_{\dot{\mathfrak{u}}_q})^* \otimes (\mathrm{St}\,|_{\dot{\mathfrak{u}}_q})$. This module is projective and coevaluation provides a surjection $P \to \mathbf{1}$. Tensoring with any representation V provides a surjection $P \otimes V \to V$, and the object $P \otimes V$ is seen to be projective via the existence of the adjunction

$$\mathrm{Hom}_{\dot{\mathfrak{u}}_q}(P \otimes V, -) \cong \mathrm{Hom}_{\dot{\mathfrak{u}}_q}(V, {}^*P \otimes -) \cong \mathrm{Hom}_{\dot{\mathfrak{u}}_q}(V, P \otimes -).$$

To elaborate, $P \otimes -$ is an exact functor which takes injective values in $\mathrm{Rep}(\dot{\mathfrak{u}}_q)$ (see Lemma 2.1). Hence $P \otimes -$ sends exact sequences to split exact sequences, and we observe exactness of the above functor. We note finally that $P \otimes V$ is finite-dimensional whenever V is finite-dimensional to see that $\mathrm{rep}(\dot{\mathfrak{u}}_q)$ has enough projectives. $\qquad\square$

10. Restriction, projectivity, and injectivity for simply-connected G

We prove that the Steinberg representation is both projective and injective in $\mathrm{Rep}(G_q)$, in the simply-connected setting, and that it restricts to a simultaneously projective and injective representation in $\mathrm{Rep}(\bar{u}_q)$. We use this fact to find that the restriction map

$$\mathrm{res}_q^- : \mathrm{Rep}(G_q) \to \mathrm{Rep}(\bar{u}_q)$$

both preserves, and detects, projective and injective objects in $\mathrm{Rep}(G_q)$.

10.1. **Vibe check.** In Section 9 we considered two "induction" functors, the left and right adjoints to restriction $\mathrm{Rep}(\dot{\mathfrak{u}}_q) \to \mathrm{Rep}(\dot{\mathfrak{u}}_q^{\leq 0})$ for torally extended $\mathfrak{v}_q$-representations. We adopted the notations L.ind and R.ind for these functors.

From this point on we only consider *right adjoints* to restriction functors, and any functor which we refer to as *induction* is the right adjoint to some restriction functor.

10.2. **Exactness of induction.** We first recall a standard fact.

Lemma 10.1 ([22, §3.3]). *Let A be a Hopf algebra. For any map of A-module coalgebras $A \to C$, the right adjoint to restriction* $\mathrm{ind} : \mathrm{Corep}(C) \to \mathrm{Corep}(A)$ *is a map of* $\mathrm{Corep}(A)$-*module categories.*

Proof. Since the trivial corepresentation $\mathbf{1}$ is compact in $\mathrm{Corep}(C)$, the functor
$$\mathrm{ind}(-) = \mathrm{Hom}_C(\mathbf{1}, - \otimes A) = (- \otimes A)^C = -\,\square_C A$$
[18, Proposition 6] commutes with filtered colimits. So it suffices to establish the proposed natural isomorphism $\mathrm{ind}(- \otimes \mathrm{res}(V)) \cong \mathrm{ind}(-) \otimes V$ at finite-dimensional V. But now the claim follows by duality, since we have the sequence of natural identifications

$$\begin{aligned}
\mathrm{Hom}_C(\mathrm{res}(-), - \otimes \mathrm{res}(V)) \;\; &\cong \mathrm{Hom}_C(\mathrm{res}(- \otimes {}^*V), -)\\
&\cong \mathrm{Hom}_A(- \otimes {}^*V, \mathrm{ind}(-)) \cong \mathrm{Hom}_A(-, \mathrm{ind}(-) \otimes V).
\end{aligned}$$
$\square$

Via Tannakian reconstruction, Lemma 10.1 tells us that the right adjoints to the restriction functors
$$\mathrm{Rep}(G_q) \to \mathrm{Rep}(\dot{\mathfrak{u}}_q), \quad \mathrm{Rep}(\dot{\mathfrak{u}}_q) \to \mathrm{Rep}(\bar{u}_q), \quad \text{and} \quad \mathrm{Rep}(G_q) \to \mathrm{Rep}(\bar{u}_q)$$
are naturally $\mathrm{Rep}(G_q)$-linear, $\mathrm{Rep}(\dot{\mathfrak{u}}_q)$-linear, and again $\mathrm{Rep}(G_q)$-linear, respectively.

Proposition 10.2. *Let G be simply-connected. The right adjoint to restriction $\mathrm{Rep}(\bar{u}_q) \to \mathrm{Rep}(\dot{\mathfrak{u}}_q)$ is faithfully exact.*

Proof. We can just write down the adjoint. For any $\bar{u}_q$-representation V there is a unique lift of V to a P-graded vector space $\mathrm{lift}(V)$ whose P-grading is supported on P_l. We then take the orbit under the action of the simples k_λ associated to $\lambda \in P^*$ to get the P-graded vector space
$$\mathrm{ind}(V) = \bigoplus_{\lambda \in P^*} k_\lambda \otimes \mathrm{lift}(V).$$
This graded vector space carries a natural $\dot{\mathfrak{u}}_q$-action specified by the formulae
$$\mathbf{E}_\alpha \cdot (\zeta_\lambda \otimes v) = \zeta_{\lambda'} \otimes (\mathbf{E}_\alpha \cdot v) \quad \text{and} \quad F_\alpha \cdot (\zeta_\lambda \otimes v) = \zeta_{\lambda''} \otimes (F_\alpha \cdot v),$$
where
$$\lambda' = \lambda + \deg(v) + \alpha - \deg(\mathbf{E}_\alpha \cdot v) \quad \text{and} \quad \lambda'' = \lambda + \deg(v) - \alpha - \deg(F_\alpha \cdot v).$$
This operation is functorial, is seen directly to provide a right adjoint to restriction, and is faithful and exact by construction. $\square$

Proposition 10.3. *Let G be simply-connected. The right adjoint to restriction $\mathrm{Rep}(\dot{\mathfrak{u}}_q) \to \mathrm{Rep}(G_q)$ is exact.*

We proceed as in the proof of [4, Theorem 4.8].

Proof. Let $\mathbf{1} \xrightarrow{\sim} E$ be a bounded below resolution of $\mathbf{1}$ in $\mathrm{Rep}(G_q)$ by objects of the form $E^i = \bar{E}^i \otimes \mathrm{St}$, where $\bar{E}^i$ is any representation in $\mathrm{Rep}(G_q)$. Then E restricts to an injective resolution of the unit $\mathbf{1} \to E|_{\dot{\mathfrak{u}}_q}$ in $\mathrm{Rep}(\dot{\mathfrak{u}}_q)$. For any V in $\mathrm{Rep}(\dot{\mathfrak{u}}_q)$ we have the injective resolution $V \otimes E|_{\dot{\mathfrak{u}}_q}$ and calculate the higher derived functors of induction as
$$\begin{aligned}
\mathrm{ind}^{>0}(V) \;\; &= H^{>0}\big(\mathrm{ind}(V \otimes E|_{\dot{\mathfrak{u}}_q})\big)\\
&\cong H^{>0}\big(\mathrm{ind}(V) \otimes E\big) = \mathrm{ind}(V) \otimes H^{>0}(E) = 0.
\end{aligned}$$
Since the higher derived functors vanish, ind is exact. $\square$

We combine Propositions 10.2 and 10.3 to see that induction from the smallest quantum algebra to $\mathrm{Rep}(G_q)$ is also exact.

Theorem 10.4. *Let G be simply-connected. The right adjoint to restriction $\mathrm{Rep}(\bar{u}_q) \to \mathrm{Rep}(G_q)$ is faithfully exact.*

Proof. The restriction functor res_q^- factors through $\mathrm{Rep}(\dot{\mathfrak{u}}_q)$. Hence the right adjoint to res_q^- factors as

$$\mathrm{Rep}(\bar{u}_q) \xrightarrow{\mathrm{ind}_1} \mathrm{Rep}(\dot{\mathfrak{u}}_q) \xrightarrow{\mathrm{ind}_2} \mathrm{Rep}(G_q).$$

So exactness follows immediately from Propositions 10.2 and 10.3.

Via exactness, we can prove faithfulness by observing non-vanishing of $\mathrm{ind}(L)$ for each simple L in $\mathrm{Rep}(\bar{u}_q)$. However, each simple L in $\mathrm{Rep}(\bar{u}_q)$ is the image of some simple L' in $\mathrm{Rep}(G_q)$, by Corollary 8.6. So we have

$$0 \neq \mathrm{Hom}_{\bar{u}_q}\left(\mathrm{res}_q^-(L'), L\right) = \mathrm{Hom}_{\bar{u}_q}\left(L', \mathrm{ind}(L)\right).$$

In particular, $\mathrm{ind}(L)$ is nonzero. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 10.5. For q with odd order scalar parameters, Theorem 10.4 appears in [5, Theorem 4.8].

10.3. **Projectives and injectives.**

Proposition 10.6. *Let G be simply-connected. A G_q-representation is injective in $\mathrm{Rep}(G_q)$ if and only if its image in $\mathrm{Rep}(\bar{u}_q)$ is injective.*

Proof. First note that the induction functor $\mathrm{ind} : \mathrm{Rep}(\bar{u}_q) \to \mathrm{Rep}(G_q)$ satisfies

$$\mathrm{ind}(\mathbf{1}) = \mathbf{1} \square_{(\bar{u}_q)^*} \mathscr{O}(G_q) = {}^{\bar{u}_q^*}\mathscr{O}(G_q).$$

By Corollary 7.5 this object is just $\mathscr{O}(G_\varepsilon^*)$, and in particular lies in the semisimple subcategory $\mathrm{Rep}(G_\varepsilon^*) = \ker(\mathrm{res}_q^-)$ in $\mathrm{Rep}(G_q)$. Hence the unit map of the adjunction $\mathbf{1} \to \mathrm{ind}(\mathbf{1})$ is split. It follows that $\mathrm{ind}(\mathbf{1})$ contains $\mathbf{1}$ as a summand, and that each G_q-representation V appears as a summand in the induction $\mathrm{ind}(\mathrm{res}_q^- V) \cong \mathrm{ind}(\mathbf{1}) \otimes V$.

Suppose first that V has injective image in $\mathrm{Rep}(\bar{u}_q)$. Since V appears as a summand in $\mathrm{ind}(\mathbf{1}) \otimes V$, it suffices to show that $\mathrm{ind}(\mathbf{1}) \otimes V$ is injective in $\mathrm{Rep}(G_q)$. We have the natural identification

$$\mathrm{Hom}_{G_q}(-, \mathrm{ind}(\mathbf{1}) \otimes V) \cong \mathrm{Hom}_{G_q}(-, \mathrm{ind}(\mathrm{res}_q^- V)) \cong \mathrm{Hom}_{\bar{u}_q}(\mathrm{res}_q^- -, \mathrm{res}_q^- V),$$

from which we conclude exactness of the functor $\mathrm{Hom}_{G_q}(-, \mathrm{ind}(\mathbf{1}) \otimes V)$. It follows that $\mathrm{ind}(\mathbf{1}) \otimes V$ is injective, and hence that V is injective.

Suppose conversely that V is injective in $\mathrm{Rep}(G_q)$. To verify injectivity of $\mathrm{res}_q^-(V)$ in $\mathrm{Rep}(\bar{u}_q)$, we show that the extensions $\mathrm{Ext}_{\bar{u}_q}^{>0}(L, \mathrm{res}_q^- V)$ vanish for each simple L in $\mathrm{Rep}(\bar{u}_q)$. By Corollary 8.6, we can find a representation L' in $\mathrm{Rep}(G_q)$ with $\mathrm{res}_q^-(L') = L$, and by exactness of induction we obtain an identification

$$\mathrm{Ext}_{\bar{u}_q}^{>0}(L, \mathrm{res}_q^- V) \cong \mathrm{Ext}_{G_q}^{>0}(L', \mathrm{ind}(\mathrm{res}_q^- V)) \cong \mathrm{Ext}_{G_q}^{>0}(L', \mathrm{ind}(\mathbf{1}) \otimes V). \qquad (19)$$

Since V is injective, the product $\mathrm{ind}(\mathbf{1}) \otimes V$ is injective in $\mathrm{Rep}(G_q)$. Hence the extensions (19) vanish. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

For projectivity, we see most clearly that one can detect *finite-dimensional* projectives via restriction, though we remove this constraint in a moment.

Lemma 10.7. *The restriction functor $\mathrm{Rep}(G_q) \to \mathrm{Rep}(\bar{u}_q)$ preserves, and detects, finite-dimensional projective objects.*

Proof. Suppose that V is projective in $\mathrm{Rep}(G_q)$. Then the functor

$$\mathrm{Hom}_{\bar{u}_q}(\mathrm{res}_q^-(V), -) = \mathrm{Hom}_{G_q}(V, \mathrm{ind}(-))$$

is exact by Theorem 10.4. So $\mathrm{res}_q^-(V)$ is projective in $\mathrm{Rep}(\bar{u}_q)$.

Suppose conversely that V is finite-dimensional and has projective image in $\mathrm{Rep}(\bar{u}_q)$. We have an identification

$$\mathrm{Hom}_{G_q}(V, -) = \mathrm{Hom}_{G_\varepsilon^*}(\mathbf{1}, \mathrm{Hom}_{\bar{u}_q}(\mathrm{res}_q^- V, -)).$$

(Note that we employ finiteness of V here to ensure that, at any G_q-representation W, the G_ε^*-representation $\mathrm{Hom}_{\bar{u}_q}(\mathrm{res}_q^- V, W)$ is integrable.) This functor is exact by projectivity of $\mathrm{res}_q^-(V)$ and semisimplicity of the category $\mathrm{Rep}(G_\varepsilon^*)$. □

By completely similar arguments, with G_ε^* and Proposition 7.1 replaced by T_ε^* and Lemma 9.1, respectively, we obtain toral analogs of Proposition 10.6 and Lemma 10.7.

Proposition 10.8. *An object V in $\mathrm{Rep}(\dot{u}_q)$ is injective if and only if its image in $\mathrm{Rep}(\bar{u}_q)$ is injective. A finite-dimensional object V in $\mathrm{Rep}(\dot{u}_q)$ is projective if and only if its image in $\mathrm{Rep}(\bar{u}_q)$ is projective.*

We recall that the Steinberg representation is projective and injective in $\mathrm{Rep}(\dot{u}_q)$, by Proposition 9.11. So Proposition 10.8 implies projectivity and injectivity of St over $\bar{u}_q$. We recall also that St has simple restriction to $\bar{u}_q$, by Proposition 8.5.

Corollary 10.9. *For simply-connected G, the Steinberg representation St is projective, injective, and simple over $\bar{u}_q$.*

We now "go backwards" to observe projectivity and injectivity of St over G_q.

Corollary 10.10 ([3]). *Let G be simply-connected. The Steinberg representation St is both projective and injective in $\mathrm{Rep}(G_q)$, the subcategory $\mathrm{rep}(G_q)$ of finite-dimensional G_q-representations has enough projectives, and any projective in $\mathrm{rep}(G_q)$ is projective in $\mathrm{Rep}(G_q)$.*

Proof. The first claim follows by the three previous results. The subsequent claims are established just as in the proof of Corollary 9.12. □

We now understand infinite-dimensional projectives in $\mathrm{Rep}(G_q)$ via the following observation.

Lemma 10.11. *Let G be simply-connected. An object V in $\mathrm{Rep}(G_q)$ is projective if and only if V is a summand of $V \otimes \mathrm{St} \otimes \mathrm{St}$. Similarly, V is injective if and only if V is a summand of $V \otimes \mathrm{St} \otimes \mathrm{St}$.*

Proof. If I is an injective representation then $W \otimes I$ is injective for all W (see Lemma 2.1). So all products $W \otimes \mathrm{St}$ are injective. Furthermore, tensoring with the coevaluation map $\mathbf{1} \to \mathrm{St} \otimes \mathrm{St}^* = \mathrm{St} \otimes \mathrm{St}$ provides an injection $V \to V \otimes \mathrm{St} \otimes \mathrm{St}$ at arbitrary V. In the case that V is injective, this injection splits, so that V is a summand of the given product. Furthermore, since the class of injectives is closed under taking summands, we see that this property characterizes injectives.

For projectives, we first claim that $W \otimes \mathrm{St}$ is projective for any W. This is clear from self-duality of St, the adjunction

$$\mathrm{Hom}_{G_q}(W \otimes \mathrm{St}, -) \cong \mathrm{Hom}_{G_q}(W, - \otimes \mathrm{St}^*) \cong \mathrm{Hom}_{G_q}(W, - \otimes \mathrm{St}),$$

and the fact that $-\otimes \mathrm{St}$ is an exact endomorphism to the subcategory of injectives in $\mathrm{Rep}(G_q)$. Given this fact, we have the evaluation $\mathrm{St} \otimes \mathrm{St} = \mathrm{St}^* \otimes \mathrm{St} \to \mathbf{1}$ which provides a surjection $V \otimes \mathrm{St} \otimes \mathrm{St} \to V$ at arbitrary V. When V is projective, this surjection splits, and conversely splitting of this map implies projectivity of V. So we establish the claimed result. $\square$

As a corollary we see that the category $\mathrm{Rep}(G_q)$ is Frobenius, in the sense that its projectives and injectives agree.

10.4. Simply-connected conclusions. We collect our findings.

Theorem 10.12. *Let G be simply-connected. Consider the dominant weight $\rho_l = \frac{1}{2}\sum_{\gamma \in \Phi^+}(l_\gamma - 1)\gamma$ and the corresponding simple G_q-representation $\mathrm{St} = L(\rho_l)$.*

(1) *The representation St is both projective and injective in $\mathrm{Rep}(G_q)$.*

(2) *The representation St restricts to an object which is simultaneously projective, injective, and simple in $\mathrm{Rep}(\bar{u}_q)$.*

(3) *The category $\mathrm{Rep}(G_q)$ has enough projectives and injectives, and is Frobenius.*

(4) *The subcategory $\mathrm{rep}(G_q)$ of finite-dimensional G_q-representations has enough projectives and injectives, and is Frobenius.*

(5) *The inclusion $\mathrm{rep}(G_q) \to \mathrm{Rep}(G_q)$ preserves (and detects) projective and injective objects.*

(6) *An object in $\mathrm{Rep}(G_q)$ is projective and injective if and only if its restriction to $\mathrm{Rep}(\bar{u}_q)$ is projective and injective.*

11. RESTRICTION, PROJECTIVITY, AND INJECTIVITY FOR GENERAL G

In this final section we generalize Theorem 10.12 to arbitrary semisimple groups.

11.1. Homological algebra along the quotient $G_q^{sc} \to G_q$.

Lemma 11.1. *Let G be a semisimple algebraic group, and let G^{sc} be the simply-connected form for G.*

(1) *The categories $\mathrm{rep}(G_q)$ and $\mathrm{Rep}(G_q)$ have enough projectives and injectives.*

(2) *The inclusion functor $\mathrm{rep}(G_q) \to \mathrm{Rep}(G_q)$ preserves (and detects) projective and injective objects.*

(3) *An object in $\mathrm{Rep}(G_q)$ is projective if and only if it is injective.*

(4) *An object in $\mathrm{Rep}(G_q)$ is projective and injective if and only if its image in $\mathrm{Rep}(G_q^{sc})$ is projective and injective.*

Proof. (4) Note that $\mathrm{Rep}(G_q)$ is equal to the full subcategory of representations in $\mathrm{Rep}(G_q^{sc})$ whose P-grading is supported on the character lattice X for G. We let $F : \mathrm{Rep}(G_q) \to \mathrm{Rep}(G_q^{sc})$ denote the corresponding monoidal embedding and consider the group of characters $\Sigma = (P/X)^\vee$. This group acts naturally on G_q^{sc}-representations, and the invariants W^Σ in any G_q^{sc}-representation W produce a maximal G_q-subrepresentation in W. Indeed, taking Σ-invariants simply isolates the subrepresentation in W consisting of those vectors which are supported on the sublattice X in P.

We have the natural G_q^{sc}-linear inclusion $i_W : W^\Sigma \to W$ and projection $p_W : W \to W^\Sigma$, and restrictions along i_W and p_W provide adjunctions

$$\mathrm{Hom}_{G_q^{sc}}(F-, W) = \mathrm{Hom}_{G_q}(-, W^\Sigma) \quad \text{and} \quad \mathrm{Hom}_{G_q^{sc}}(W, F-) = \mathrm{Hom}_{G_q}(W^\Sigma, -). \tag{20}$$

Since taking Σ-invariants is an exact functor, the above adjunctions imply that a G_q-representation P has projective (resp. injective) image in $\mathrm{Rep}(G_q^{sc})$ whenever P is projective (resp. injective) in $\mathrm{Rep}(G_q)$.

Conversely, fully faithfulness of the embedding F implies that a given representation P over G_q is projective (resp. injective) in $\mathrm{Rep}(G_q)$ provided it has projective (resp. injective) image in $\mathrm{Rep}(G_q^{sc})$. We therefore establish (4).

(1) Consider any G_q-representation V and any surjection $P \to FV$ from a projective in $\mathrm{Rep}(G_q^{sc})$. By Theorem 10.12, such P exists and can be taken to be finite-dimensional when V is finite-dimensional. By the adjunction (20) we find that the invariants P^Σ are projective in $\mathrm{Rep}(G_q)$, and the induced surjection $P^\Sigma \to V$ provides a projective covering of V. One similarly argues with injectives, so that we obtain (1). Properties (2) and (3) are now inherited from $\mathrm{Rep}(G_q^{sc})$ via (1), (4), and Theorem 10.12. $\square$

For the smallest quantum algebra, we have the inclusion $X \to P$ and recover X^* as the intersection $X^* = X \cap P^*$. So this inclusion reduces to an inclusion on the quotients $X/X^* \to P/P^*$. We therefore locate $\mathrm{Rep}(\bar{u}_q)$ in its simply-connected form $\mathrm{Rep}(\bar{u}_q^{sc})$ as the full subcategory of $\bar{u}_q^{sc}$-representations whose P/P^*-grading is supported on the subgroup X/X^*.

Lemma 11.2. *Consider the smallest quantum algebra $\bar{u}_q^{sc}$ for G_q^{sc}, and the inclusion $\mathrm{Rep}(\bar{u}_q) \to \mathrm{Rep}(\bar{u}_q^{sc})$. A $\bar{u}_q$-representation is projective (resp. injective) if and only if its image in $\mathrm{Rep}(\bar{u}_q^{sc})$ is projective (resp. injective).*

Proof. Since the inclusion $\mathrm{Rep}(\bar{u}_q) \to \mathrm{Rep}(\bar{u}_q^{sc})$ is exact and fully faithful, any $\bar{u}_q$-representation V whose image in $\mathrm{Rep}(\bar{u}_q^{sc})$ is projective (resp. injective) must be projective (resp. injective) in $\mathrm{Rep}(\bar{u}_q)$.

For the converse, take $\mathrm{K}^\vee = \mathrm{coker}(X/X^* \to P/P^*)$ and consider the central subgroup $\mathrm{K} = (\mathrm{K}^\vee)^\vee \subseteq \bar{u}_q^{sc}$. For any $\bar{u}_q$-representation V and $\bar{u}_q^{sc}$-representation W, we have natural identifications

$$\mathrm{Hom}_{\bar{u}_q^{sc}}(V, W) = \mathrm{Hom}_{\bar{u}_q}(V, W^\mathrm{K}) \quad \text{and} \quad \mathrm{Hom}_{\bar{u}_q^{sc}}(W, V) = \mathrm{Hom}_{\bar{u}_q}(W^\mathrm{K}, V).$$

These identifications are obtained by composing with the natural inclusion $W^\mathrm{K} \to W$ and projection $W \to W^\mathrm{K}$ respectively. Since K is finite, taking K-invariants is an exact functor, and we conclude that V is projective (resp. injective) in $\mathrm{Rep}(\bar{u}_q^{sc})$ whenever V is projective (resp. injective) in $\mathrm{Rep}(\bar{u}_q)$. $\square$

11.2. **Semisimple algebraic groups.** We apply Lemmas 11.1 and 11.2 to obtain a version of Theorem 10.12 for general G.

Theorem 11.3. *Let G be a semisimple algebraic group and q be an arbitrary quantum parameter. The following hold:*

(1) *The category $\mathrm{Rep}(\bar{u}_q)$ is Frobenius.*

(2) *The category $\mathrm{Rep}(G_q)$ has enough projectives and injectives, and is Frobenius.*

(3) *The subcategory* $\mathrm{rep}(G_q)$ *of finite-dimensional representations has enough projectives and injectives, and is Frobenius.*

(4) *The inclusion* $\mathrm{rep}(G_q) \to \mathrm{Rep}(G_q)$ *preserves (and detects) projective and injective objects.*

(5) *An object in* $\mathrm{Rep}(G_q)$ *is projective and injective if and only if its restriction to* $\mathrm{Rep}(\bar{u}_q)$ *is projective and injective.*

Proof. Points (2)–(4) were already covered in Lemma 11.1. Point (5) follows from Theorem 10.12 (6) in conjunction with Lemma 11.1 (5) and Lemma 11.2. Point (1) follows from (2), (5), and Corollary 8.6 in the simply-connected case. The general case then follows by Lemma 11.2. Alternatively, we observe (1) via Skryabin's Theorem [53, Theorem 6.1]. □

11.3. **An incomplete accounting of the literature.** As we noted in the preamble to Section 9, the Steinberg representation has a long and storied history in both quantum and modular representation theory. We recall some of the quantum hits below.

In [49] Parshall and Wang show that, for $G = \mathrm{SL}(n)$ and q of odd order, the Steinberg representation is projective and injective in $\mathrm{Rep}(\mathrm{SL}(n)_q)$, and also in $\mathrm{Rep}(\bar{u}_q)$ [49, Theorem 9.10.3, Corollary 9.10.4]. Their proof relies on an analysis of a gaggle of induction functors, between various quantum groups, and a reciprocity theorem which essentially counts the lengths of indecomposable injectives in $\mathrm{Rep}(\dot{u}_q)$ [49, Proposition 9.8.4]. As the authors note, their methods are directly inspired by known phenomena in modular representation theory. See for example [34, Ch 10 & 11].

In [4] Andersen, Polo, and Wen prove that the Steinberg representation is projective and injective in $\mathrm{Rep}(G_q)$, but now for arbitrary simply-connected G and q of a prime power order [4, Lemma 6.6, Theorem 9.8]. In this context the authors rely on a rigorous analysis of induction from the quantum Borel $H^*(G_q/B_q, -)$, and a linkage principle for G_q-representations [4, § 8]. In a subsequent text [5] Andersen, Polo, and Wen prove projectivity and injectivity of St over the small quantum group, under essentially the same restrictions on G and q [5, Theorem 4.3]. In [5] it is also shown that the induction functor from the small quantum group $\mathrm{Rep}(\bar{u}_q) \to \mathrm{Rep}(G_q)$ is exact and faithful [5, Theorem 4.8].

In [6] Andersen and Wen again work at q of prime power order and provide analyses which parallel those of [4, 5], but now in mixed characteristic. In [2] Andersen addresses induction and tilting modules for simply-connected G at q of odd order which is greater than the Coxeter number. In [3] Andersen returns to this topic, and proves a linkage principle for the big quantum group G_q at simply-connected G and arbitrary q. As suggested in [3, pg 14], one can argue from the strong linkage principle that St is both projective and injective in $\mathrm{Rep}(G_q)$ [1], though the details are not entirely trivial and do not appear in the referenced text.

The papers [5, 6, 3] provide the most recent treatments of these topics which are relatively general and completely rigorous, as far as I know. Since [5, 6] it has often been proposed, either implicitly or explicitly, that the methods of these texts generalize immediately to most quantum parameters, most odd quantum parameters, or all parameters q. (Consider, for example, the curious appearance of the Steinberg representation in [47, proof of Lemma 4.1].)

In reconsidering this topic however, the transition from the odd order case to the even order case did not seem to be completely transparent. Hence we produce this text.

APPENDIX A. BRAID GROUP ACTION ON $\mathfrak{v}_q$

For this section $\mathfrak{v}_q$ specifically denotes the subalgebra in U_q generated by the simple root vectors E_α and F_α with $l_\alpha > 1$, the grouplikes K_ν, and the distinguished generators from Propositions 6.2 and 6.4 at low order q.

Our goal is to show that the subalgebra $\mathfrak{v}_q$ is preserved under the action of the braid group operators on U_q. It will follow that $\mathfrak{v}_q$ is the smallest subalgebra in U_q which contains all of the simple root vectors E_α and F_α with $l_\alpha > 1$, all of the grouplikes K_ν, and is stable under the action of the braid group $B_\mathfrak{g}$, i.e. that $\mathfrak{v}_q$ is identified with the subalgebra of (9).

Let us be clear here that our braid group $B_\mathfrak{g}$, for $\mathfrak{g} = \mathrm{Lie}(G)$, acts on U_q via the specific operators

$$T_\alpha = T''_{\alpha,1}$$

from [43, §37].

Remark A.1. The reader should note that the usual braid group [8] (only) appears in type A_n, from the perspective of the works [43, 44]. The "braid group" we are referring to here is a group $B_\mathfrak{g}$ which depends on the Dynkin type of $\mathfrak{g}$. Specifically, the operators T_α in $B_\mathfrak{g}$ are only supposed to satisfy the relations

$$T_{\alpha_m} \ldots T_{\alpha_1} = T_{\beta_m} \ldots T_{\beta_1} \ \text{ whenever } \prod_i \sigma_{\alpha_i} = \prod_i \sigma_{\beta_i} \text{ and } \mathrm{length}(\sigma) = m.$$

A.1. **Reducing to almost-simple G at the lacing number.** When each l_i is greater than the lacing number for its associated almost-simple factor in G, the algebra $\mathfrak{v}_q$ is seen to be stable under the action of $B_\mathfrak{g}$ by direct inspection. Namely, for each pair of simple roots α and β the explicit expression of $T_\beta(E_\alpha)$ from [43] shows that $T_\beta(E_\alpha)$ is in the subalgebra generated by the simple E_ν in U_q. The same is true of the F's.

According to this information, we need only check the case where some of the l_i's are less than or equal to the lacing numbers for their associated factors in G. The problem then reduces to considering almost-simple G at such low order parameters. We deal with these particular almost-simple cases below.

A.2. **Some lemmas.**

Lemma A.2. *If β is a simple root with $q_\beta = -1$, then $T_\beta = T_\beta^{-1}$.*

Proof. In this case $\mathsf{K}_\beta = \mathsf{K}_\beta^{-1}$, and K_β centralizes both E_β and F_β. So the result follows from the explicit expression of T_β^{-1} [44, §37.1.3] and the fact that $(-1)^r q_\beta^{\pm r} = 1$ at every integer index r. $\square$

We translate between computations for the E's and F's via a "symmetry" on the quantum enveloping algebra. Specifically, we observe the following braid group equivariant isomorphism.

Lemma A.3 ([43]). *At an arbitrary parameter q, there is an anti-algebra isomorphism $\phi : U_{q^{-1}} \to U_q$,*

$$\phi(E_\alpha^{(n)}) = F_\alpha^{(n)}, \ \ \phi(F_\alpha^{(n)}) = E_\alpha^{(n)}, \ \ \phi(\mathsf{K}_\gamma) = \mathsf{K}_\gamma^{-1},$$

with $T_\beta \phi(x) = \phi T_\beta(x)$ at each simple β and $x \in U_q$.

This anti-algebra isomorphism restricts to an isomorphism between $\mathfrak{v}_{q^{-1}}$ and $\mathfrak{v}_q$.

Proof. The first claim is immediate from the explicit expressions of the braid group operators [44, §37.1.3]. As for $\mathfrak{v}_q$, the distinguished generators are obtained by applying braid group operators the the E_α and F_α with $l_\alpha > 1$. So one sees that ϕ maps the generators for $\mathfrak{v}_{q^{-1}}$ bijectively to the generators for $\mathfrak{v}_q$. $\qquad\square$

A.3. Types B, C and F.

Lemma A.4. *When G is of type B_n, C_n, or F_4, and $l = 2$, the algebra $\mathfrak{v}_q$ is stable under the action of the braid group $B_{\mathfrak{g}}$.*

Proof. The main arguments for types B_n and C_n already appear in the arguments for type F_4. So we only cover the F_4 case.

Enumerate the simple roots as $\{\beta_4, \beta_3, \alpha_2, \alpha_1\}$, with the β_i long, the α_i short, and consecutive roots neighbors in the Dynkin diagram. Take $E_i = E_{\alpha_i}$ at $i < 3$ and $E_j = E_{\beta_j}$ at $j \geq 3$. Consider the subalgebras $\mathfrak{v}_q^2$, $\mathfrak{v}_q^3$, $\mathfrak{v}_q^4$ in $\mathfrak{v}_q$ defined by

$$\mathfrak{v}_q^2 = \text{the subalg gen'd by all } K\text{'s, } E_1, E_2, F_1, \text{ and } F_2.$$

$$\mathfrak{v}_q^3 = \text{the subalg gen'd by } \mathfrak{v}_q^2 \text{ and } E_{32}, F_{32},$$

$$\mathfrak{v}_q^4 = \text{the subalg gen'd by } \mathfrak{v}_q^3 \text{ and } E_{432}, F_{432} = \mathfrak{v}_q.$$

Take also $\mathfrak{v}_q^5 = \mathfrak{v}_q$. It suffices to show that $T_\nu(\mathfrak{v}_q^j) \subseteq \mathfrak{v}_q^{j+1}$ for each simple ν and all $j \leq 4$.

We let $T_i = T_{\alpha_i}$, or T_{β_i} when appropriate. In the case $j = 2$, $\mathfrak{v}_q^2$ is immediately seen to be stable under the actions of T_1 and T_2, and is invariant under T_4. So we need only check T_3, and since the K's are stable under the T_i we need only check that T_3 applied to the E's and F's lie in $\mathfrak{v}_q^3$. The vectors E_1 and F_1 are T_3 invariant, and

$$T_3(E_2) = E_{32}, \ T_3(F_2) = F_{32} \ \in \ \mathfrak{v}_q^3.$$

We're done.

For the T_i applied to $\mathfrak{v}_q^3$, we need only check the values of these operations on the new vectors E_{32} and F_{32}. One sees directly from the definitions, or the braid group relations, that $T_i T_j(E_r) = T_j T_i(E_r)$ and $T_i T_j(F_r) = T_j T_i(F_r)$ whenever $|i - j| > 1$. So we have

$$T_1(E_{32}) = T_1 T_3(E_2) = T_3 T_1(E_2) \in T_3(\mathfrak{v}_q^2) \subseteq \mathfrak{v}_q^3 \subseteq \mathfrak{v}_q^4,$$

and similarly find $T_1(F_{32}) \in \mathfrak{v}_q^4$. Via the rank 2 calculation [44, §39.2.3 (b)], and the fact that $T_3^{-1} = T_3$ by Lemma A.2, we have also

$$T_2(E_{32}) = T_3 T_3 T_2 T_3(E_2) = T_3(E_2) = E_{32} \ \text{ and similarly } \ T_2(F_{32}) = F_{32} \in \mathfrak{v}_q^3.$$

Here we've used [44, §39.2.3] specifically for the calculation $T_3 T_2 T_3(E_2) = E_2$.

For T_3, we have $T_3^2 = id$ by Lemma A.2 so that

$$T_3(E_{32}) = T_3^2(E_2) = E_2 \ \text{ and } \ T_3(F_{32}) = T_3^2(F_2) = F_2.$$

For T_4 we have simply $T_4(E_{32}) = E_{432}$ and $T_4(F_{32}) = F_{432}$. These elements are all in $\mathfrak{v}_q^4$, so that $T_i(\mathfrak{v}_q^3) \subseteq \mathfrak{v}_q^4$ at all i.

For $\mathfrak{v}_q^4$ we need to check

$$T_i(E_{432}), \ T_i(F_{432}) \ \in \mathfrak{v}_q^4 \ \text{ for all } i.$$

For $i = 1$ or 2 we have

$$T_i(E_{432}) = T_i T_4(E_{32}) = T_4 T_i(E_{32}) \in T_4(\mathfrak{v}_q^3) \subseteq \mathfrak{v}_q^4.$$

For $i = 4$ we have $T_4 = T_4^{-1}$ by Lemma A.2 so that $T_4(E_{432}) = T_4^2(E_{32}) = E_{32} \in \mathfrak{v}_q^4$.
For $i = 3$ we apply the braid group relations [44, Theorem 39.4.3] to get

$$T_3(E_{432}) = T_3 T_4 T_3(E_2) = T_4 T_3 T_4(E_2) = T_4(E_{32}) = E_{432}.$$

The results for F_{432} are obtained similarly. $\qquad\square$

A.4. **Type G_2.**

Lemma A.5. *For G of type G_2, and $l = 3$, the algebra $\mathfrak{v}_q$ is stable under the braid group action.*

Proof. Write $\Delta = \{\beta, \alpha\}$ with α short and β long. The algebra $\mathfrak{v}_q$ is generated by E_α, $E_{\beta+\alpha} = T_\beta(E_\alpha)$, and the corresponding F's. We have $T_\beta(E_\alpha) = E_{\beta+\alpha} \in \mathfrak{v}_q$ already and

$$T_\beta(E_{\beta+\alpha}) = T_\beta^2(E_\alpha) = E_\alpha \ \in \ \mathfrak{v}_q$$

by Lemma A.2, since $q_\beta = -1$. For α we have $T_\alpha(E_\alpha) = -F_\alpha K_\alpha$ and

$$T_\alpha(E_{\beta+\alpha}) = E_\alpha^{(2)} E_\beta - q_\alpha^{-2} E_\alpha E_\beta E_\alpha + E_\beta E_\alpha^{(2)} \ \in \ \mathfrak{v}_q,$$

by [44, §39.2.2]. This shows $T_\nu(\mathfrak{v}_q^+) \subseteq \mathfrak{v}_q$ at all simple ν and, via symmetry, we obtain $T_\nu(\mathfrak{v}_q^-) \subseteq \mathfrak{v}_q$ as well. $\qquad\square$

Lemma A.6. *For G of type G_2, and $l = 2$, the algebra $\mathfrak{v}_q$ is stable under the braid group action.*

Proof. We recall

$$E_{2\alpha+\beta} = E_\alpha^{(2)} E_\beta + E_\beta E_\alpha^{(2)} + E_\alpha E_\beta E_\alpha.$$

For the simple root vectors we have

$$T_\alpha(E_\alpha) = -F_\alpha K_\alpha, \quad T_\beta(E_\alpha) = E_{\beta+\alpha} = E_\beta E_\alpha - q_\alpha E_\alpha E_\beta, \quad T_\beta(E_\beta) = -F_\beta K_\beta.$$

We have $[3]_{q_\alpha} = -1$ so that $E_\alpha^{(3)} = -E_\alpha^{(2)} E_\alpha$, which gives

$$T_\alpha(E_\beta) = E_\alpha^{(3)} E_\beta + q_\alpha E_\alpha^{(2)} E_\beta E_\alpha - E_\alpha E_\beta E_\alpha^{(2)} - q_\alpha E_\beta E_\alpha^{(3)}$$
$$= -E_\alpha E_{2\alpha+\beta} + q_\alpha E_{2\alpha+\beta} E_\alpha.$$

We are only left to check the values of T_α and T_β on $E_{2\alpha+\beta}$.

We have

$$E_\alpha = T_\beta T_\alpha T_\beta T_\alpha T_\beta(E_\alpha) = T_\beta T_\alpha(E_{2\alpha+\beta})$$

by the rank 2 calculation [44, §39.2.2 (b)] and, using the explicit formula for the inverse T_β^{-1} [44, §37.1], we the obtain

$$T_\alpha(E_{2\alpha+\beta}) = T_\beta^{-1}(E_\alpha) = -q_\alpha E_\beta E_\alpha + E_\alpha E_\beta.$$

For $T_\beta(E_{2\alpha+\beta})$ we have

$$T_\beta(E_\alpha^{(2)}) = E_\beta^{(2)} E_\alpha^{(2)} - q_\alpha E_\beta E_\alpha^{(2)} E_\beta - E_\alpha^{(2)} E_\beta^{(2)}$$

[44, §37.1] and $T_\beta(E_\beta) = \mathsf{K}_\beta F_\beta$ so that

$$T_\beta(E_{2\alpha+\beta} - E_\alpha E_\beta E_\alpha)$$

$$= \mathsf{K}_\beta(F_\beta T_\beta(E_\alpha^{(2)}) - T_\beta(E_\alpha^{(2)})F_\beta)$$

$$= -\mathsf{K}_\beta([E_\beta^{(2)}, F_\beta]E_\alpha^{(2)} - q_\alpha \begin{bmatrix} \mathsf{K}_\beta; 0 \\ 1 \end{bmatrix} E_\alpha^{(2)} E_\beta - q_\alpha E_\beta E_\alpha^{(2)} \begin{bmatrix} \mathsf{K}_\beta; 0 \\ 1 \end{bmatrix} - E_\alpha^{(2)}[E_\beta^{(2)}, F_\beta])$$

$$= -\mathsf{K}_\beta([E_\beta^{(2)}, F_\beta]E_\alpha^{(2)} - E_\alpha^{(2)}[E_\beta^{(2)}, F_\beta]) + q_\alpha \mathsf{K}_\beta \begin{bmatrix} \mathsf{K}_\beta; 0 \\ 1 \end{bmatrix} (E_\alpha^{(2)} E_\beta + E_\beta E_\alpha^{(2)}).$$

Now,

$$[E_\beta^{(2)}, F_\beta] = \begin{bmatrix} \mathsf{K}_\beta; -1 \\ 1 \end{bmatrix} E_\beta$$

so that the expression reduces to

$$-q_\alpha \mathsf{K}_\beta (\begin{bmatrix} \mathsf{K}_\beta; 0 \\ t \end{bmatrix} - q_\alpha^{-3} \begin{bmatrix} \mathsf{K}_\beta; 1 \\ t \end{bmatrix})(E_\alpha^{(2)} E_\beta + E_\beta E_\alpha^{(2)})$$

$$= E_\alpha^{(2)} E_\beta + E_\beta E_\alpha^{(2)} = E_{2\alpha+\beta} - E_\alpha E_\beta E_\alpha$$

by [43, §6.4 (b4)]. Hence $T_\beta(E_{2\alpha+\beta}) = E_{2\alpha+\beta} - E_\alpha E_\beta E_\alpha - q_\alpha \mathsf{K}_\beta E_{\beta+\alpha} F_\beta E_{\beta+\alpha}$.

The above arguments establish inclusions $T(\mathfrak{v}_q^+) \subseteq \mathfrak{v}_q$ for all braid group operators T, and by symmetry we have $T(\mathfrak{v}_q^-) \subseteq \mathfrak{v}_q$ as well. $\square$

Appendix B. Normality below the lacing number

We cover the details for the proof of Lemma 7.3 at low order parameters.

B.1. **Completed proof of 7.3.** As argued in the early parts of the proof, we may assume G is almost-simple and that $l > 1$. We first cover the case where G is not of type G_2 at $l = 2$.

Proof away from G_2 at $l = 2$. We have that l is equal to the lacing number for G, and suppose additionally that the scalar parameter for G is of order > 3 in type G_2. Equivalently, we assume $q_\beta = -1$ whenever β is a long simple root. This implies $K_\beta = K_\beta^{-1}$.

It is shown in [39, Lemma 6.6] that $\mathfrak{v}_q^+$ remains a normal (braided) Hopf subalgebra in U_q^+ when l is equal to the lacing number. By considering the Hopf isomorphism $U_{q^{-1}}^{\geq 0} \to U_q^{\leq 0}$ which exchanges E's and F's, we also have that $\mathfrak{v}_q^-$ is normal in U_q^-. Such Hopf normality implies

$$E_\beta^{(l_\beta)} E_\alpha + (-1)^{l_\beta} q(\beta, \alpha)^{l_\beta} E_\alpha E_\beta^{(l_\beta)} \in \mathfrak{m}^+ \quad \Rightarrow \quad [E_\beta^{(l_\beta)}, E_\alpha] \in \widehat{U}_q \mathfrak{m}^+, \qquad (21)$$

where $\mathfrak{m}^+$ is the subspace of positive degree elements in $\mathfrak{v}_q^+$. But now one sees, by scaling by grouplikes, that $\widehat{U}_q \mathfrak{m}^+ \subseteq \widehat{U}_q \mathfrak{m}$. So we locate the commutator (21) in $\widehat{U}_q \mathfrak{m}$. We similarly find that $[F_\beta^{(l_\beta)}, F_\alpha] \in \widehat{U}_q \mathfrak{m}$ at all $\alpha \in \Delta_l$, $\beta \in \Delta$. So we need only check the commutators of the E's with the F's in (16).

In considering these commutators we have three cases to check; when β is short and has no long neighbors, when β is short but has a long neighbor, and when β is long. In the last case $l_\beta = 1$. Let us list the simple roots as

$$\Delta = \{\beta_n, \ldots, \beta_{r+1}, \alpha_r, \ldots, \alpha_1\}$$

$$\Delta_l = \{\beta_n + \cdots + \beta_{r+1} + \alpha_r, \ldots, \beta_{r+1} + \alpha_r, \alpha_r, \ldots, \alpha_1\}$$

with neighbors appearing in consecutive order, all the α_i short, and all β_j long. Take

$$E_i = E_{\alpha_i} \text{ or } E_{\beta_i} \text{ where appropriate, } E_{r,r} = E_r \text{ and } E_{r+j,r} = E_{\beta_{r+j}+\cdots+\beta_{r+1}+\alpha_r}.$$

Adopt a similar notation for the F's.

Recall that $q_\beta = -1$ whenever β is long. When β is short and has no long neighbors $[F_\beta^{(l_\beta)}, E_{r+j,r}] = [E_\beta^{(l_\beta)}, F_{r+j,r}] = 0$ for all positive j. So the computation reduces to a type A_n calculation at $l = 2$, which we have already addressed in Section 7.1.

When β is short and has a long neighbor we have $\beta = \alpha_r$ necessarily. The generic commutator relation [43, §6.5 (a2)] shows that $[E_j, F_\beta^{(l_\beta)}]$ lies in $\widehat{U}_q\mathfrak{m}$ for all j. For $E_{r+1,r}$ we have

$$[E_{r+1,r}, F_\beta^{(l_\beta)}] = E_{r+1}[E_r, F_\beta^{(l_\beta)}] + [E_r, F_\beta^{(l_\beta)}]E_{r+1}.$$

Now

$$[E_r, F_\beta^{(l_\beta)}] = \begin{bmatrix} K_\beta; m \\ 1 \end{bmatrix} F_\beta^{(l_\beta-1)}$$

for some integer m [43, §6.5 (a2), (a6)]. Since F_β commutes with E_{r+1} in this case we get

$$[E_{r+1,r}, F_\beta^{(l_\beta)}] = (\begin{bmatrix} K_\beta; m' \\ 1 \end{bmatrix} + \begin{bmatrix} K_\beta; m \\ 1 \end{bmatrix})E_{r+1}F_\beta^{(l_\beta-1)}.$$

Since $F_\beta^{(l_\beta-1)}$ is in $\mathfrak{m} \subseteq \bar{u}_q$ this element is in $\widehat{U}_q\mathfrak{m}$, as desired. One now sees by induction that

$$[E_{r+j,r}, F_\beta^{(l_\beta)}] \in \widehat{U}_q F_\beta \subseteq \widehat{U}_q\mathfrak{m}$$

for all positive j, when $\beta = \alpha_r$.

We are left, finally, to deal with the case where β is long, so that $l_\beta = 1$. We have $F_\beta^{(l_\beta)} = F_{r+j}$ for some positive j. Now, when $j > j' \geq 0$ we have immediately $[F_{r+j}, E_{r+j',r}] = 0$. When $j' = j$ we have

$$[F_{r+j}, E_{r+j,r}] = -[F_{r+j}, E_{r+j}E_{r+j-1,r} + E_{r+j-1,r}E_{r+j}]$$

$$= -\begin{bmatrix} K_{r+j}; 0 \\ 1 \end{bmatrix} E_{r+j-1,r} - E_{r+j-1,r}\begin{bmatrix} K_{r+j}; 0 \\ 1 \end{bmatrix}. \tag{22}$$

Since $q_\beta = -1$, we have the commutativity relations

$$E_{r+j-1}\begin{bmatrix} K_{r+j}; 0 \\ 1 \end{bmatrix} = \begin{bmatrix} K_{r+j}; 1 \\ 1 \end{bmatrix} E_{r+j-1} = -\begin{bmatrix} K_{r+j}; 0 \\ 1 \end{bmatrix} E_{r+j-1} + K_\beta E_{r+j-1}$$

[43, §6.4 and 6.5 (b5), (a5)] which reduces (22) to

$$[F_{r+j}, E_{r+j,r}] = -K_{r+j}E_{r+j-1,r} \in \mathfrak{m}.$$

For $r + j + 1$ the above expression gives also

$$[F_{r+j}, E_{r+j+1,r}] = -E_{r+j+1}K_{r+j}E_{r+j-1,r} - K_{r+j}E_{r+j-1,r}E_{r+j+1}$$

$$= -(q_{\beta_{r+j}}^{-1} + 1)K_{r+j}E_{r+j-1,r}E_{r+j+1} = 0,$$

since $q_{\beta_{r+j}} = -1$. By induction one now finds $[F_{r+j}, E_{r+j+k,r}] = 0$ at all $k \geq 1$. We have now checked all cases and obtain the desired inclusion

$$[F_\beta^{(l_\beta)}, E_\alpha] \in \widehat{U}_q\mathfrak{m}$$

for all $\beta \in \Delta$ and $\alpha \in \Delta_l$. The cases with the E's and F's swapped follow by symmetry.

We have now established the desired inclusions (16) whenever G is not of type G_2 at a 3-rd root of unity. The final case of G_2 at a 3-rd root of 1 is established via completely similar arguments, though some signs are changed. $\square$

We now cover the special case of G_2 at a 4-th root of 1.

Proof for G_2 at $l = 2$. As in the analysis at (21), we can use Lentner's normality result [39, Lemma 6.6] to find that

$$E_\alpha E_\beta^{(l_\nu)}, \ F_\alpha F_\beta^{(l_\nu)} \ \in \widehat{U}_q \mathfrak{m}$$

whenever $\alpha \in \Delta_l$ and $\beta \in \Delta$. So we need only check the products $E_\alpha F_\beta^{(l_\nu)}, F_\alpha E_\beta^{(l_\nu)}$.

In this special setting $l_\beta = 2$ at all simple β. We have

$$[E_\beta^{(l_\beta)}, F_\alpha], \ [F_\beta^{(l_\beta)}, E_\alpha] \ \in \ \widehat{U}_q \mathfrak{m}$$

whenever $\alpha, \beta \in \Delta$ by the generic commutativity relation [44, §6.5 (a2)]. For the unique non-simple root $2\alpha + \beta$ in Δ_l we have

$$E_{2\alpha+\beta} = E_\alpha^{(2)} E_\beta + E_\beta E_\alpha^{(2)} + E_\alpha E_\beta E_\alpha \tag{23}$$

and compute

$$[F_\beta^{(2)}, E_\alpha^{(2)}] = [F_\beta^{(2)}, E_\alpha] = 0, \quad [F_\beta^{(2)}, E_\beta] = \begin{bmatrix} K_\beta; -1 \\ 1 \end{bmatrix} F_\beta, \quad [F_\alpha^{(2)}, E_\beta] = 0$$

$$[F_\alpha^{(2)}, E_\alpha] = \begin{bmatrix} K_\alpha; -1 \\ 1 \end{bmatrix} F_\beta, \quad \text{and} \quad [F_\alpha^{(2)}, E_\alpha^{(2)}] = \begin{bmatrix} K_\alpha; m \\ 1 \end{bmatrix} F_\alpha E_\alpha - \begin{bmatrix} K_\alpha; 0 \\ 2 \end{bmatrix},$$

where m is some integer [43, §6.5]. Hence applying the derivation $[F_\beta^{(2)}, -]$ to the expression (23) yields an elements

$$[F_\beta^{(2)}, E_{2\alpha+\beta}] \ \in \ \widehat{U}_q F_\beta \ \subseteq \ \widehat{U}_q \mathfrak{m}.$$

This implies $E_{2\alpha+\beta} F_\beta^{(2)} \in \widehat{U}_q \mathfrak{m}$.

As for the derivation $[F_\alpha^{(2)}, -]$, we have

$$E_\beta \begin{bmatrix} K_\alpha; 0 \\ 2 \end{bmatrix} = \begin{bmatrix} K_\alpha; 2 \\ 2 \end{bmatrix} E_\beta$$

[43, §6.5] so that we find again

$$[F_\alpha^{(2)}, E_{2\alpha+\beta}] \ \in \ \widehat{U}_q \mathfrak{m} \ \Rightarrow \ E_{2\alpha+\beta} F_\alpha^{(2)} \in \widehat{U}_q \mathfrak{m}.$$

We employ symmetry to obtain $F_{2\alpha+\beta} E_\beta^{(2)}, \ F_{2\alpha+\beta} E_\alpha^{(2)} \in \widehat{U}_q \mathfrak{m}$ as well. This completes the proof. $\square$

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