SMALL QUANTUM GROUPS ASSOCIATED TO BELAVIN-DRINFELD TRIPLES

CRIS NEGRON

ABSTRACT. For a simple Lie algebra $\mathfrak{g}$ of type A, D, E we show that any Belavin-Drinfeld triple on the Dynkin diagram of $\mathfrak{g}$ produces a collection of Drinfeld twists for Lusztig's small quantum group $u_q(\mathfrak{g})$. These twists give rise to new finite-dimensional factorizable Hopf algebras, i.e. new small quantum groups. For any Hopf algebra constructed in this manner, we identify the group of grouplike elements, identify the Drinfeld element, and describe the irreducible representations of the dual in terms of the representation theory of the parabolic subalgebra(s) in $\mathfrak{g}$ associated to the given Belavin-Drinfeld triple. We also produce Drinfeld twists of $u_q(\mathfrak{g})$ which express a known algebraic group action on its category of representations, and pose a subsequent question regarding the classification of all twists.

INTRODUCTION

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ of type A, D, E, and let Γ be its Dynkin diagram. A Belavin-Drinfeld triple on Γ is a choice of two subgraphs Γ_1 and Γ_2 and an isomorphism $T : \Gamma_1 \to \Gamma_2$ satisfying a certain nilpotence condition. In [5, Ch. 6] Belavin and Drinfeld showed that such a triple gives rise to solutions to the classical Yang-Baxter equation in $\mathfrak{g} \otimes \mathfrak{g}$, and in [14] Etingof, Schedler, and Schiffmann showed that any Belavin-Drinfeld triple gives rise to (Drinfeld) twists of the Drinfeld-Jimbo quantum group $U_\hbar(\mathfrak{g})$. Such a twist J of $U_\hbar(\mathfrak{g})$ produces a new quantum group $U_\hbar(\mathfrak{g})^J$ and new R-matrix, i.e. solution to the Yang-Baxter equation (see Section 2). These new solutions to the Yang-Baxter equation quantize the classical solutions of Belavin and Drinfeld, in the sense described in [12, 14]. Furthermore, one can show that any twist of the Drinfeld-Jimbo quantum group, over $\mathbb{C}[\![\hbar]\!]$, arises as one of the quantizations of [14], up to gauge equivalence.

Here we follow the methods of [14, 4] to produce twists of Lusztig's small quantum group $u_q(\mathfrak{g})$ from Belavin-Drinfeld triples. We also produce explicit twisted automorphisms of $u_q(\mathfrak{g})$ which arise out of an algebraic group action on its category of representations. The action we consider here first appeared in the work of Arkhipov and Gaitsgory [3], but can also be derived from De Concini and Kac's earlier quantum coadjoint action [8], as is explained in Section 9 below. Using the Belavin-Drinfeld twists, and those twists associated to the algebraic group action, we propose a question regarding the classification of all twists of the small quantum group.

Belavin-Drinfeld triples and twists of $u_q(\mathfrak{g})$. Recall that the small quantum group is a finite dimensional quasitriangular Hopf algebra produced from the Cartan

Date: September 17, 2017.
This work was supported by NSF Postdoctoral Research Fellowship DMS-1503147.

data for g and a primitive lth root of unity q.[1] In addition to a triple (Γ_1, Γ_2, T) for g we need one more piece of data $\mathtt{S}$. The element $\mathtt{S}$ is a choice of solution to a certain equation involving T, which we describe below (see Section 3). Given any Belavin-Drinfeld tiple (Γ_1, Γ_2, T) we will have $\max\{1, l|\Gamma-\Gamma_1|(|\Gamma-\Gamma_1|-1)/2\}$ such solutions $\mathtt{S}$. We show

Theorem I (3.1)**.** *Any Belavin-Drinfeld triple (Γ_1, Γ_2, T) for g and solution $\mathtt{S}$ produces a twist $J = J_{T,\mathtt{S}}$ for the small quantum group $u_q(\mathsf{g})$, and an associated Hopf algebra $u_q(\mathsf{g})^J$.*

The twist $J_{T,\mathtt{S}}$ is given explicitly by the formula

$$J_{T,\mathtt{S}} = (T_+ \otimes 1)(R) \dots (T_+^n \otimes 1)(R)\mathtt{S}^{-1}\Omega_{L^\perp}^{-1/2}(T^n \otimes 1)(\Omega)^{-1} \dots (T \otimes 1)(\Omega)^{-1}$$

where R is the R-matrix for $u_q(\mathsf{g})$, Ω is an element representing the negated Killing form, and T_+ is an extension of T to an endomorphism of the positive quantum Borel in $u_q(\mathsf{g})$. The above theorem is a non-dynamical analog of [13, Sect. 5.2], and a discrete version of [14, Cor. 6.1].

Recall that for any twist J of a Hopf algebra H we will have a canonical equivalence between the associated tensor categories of finite dimensional representations $\mathrm{rep}(H) \overset{\sim}{\to} \mathrm{rep}(H^J)$. In addition to studying the relationship between Belavin-Drinfeld triples and solutions to the Yang-Baxter equation (i.e. R-matrices) for finite dimensional Hopf algebras, we want to study variances of Hopf structures under tensor-equivalence. With this purpose in mind we give an in depth study of the Hopf algebras $u_q(\mathsf{g})^J$ arising from our twists.

For the remainder of the introduction fix $J = J_{T,\mathtt{S}}$ the twist associated to some Belavin-Drinfeld data (Γ_1, Γ_2, T) and $\mathtt{S}$. Using the new R-matrix for $u_q(\mathsf{g})^J$, in conjunction with the frameworks of [26], we identify the grouplike elements of $u_q(\mathsf{g})^J$, show that the Drinfeld element for $u_q(\mathsf{g})^J$ is equal to that of the untwisted algebra $u_q(\mathsf{g})$, verify invariance of the traces of the powers of the antipode under the twists $J = J_{T,\mathtt{S}}$, and classify irreducible representations of the dual. (See Corollaries 7.8, 8.2, 8.3, and Theorem 7.4 below.) Our analyses of the Drinfeld element and antipode give positive answers to some general questions of [24] and [28] in the particular case of Belavin-Drinfeld twists of the small quantum group. We describe our result on irreducibles more explicitly below.

In the statement of the following theorem we let p^{ss} be the semisimple Lie algebra associated to the union of Dynkin diagrams Γ_1 appearing in the Belavin-Drinfeld triple (Γ_1, Γ_2, T).

Theorem II (7.4)**.** *There is an abelian group $\mathcal{L}$ of order $l(|\Gamma - \Gamma_1|)$ and bijection*

$$\mathrm{Irrep}\,(\mathbb{C}[\mathcal{L}] \otimes u_q(\mathsf{p}^{ss})) \overset{\cong}{\to} \mathrm{Irrep}\,\left((u_q(\mathsf{g})^J)^*\right)$$

induced by a surjective algebra map $(u_q(\mathsf{g})^J)^* \to \mathbb{C}[\mathcal{L}] \otimes u_q(\mathsf{p}^{ss})$.

By comparison, for the untwisted algebra $u_q(\mathsf{g})$ we have that $\mathrm{Irrep}\,(u_q(\mathsf{g})^*) = (\mathbb{Z}/l\mathbb{Z})^{|\Gamma|}$, and the representation theory of the dual is rather banal from the perspective or irreducibles and the fusion rule. After twisting $u_q(\mathsf{sl}_{n+1})$, for example, we can have a copy of the rather rich category $\mathrm{rep}\,(u_q(\mathsf{sl}_n))$ in the category of representations for the dual $(u_q(\mathsf{sl}_{n+1})^J)^*$. This will specifically be the case for (what we call) maximal triples on A_n. One should compare this result to [13, Thm. 5.4.1].

[1] We will need l to be coprime to a small number of integers throughout this work.

The Arkhipov-Gaitsgory action and twisted automorphisms. Take $\mathbb{G}$ the connected, simply connected, semisimple algebraic group with Lie algebra g. In [3] Arkhipov and Gaitsgory show that the category $\mathrm{rep}(u_q(\mathsf{g}))$ is tensor equivalent to a de-equivariantization of the category of corepresentations of the quantum function algebra $\mathscr{O}_q(\mathbb{G})$. The de-equivariantization is a certain (non-full) monoidal subcategory in $\mathrm{Coh}(\mathbb{G})$ which inherits a natural action of $\mathbb{G}$ by left translation (see [2, 15]). From the aforementioned equivalence we then get an action of $\mathbb{G}$ on $\mathrm{rep}(u_q(\mathsf{g}))$.

According to general principles, any autoequivalence of $\mathrm{rep}(u_q(\mathsf{g}))$ should be expressible as a twisted automorphism (ϕ, J), i.e. a pair of a twist J and a Hopf isomorphism $\phi : u_q(\mathsf{g}) \to u_q(\mathsf{g})^J$. Hence, the action of $\mathbb{G}$ should generate twists of $u_q(\mathsf{g})$.

In Section 9 we show that any simple root α of g, or its negation $-\alpha$, has an associated 1-parameter family of twisted automorphisms $(\exp^{\lambda}_{\pm\alpha}, J^{\lambda}_{\pm\alpha})$, which then give a 1-parameter subgroup $\omega_{\pm\alpha}$ in the group of autoequivalences of $\mathrm{rep}(u_q(\mathsf{g}))$. We identify these 1-parameter subgroups $\omega_{\pm\alpha}$ with the action of Arkhipov and Gaitsgory.

Proposition (9.4)**.** *For $\gamma_{\pm\alpha} : \mathbb{C} \to \mathbb{G}$ the 1-parameter subgroup given by exponentiating the root space $\mathsf{g}_{\pm\alpha}$, we have a diagram*

$$\begin{array}{ccc} & \mathbb{G} & \\ {\scriptstyle \gamma_{\pm\alpha}} \nearrow & & \searrow {\scriptstyle \text{AG actn}} \\ \mathbb{C} & \xrightarrow{\quad \omega_{\pm\alpha} \quad} & \mathrm{Aut}(\mathrm{rep}(u_q(\mathsf{g}))), \end{array}$$

where $\omega_{\pm\alpha}$ is the 1-parameter subgroup specified by the twisted automorphisms $(\exp^{\lambda}_{\pm\alpha}, J^{\lambda}_{\pm\alpha})$.

This result allows us to produce an explicit action of $\mathbb{G}$ on the collection $\mathrm{TW}(u_q(\mathsf{g}))$ of gauge equivalence classes of twists. We let $\mathrm{BD}(u_q(\mathsf{g})) \subset \mathrm{TW}(u_q(\mathsf{g}))$ denote the subcollection of Belavin-Drinfeld twists $\{J_{T,\mathsf{s}}\}_{T,\mathsf{s}}$. We pose the following question, which is also raised in [7].

Question (9.5)**.** *Do the Belavin-Drinfeld twists and the 1-parameter subgroups $\{(\exp^{\lambda}_{\pm\alpha}, J^{\lambda}_{\pm\alpha})\}_{\lambda,\alpha}$ generate all twists of $u_q(\mathsf{g})$? Equivalently, is the inclusion $\mathrm{BD}(u_q(\mathsf{g}))\cdot \mathbb{G} \to \mathrm{TW}(u_q(\mathsf{g}))$ an equality?*

As was stated above, for the Drinfeld-Jimbo algebra $U_\hbar(\mathsf{g})$ one can show that the Belavin-Drinfeld twists are the only twists, up to gauge equivalence. So the appearance of $\mathbb{G}$ here already marks a deviation from the generic setting.

We note that a classification of twists for $u_q(\mathsf{sl}_2)$ is know, and can be deduced from Mombelli's work [22]. However, even for $\mathsf{g} = \mathsf{sl}_3$ the problem is completely open.

Organization. Sections 1 and 2 are dedicated to background. In Section 3 we introduce and prove Theorem I. In Sections 4 and 5 we analyze relations between Radford's left and right subalgebras $R^J_{(l)}$ and $R^J_{(r)}$ in $u_q(\mathsf{g})^J$ and the quantum parabolics associated to Γ_1 and Γ_2. We prove an explicit description of the $R^J_{(*)}$ in Section 6, which leads to the proof of Theorem II in Section 7. In Section 8 we discuss the Drinfeld element and antipode of such a twist $u_q(\mathsf{g})^J$. Section 9 is dedicated to the action of the algebraic group $\mathbb{G}$ on $\mathrm{rep}(u_q(\mathsf{g}))$.

Acknowledgements. Thanks to Pavel Etingof for many helpful conversations and for providing a simplification of the proof of Proposition 8.1. Thanks also to the referee for their helpful comments.

1. The small quantum group, Belavin-Drinfeld triples, and associated subgroups in the Cartan

We introduce the small quantum group $u_q(\mathfrak{g})$, then give some information on the Cartan subgroup $G = G(u_q(\mathfrak{g}))$ and Belavin-Drinfeld tiples.

1.1. **The small quantum group.** Take $\mathfrak{g}$ a simple and simply laced Lie algebra, i.e. a Lie algebra of type A, D, E. Let Φ be a root system for $\mathfrak{g}$ (in the dual of some Cartan), Γ be a choice of simple roots, and l be an odd integer coprime to the determinant of the Cartan matrix for $\mathfrak{g}$. We let $Q = \mathbb{Z}\cdot\Gamma$ denote the root lattice and $(?,?)$ be the scaling of the Killing form so that each (α,β) is the Cartan integer for simple roots α,β. Take q a primitive lth root of unity.

The small quantum group $u_q(\mathfrak{g})$ is the Hopf algebra

$$u_q(\mathfrak{g}) = \mathbb{C}\langle K_\alpha, E_\alpha, F_\alpha : \alpha\in\Gamma\rangle/(\text{Rels}),$$

where Rels is the set of relations

$$[K_\alpha, K_\beta] = 0,\quad K_\alpha E_\beta = q^{(\alpha,\beta)}E_\beta K_\alpha,\quad K_\alpha F_\beta = q^{-(\alpha,\beta)}F_\beta K_\alpha,$$

$$[E_\alpha, F_\beta] = \delta_{\alpha,\beta}\frac{K_\alpha - K_\alpha^{-1}}{q-q^{-1}},$$

$$[E_\alpha, E_\beta] = [F_\alpha, F_\beta] = 0 \ \text{ when } \ (\alpha,\beta) = 0,$$

$$E_\alpha^2 E_\beta - (q+q^{-1})E_\alpha E_\beta E_\alpha + E_\beta E_\alpha^2, \ \text{ when } \ (\alpha,\beta) = -1,$$

$$F_\alpha^2 F_\beta - (q+q^{-1})F_\alpha F_\beta F_\alpha + F_\beta F_\alpha^2 \ \text{ when } \ (\alpha,\beta) = -1.$$

$$K_\alpha^l = 1,\quad E_\mu^l = F_\mu^l = 0\ \ \forall\, \mu\in\Phi^+. \tag{1}$$

We will explain the (currently opaque) relations (1) more clearly below. The coproduct is given by

$$\Delta(K_\alpha) = K_\alpha\otimes K_\alpha,\quad \Delta(E_\alpha) = E_\alpha\otimes 1 + K_\alpha\otimes E_\alpha,\quad \Delta(F_\alpha) = F_\alpha\otimes K_\alpha^{-1} + 1\otimes F_\alpha$$

and the antipode is given by

$$S(K_\alpha) = K_\alpha^{-1},\quad S(E_\alpha) = -K_\alpha^{-1}E_\alpha,\quad S(F_\alpha) = -F_\alpha K_\alpha.$$

We let G denote the group of grouplikes in $u_q(\mathfrak{g})$, u^+ and u^- denote the subalgebras generated by the E_α and F_α respectively, and u_+ and u_- denote the positive and negative quantum Borels in $u_q(\mathfrak{g})$. Note that G is generated by the K_α and that under the adjoint action of G on $u^\pm$ we will have $u_\pm = u^\pm \rtimes \mathbb{C}[G]$. Note also that $u_q(\mathfrak{g})$ and the $u_\pm$ are graded by the root lattice $Q = \mathbb{Z}\cdot\Gamma$, where the generators E_α, F_α, and K_α have degrees α, $-\alpha$, and 0 respectively.

There is an obvious group map $Q\to G$ which sends an element $\gamma = \sum_\alpha n_\alpha\alpha$ in Q to the product $K_\gamma := \prod_\alpha K_\alpha^{n_\alpha}$. This mapping induces an identification $Q/lQ \cong G$. We let K_γ to denote the image of an element $\gamma\in Q$ (or $\gamma\in Q/lQ$) in G throughout.

We would like to employ Lusztig's standard basis for $u_q(\mathfrak{g})$, which we review here. Recall that for a reduced expression $w = \sigma_{\alpha_1}\dots\sigma_{\alpha_t}$ of the longest word w in the Weyl group, in terms of the simple reflections, we have $\text{length}(w) = |\Phi^+|$ and

$$\Phi^+ = \{\sigma_{\alpha_1}\dots\sigma_{\alpha_{i-1}}(\alpha_i) : 1\le i\le \text{length}(w)\}. \tag{2}$$

(See e.g. [30].) For each simple root α there is an automorphism B_α of u_q so that the B_α together give an action of the braid group $B(\Gamma)$ on u_q [21].[2] Now for each $\mu \in \Phi^+$ we take

$$E_\mu = B_{\alpha_1} \dots B_{\alpha_{i-1}}(E_{\alpha_i}), \quad \text{and} \quad F_\mu = B_{\alpha_1} \dots B_{\alpha_{i-1}}(F_{\alpha_i}),$$

where $\mu = \sigma_{\alpha_1} \dots \sigma_{\alpha_{i-1}}(\alpha_i)$. The E_μ and F_μ defined here are the elements appearing in the above relations (1).

Theorem 1.1 ([21])**.** *For each $\mu \in \Phi^+$, the element E_μ (resp. F_μ) is homogeneous of degree μ (resp. $-\mu$) with respect to the root lattice grading on $u_q(\mathfrak{g})$. Furthermore, the collection of elements*

$$\{\prod_{\mu\in\Phi^+} E_\mu^{n_\mu} : 0 \leq n_\mu \leq l-1\}, \quad \{\prod_{\nu\in\Phi^+} F_\nu^{m_\nu} : 0 \leq m_\nu \leq l-1\},$$

give $\mathbb{C}$-bases for u^+ and u^- respectively, and

$$\{(\prod_{\nu\in\Phi^+} F_\nu^{m_\nu})(\prod_{\mu\in\Phi^+} E_\mu^{n_\mu}) : 0 \leq n_\mu, m_\nu \leq l-1\}$$

gives a $\mathbb{C}[G]$-basis for $u_q(\mathfrak{g})$.

Homogeneity of the E_μ is equivalent to the statement that E_μ is a linear combination of permutations of the monomial $E_{\alpha_1} \dots E_{\alpha_k}$, where $\mu = \alpha_1 + \dots + \alpha_k$ with the $\alpha_i \in \Gamma$. The analogous statement holds for the F_ν as well. We note that the homogeneity is not covered in [21], but can easily be seen from the fact that each braid group operator B_α is such that $\deg(B_\alpha(a)) = \sigma_\alpha(\deg(a))$, for homogeneous $a \in u_q$. From the identifications $u_\pm = u^\pm \rtimes \mathbb{C}[G]$ the $\mathbb{C}$-bases for $u^\pm$ produce $\mathbb{C}[G]$-bases for the quantum Borels.

We recall finally that $u_q(\mathfrak{g})$ is quasi-triangular. The R-matrix is

$$R = \prod_{\mu\in\Phi^+} \left(\sum_{n=0}^{l-1} q^{-n(n+1)/2} \frac{(1-q^2)^n}{[n]_q!} E_\mu^n \otimes F_\mu^n \right) \Omega,$$

where $[n]_q!$ is the standard q-factorial, $\Omega \in \mathbb{C}[G] \otimes \mathbb{C}[G]$ is given by

$$\Omega = \frac{1}{|G|} \sum_{\beta,\gamma\in Q/lQ} q^{(\beta,\gamma)} K_\beta \otimes K_\gamma,$$

and the product is ordered with respect to the ordering on Φ^+ given by (2) (see [29]). As we will see below Ω can be identified with the a negated Killing form on the character group of G.

1.2. **Belavin-Drinfeld triples and subgroups of G.** We recall some information from [13]. To ease notation we let $\mathcal{G}$ denote the character group of G, $\mathcal{G} = G^\vee$.

The Killing form on Q induces a $\mathbb{C}^\times$-valued form on G given by the formula $(K_\alpha, K_\beta) = q^{(\alpha,\beta)}$. Our assumption that l is coprime to the determinant of the Cartan matrix for $\mathfrak{g}$ ensures non-degeneracy of this form. Non-degeneracy of the form allows us to identify Q/lQ with the character group of G by taking an element γ of the root lattice to the function $K_\alpha \mapsto q^{(\gamma,\alpha)}$.

Throughout this work we identify elements of the (truncated) root lattice with characters on G via the Killing form. The isomorphism $G \to \mathcal{G}$ induced by the Killing form is such that $K_\alpha \mapsto \alpha$.

[2]Our B_α are the T_α from [21].

Now, the Cartan part Ω of the R-matrix for $u_q(\mathfrak{g})$ provides a form on the character group $\mathcal{G}$,

$$\Omega : \mathcal{G} \times \mathcal{G} \to \mathbb{C}^\times, \quad (\mu, \nu) \mapsto (\mu \otimes \nu)(\Omega) = \frac{1}{|G|} \sum_{\beta,\gamma \in Q/lQ} q^{(\beta,\gamma)} \mu(K_\beta)\nu(K_\gamma).$$

When μ and ν are elements of the root lattice the above expression reduces to

$$\begin{aligned} \Omega(\mu,\nu) = \tfrac{1}{|G|} \textstyle\sum_{\beta,\gamma} q^{(\beta,\gamma)} \mu(K_\beta)\nu(K_\gamma) \quad &= \tfrac{1}{|G|} \textstyle\sum_{\beta,\gamma} q^{(\beta,\gamma)+(\beta,\mu)+(\nu,\gamma)} \\ &= q^{-(\mu,\nu)} \tfrac{1}{|G|} \textstyle\sum_{\beta,\gamma} q^{(\beta+\nu,\gamma+\mu)} \\ &= q^{-(\mu,\nu)} \tfrac{1}{|G|} \textstyle\sum_{\sigma,\tau} q^{(\sigma,\tau)} = q^{-(\mu,\nu)}. \end{aligned}$$

So we see that Ω is identified with the negated Killing form. In particular Ω is non-degenerate.

The following structure was introduced by Belavin and Drinfeld in [5].

Definition 1.2. A Belavin-Drinfeld triple (BD triple) on Γ is a choice of two subsets $\Gamma_1, \Gamma_2 \subset \Gamma$ and inner product preserving bijection $T : \Gamma_1 \to \Gamma_2$ which satisfies the following nilpotence condition: for each $\alpha \in \Gamma_1$ there exists $n \geq 1$ with $T^n(\alpha) \in \Gamma - \Gamma_1$.

We often take $T\alpha = T(\alpha)$. Having fixed some BD triple (Γ_1, Γ_2, T) we can define a number of subgroups in $\mathcal{G}$ and G. We take

$$\mathcal{L} = (\mathbb{Z}/l\mathbb{Z} \cdot \{\alpha - T\alpha : \alpha \in \Gamma_1\})^\perp,$$

where the perp is calculated with respect to the Killing form, and

$$L = \left(\mathbb{Z}/l\mathbb{Z} \cdot \{K_\alpha K_{T\alpha}^{-1} : \alpha \in \Gamma_1\}\right)^\perp.$$

We take also $\mathcal{G}_i = \mathbb{Z}/l\mathbb{Z} \cdot \Gamma_i$ and the $G_i = \mathbb{Z}/l\mathbb{Z} \cdot \{K_\alpha : \alpha \in \Gamma_i\}$.

We assume that l is such that restrictions of the form to $\mathbb{Z}/l\mathbb{Z} \cdot \{\alpha - T\alpha : \alpha \in \Gamma_1\}$ and $\mathcal{G}_i$ are non-degenerate. To find such an l one simply considers the determinants of the (integer) matrices $[(\alpha - T\alpha, \beta - T\beta)]_{\alpha,\beta \in \Gamma_1}$ and $[(\alpha, \beta)]_{\alpha,\beta \in \Gamma_1}$ and chooses l coprime to these determinants. This will give $\mathcal{L}^\perp = \mathbb{Z}/l\mathbb{Z} \cdot \{\alpha - T\alpha : \alpha \in \Gamma_1\}$ and split $\mathcal{G}$ and G as $\mathcal{G} = \mathcal{L} \times \mathcal{L}^\perp = \mathcal{G}_i \times \mathcal{G}_i^\perp$, $G = L \times L^\perp = G_i \times G_i^\perp$. We also assume l is such that the restriction of the form to $\mathcal{G}_i \times \mathcal{L}^\perp$ is non-degenerate, which we can do by [14, Lem 3.1], and which can be checked by considering the determinant of the corresponding matrix. The following lemma was covered in [13, Sect. 5.2] (see also [14, Cor 3.2]).

Lemma 1.3. *Under the above assumptions on l, there are splittings $\mathcal{G} = \mathcal{G}_1 \times \mathcal{L}$ and $\mathcal{G} = \mathcal{G}_2 \times \mathcal{L}$, and a unique extension of $T : \Gamma_1 \to \Gamma_2$ to a group automorphism $T : \mathcal{G} \to \mathcal{G}$ with $T|\mathcal{L} = id_{\mathcal{L}}$. This automorphism preserves the form on $\mathcal{G}$.*

We will denote this extension of T to an automorphism on $\mathcal{G}$ simply by T. By a further abuse of notation we let T also denote the induced automorphism on the dual. That is, $T : G \to G$ is the map $K_\alpha \mapsto K_{T(\alpha)}$. Preservation of the form means specifically $(T\mu, T\nu) = (\mu, \nu)$ for each $\mu, \nu \in \mathcal{G}$ and $(T \otimes T)(\Omega) = \Omega$.

Throughout this work we make copious use of the dualities

$$G \leftrightsquigarrow \mathcal{G}, \quad L \leftrightsquigarrow \mathcal{L}, \quad G_i \leftrightsquigarrow \mathcal{G}_i, \quad L^\perp \leftrightsquigarrow \mathcal{L}^\perp, \quad G_i^\perp \leftrightsquigarrow \mathcal{G}_i^\perp.$$

By this we mean both that the duality functor $(?)^\vee$ sends the group on the left to the group on the right, and vice-versa, and that for any K_μ in the group on the left the function $(K_\mu, ?)$ will be an element in the corresponding group on the right, and vice-versa.

2. Twists and R-matrices

A (Drinfeld) twist of a Hopf algebra H is a unit $J \in H \otimes H$ which satisfies the dual cocycle condition

$$(\Delta \otimes 1)(J)(J \otimes 1) = (1 \otimes \Delta)(J)(1 \otimes J)$$

and $(\epsilon \otimes 1)(J) = (1 \otimes \epsilon)(J) = 1$. From such a J we can define a new Hopf algebra H^J which is equal to H as an algebra and has the new comultiplication

$$\Delta^J(h) = J^{-1}\Delta(h)J.$$

The antipode on H^J is given by

$$S_J(h) = Q_J^{-1}S(h)Q_J,$$

where $Q_J = m((S \otimes 1)(J))$ and $Q_J^{-1} = m((1 \otimes S)(J^{-1}))$ and m is multiplication. (See e.g. [15, 27].)

Recall that a quasitriangular Hopf algebra is a Hopf algebra H with a unit $R \in H \otimes H$ satisfying $R\Delta(h)R^{-1} = \Delta^{op}(h)$ for all $h \in H$, as well as the relations $(\Delta \otimes 1)(R) = R_{13}R_{23}$ and $(1 \otimes \Delta)(R) = R_{13}R_{12}$. We have the additional relations

$$(\epsilon \otimes 1)(R) = (1 \otimes \epsilon)(R) = 1, \quad (S \otimes 1)(R) = (1 \otimes S^{-1})(R) = R^{-1}$$

and $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ [10]. When H is quasitriangular with R-matrix R, the twist H^J will naturally be quasitriangular with new R-matrix

$$R^J = J_{21}^{-1}RJ.$$

2.1. Bicharacters and twists on group rings.

Let Λ be a finite abelian group. We call an element $B \in \mathbb{C}[\Lambda] \otimes \mathbb{C}[\Lambda] = \mathbb{C}[(\Lambda^\vee \times \Lambda^\vee)^\vee]$ a (symmetric, antisymmetric, etc.) bicharcter if its restriction $B : \Lambda^\vee \times \Lambda^\vee \to \mathbb{C}$ is a (symmetric, antisymmetric, etc.) bicharacter. An easy direct check verifies

Lemma 2.1. *Any bicharacter $B \in \mathbb{C}[\Lambda] \otimes \mathbb{C}[\Lambda]$ is a twist for $\mathbb{C}[\Lambda]$.*

Indeed, up to so-called gauge equivalence, every twist of the group ring of an abelian group of odd order is given by an antisymmetric bicharacter (see e.g. [16, **?**]).

3. Twists from Belavin-Drinfeld triples

For the remainder of this study we fix $\mathfrak{g}$ a simply laced simple Lie algebra with root system Φ and a choice of simple roots Γ. We take l as in Section 1.2 and $u_q = u_q(\mathfrak{g})$.

Let (Γ_1, Γ_2, T) be a BD triple. Following [14, 13], we extend the group maps $T^{\pm 1} : G \to G$ constructed in Lemma 1.3 to Hopf endomorphisms of the quantum Borels $T_\pm : u_\pm \to u_\pm$ defined by

$$T_+(E_\alpha) = \begin{cases} E_{T\alpha} & \text{when } \alpha \in \Gamma_1 \\ 0 & \text{when } \alpha \in \Gamma - \Gamma_1, \end{cases} \quad T_-(F_\beta) = \begin{cases} E_{T^{-1}\beta} & \text{when } \beta \in \Gamma_2 \\ 0 & \text{when } \beta \in \Gamma - \Gamma_2. \end{cases}$$

There will be a unique minimal positive integer n such that $T_\pm^n|I_\pm = 0$, where $I_\pm$ is the ideal in $u_\pm$ generated by all the E_α, or F_α. We call this integer the *nilpotence degree* of $T_\pm$.

We will be interested in antisymmetric bicharacters $\mathtt{S}$ in $\mathbb{C}[G] \otimes \mathbb{C}[G]$ solving the following equation:

$$\mathtt{S}^2(\alpha - T\alpha, ?) = \Omega(\alpha + T\alpha, ?) \;\; \forall\, \alpha \in \Gamma_1. \tag{EQ–S}$$

We verify below that such solutions always exist, and that there are exactly $|L\wedge_{\mathbb{Z}} L|$ of them, which is expected from [5, 14].

This section is dedicated to a proof of the following theorem.

Theorem 3.1. *Consider any Belavin-Drinfeld triple* (Γ_1,Γ_2,T) *and solution* $\mathtt{S}$ *to* (EQ–S)*. The element*

$$J_{T,\mathtt{S}}=(T_+\otimes 1)(R)\dots(T_+^n\otimes 1)(R)\mathtt{S}^{-1}\Omega_{L^\perp}^{-1/2}(T^n\otimes 1)(\Omega)^{-1}\dots(T\otimes 1)(\Omega)^{-1}$$

is a twist for the small quantum group $u_q(\mathfrak{g})$*, where* n *is the nilpotence degree of* T_+*.*

This result is a non-dynamical version of [13, Sect. 5.2], and a discrete version of [14, Thm. 6.1]. To clarify our previous point, we have

Lemma 3.2. *Antisymmetric bicharacter solutions* $\mathtt{S}$ *to equation* (EQ–S) *always exist, and there are exactly* $|L\wedge_{\mathbb{Z}} L|$ *such solutions.*

Proof. We decompose G as $L\oplus L^\perp$ to get $G\wedge_{\mathbb{Z}} G=(L^\perp\wedge_{\mathbb{Z}} G)\oplus(L\wedge_{\mathbb{Z}} L)$. Since

$$L^\perp=(\mathcal{L}^\perp)^\vee=(\mathbb{Z}/l\mathbb{Z}\cdot\{\alpha-T\alpha:\alpha\in\Gamma_1\})^\vee$$

we see that the equation (EQ–S) specifies uniquely an element $\mathtt{S}_0$ in $L^\perp\wedge_{\mathbb{Z}} G$, which we extend to a bicharacter on $\mathcal{G}$ which vanishes on $\mathcal{L}\times\mathcal{L}$. Whence we have found a solution to (EQ–S). We can add arbitrary elements of $L\wedge_{\mathbb{Z}} L$ to arrive at the set of all solutions $\mathtt{S}_0+L\wedge_{\mathbb{Z}} L$. □

One should note that when $\mathrm{rank}(L)=|\Gamma|-1$, the solution $\mathtt{S}$ will be unique. Using our nilpotence assumption on T one sees that, up to an automorphism of the Dynkin diagram, this occurs only in type A for the triple

$$\Gamma=A_n,\ \Gamma=\{\text{first } n-1 \text{ roots}\},\ \Gamma_2=\{\text{last } n-1 \text{ roots}\},\ T(\alpha_i)=\alpha_{i+1}.$$

We will call this the maximal triple on A_n.

We will need the following basic property of the R-matrix.

Lemma 3.3. *The* R*-matrix for* $u_q(\mathfrak{g})$ *satisfies* $(T_+\otimes 1)(R)=(1\otimes T_-)(R)$*.*

Proof. Any element $W\in u_+\otimes u_-$ is uniquely specified by the corresponding function $W:u_+^*\otimes u_-^*\to\mathbb{C}$ and subsequent map $t_W:u_+^*\to u_-$, $f\mapsto(f\otimes 1)(W)$. By [26, Prop. 2] and the fact that $T_\pm$ is a Hopf map, we see that when $W=(T_+\otimes 1)(R)$ or $(1\otimes T_-)(R)$ the $t_W:u_+^*\to u_-$ are algebra morphisms. Since u_+ is cordically graded, u_+^* is generated in degrees 0 and -1 as an algebra, with $(u_+^*)_0=\mathbb{C}[G]^*$ and $(u_+^*)_{-1}=(\sum_\alpha\mathbb{C}[G]E_\alpha)^*$, and we see that the t_W are determined by the restrictions $t_W|(u_+^*)_0$ and $t_W|(u_+^*)_{-1}$. These restrictions are in turn determined by the homogeneous pieces

$$(T_+\otimes 1)(R)_0=(T\otimes 1)(\Omega),\quad (1\otimes T_-)(R)_0=(1\otimes T^{-1})(\Omega)$$

and

$$(T_+\otimes 1)(R)_1=(q^{-1}-q)(\sum_{\alpha\in\Gamma_1}E_{T\alpha}\otimes F_\alpha)(T\otimes 1)(\Omega),$$

$$(1\otimes T_-)(R)_1=(q^{-1}-q)(\sum_{\beta\in\Gamma_2}E_\beta\otimes F_{T^{-1}\beta})(1\otimes T^{-1})(\Omega),$$

where we grade $u_+\otimes u_-$ by the degree on u_+. By T-invariance of the form Ω, it follows that $(T_+\otimes 1)(R)_0=(1\otimes T_-)(R)_0$ and $(T_+\otimes 1)(R)_1=(1\otimes T_-)(R)_1$. Whence we have the proposed equality. □

3.1. **General outline.** In order to prove Theorem 3.1 we will basically repeat the arguments of [14, 13], and so only sketch some of the unoriginal details.

Fix a triple (Γ_1, Γ_2, T). Following the suggestions of [14, Remark 6.1], and the general approach of [4], we will show that J is a twist by showing that both

$$(\Delta \otimes 1)(J)(J \otimes 1) \text{ and } (1 \otimes \Delta)(J)(1 \otimes J)$$

solve a certain "mixed ABRR" equation. Solutions to this equation with a specified "initial condition" are shown to be unique, so that we will have

$$(\Delta \otimes 1)(J)(J \otimes 1) = (1 \otimes \Delta)(J)(1 \otimes J).$$

Remark 3.4. (1) The letters ABRR throughout refer to the motivating work of Arnaudon, Buffenoir, Ragoucy, and Roche [4].
(2) Our presentation is slightly more complicated than that of [14]. This is a result of our choice to avoid the use of dynamical twists.

3.2. **Discrete ABRR in 2-components.** For a given solution $\mathtt{S}$ let Z denote the restriction of $\mathtt{S}$ to $\mathcal{L} \times \mathcal{L}^{\perp}$, Σ denote the restriction to $\mathcal{L}^{\perp} \times \mathcal{L}$, and take

$$Q = \mathrm{Z}\Sigma = \mathtt{S}| \left((\mathcal{L} \times \mathcal{L}^{\perp}) + (\mathcal{L}^{\perp} \times \mathcal{L})\right).$$

We view Z, Σ, and Q as bicharacters on $\mathcal{G}$ by letting them vanish on all other factors of $\mathcal{G} \times \mathcal{G}$. We also let Ω_L denote the restriction $\Omega|(\mathcal{L} \times \mathcal{L})$.

Definition 3.5. We define A^2_L and A^2_R the be the linear endomorphisms of $u_+ \otimes u_-$ defined by

$$A^2_L(\xi) = (T_+ \otimes 1)(R\xi Q)Q^{-1}\Omega_L^{-1}, \quad A^2_L(\xi) = (1 \otimes T_-)(R\xi Q)Q^{-1}\Omega_L^{-1}.$$

The left and right 2-component ABRR equations are the equations $A^2_L(X) = X$ and $A^2_R(X) = X$ respectively.

We note that Q can be replaced with Σ and Z in the expressions for A^2_L and A^2_R respectively. These alternate expressions are preferable for some calculations.

Since R decomposes as a sum $R = \Omega + R_+$, where R_+ is in the nilpotent ideal $I_+ \otimes I_-$, we get a corresponding decomposition of A^2_L as

$$A^2_L(\xi) = (T_+ \otimes 1)(\Omega\xi Q)Q^{-1}\Omega_L^{-1} + (T_+ \otimes 1)(R_+\xi Q)Q^{-1}\Omega_L^{-1}.$$

From this one finds that we can solve the left 2-component ABRR equation provided we can solve to the equation $(T \otimes 1)(\Omega X_0 Q)Q^{-1}\Omega_L^{-1} = X_0$ in $\mathbb{C}[G] \otimes \mathbb{C}[G]$. The analogous stamentement holds for the right ABRR equation. Whence we have the following discrete analog of [14, Cor. 4.1].

Lemma 3.6 (cf. [14]). *Any solution $B \in \mathbb{C}[G] \otimes \mathbb{C}[G]$ to the equation*

$$(T \otimes 1)(\Omega X_0 Q)Q^{-1}\Omega_L^{-1} = X_0 \quad (\textit{resp. } (1 \otimes T^{-1})(\Omega X_0 Q)Q^{-1}\Omega_L^{-1} = X_0) \tag{3}$$

extends uniquely to a solution $J \in B + I_+ \otimes I_-$ to the ABRR equation $A^2_L(X) = X$ (resp. $A^2_R(X) = X$).

In the proof of the following lemma we use the fact that for any element $K_\mu \in \mathbb{C}[G]$, and $\nu \in \mathcal{G}$, we have

$$(T^{\pm 1}(K_\mu))(\nu) = K_\mu(T^{\mp 1}\nu).$$

This follows from the easy sequence

$$(T^{\pm 1}(K_\mu))(\nu) = K_{T^{\pm 1}\mu}(\nu) = (T^{\pm 1}\mu, \nu) = (\mu, T^{\mp 1}\nu) = K_\mu(T^{\mp 1}\nu).$$

Lemma 3.7. *There are unique solutions* $J_L, J_R \in \mathtt{S}^{-1}\Omega_{L^\perp}^{-1/2} + I_+ \otimes I_-$ *to the left and right* 2*-component ABRR equations, respectively.*

Proof. We are claiming first that $\mathtt{S}^{-1}\Omega_L^{-1/2}$ solves the degree 0 ABRR equations from the previous lemma. Reorganizing, and applying $T^{-1}\otimes 1$, we see that $\mathtt{S}^{-1}\Omega_L^{-1/2}$ solves ABRR on the left, say, if and only if the equation

$$\Omega\Omega_{L^\perp}^{-1/2}(T^{-1}\otimes 1)(\Omega_L^{-1}\Omega_{L^\perp}^{1/2}) = \mathtt{S}Q^{-1}(T^{-1}\otimes 1)(\mathtt{S}^{-1}Q)$$

is satisfied. Using the fact that $T|L = id_L$ and $\Omega = \Omega_L\Omega_{L^\perp}$ we reduce to

$$\Omega_{L^\perp}^{1/2}(T^{-1}\otimes 1)(\Omega_{L^\perp}^{1/2}) = \mathtt{S}Q^{-1}(T^{-1}\otimes 1)(\mathtt{S}^{-1}Q).$$

Applying to arbitrary elements $\mu,\nu \in \mathcal{G}$ gives the equivalent equation

$$\Omega_{L^\perp}^{1/2}(\mu + T\mu, \nu) = \mathtt{S}Q^{-1}(\mu - T\mu, \nu). \tag{4}$$

By writing μ as a sum of elements in $\mathcal{L}$ and $\mathcal{L}^\perp$ we see that the above equation holds if and only if it holds when $\mu \in \mathcal{L}$, or $\mu \in \mathcal{L}^\perp$. When $\mu \in \mathcal{L}$ both sides of equation (4) vanish since $T|\mathcal{L} = id_{\mathcal{L}}$. Suppose now $\mu \in \mathcal{L}^\perp$. When $\nu \in \mathcal{L}$ both sides of the equation vanish by the definition of Q, and when $\nu \in \mathcal{L}^\perp$ the equation reduces to

$$\Omega^{1/2}(\mu + T\mu, \nu) = \mathtt{S}(\mu - T\mu, \nu),$$

which holds by equation (EQ–S). The check on the right is similar. □

Lemma 3.8. *The elements* J_L *and* J_R *from Lemma 3.7 are equal. Rather, there is a unique simultaneous solution* J *to both the left and right* 2*-component ABRR equations in* $\mathtt{S}^{-1}\Omega_{L^\perp}^{-1/2} + I_+ \otimes I_-$.

Proof. One shows that the operators A_L^2 and A_R^2 commute, then proceeds as in [14, Cor. 4.1]. □

We find now

Lemma 3.9 (cf. [14, Prop. 3.3])**.** *Our proposed twist* $J_{T,\mathtt{S}}$ *solves both the left and right* 2*-component ABRR equations.*

Proof. Let J denote the solution from Lemma 3.8. We have $J = B + J_+$, where $B = \mathtt{S}^{-1}\Omega_{L^\perp}^{-1/2}$ and $J_+ \in I_+ \otimes I_-$. From the appearance of T_+ in A_L^2, and the fact that $J = A_L^2(J)$, we have

$$J = (A_L^2)^n(J) = (A_L^2)^n(B) + (A_L^2)^n(J_+) = (A_L^2)^n(B),$$

where n is the nilpotence degree of T_+. One establishes the equality

$$(A_L^2)^k(B) = (T_+\otimes 1)(R)\dots(T_+^k\otimes 1)(R)B(T^k\otimes 1)(\Omega)^{-1}\dots(T\otimes 1)(\Omega)^{-1}$$

by induction on k, using the fact that $(T\otimes 1)(\Omega BQ)Q^{-1}\Omega_L^{-1} = B$. This gives $(A_L^2)^n(B) = J_{T,\mathtt{S}}$. □

3.3. **The 3-component and mixed ABRR equations.** For any element $\xi \in u_q \otimes u_q$ take $\xi_{12,3} = (\Delta \otimes 1)(\xi)$ and $\xi_{1,23} = (1 \otimes \Delta)(\xi)$. So the dual cocycle equation for a twist now appears as $J_{12,3}J_{12} = J_{1,23}J_{23}$, where J_{12} and J_{23} are $J \otimes 1$ and $1 \otimes J$ respectively.

Definition 3.10. Take A^3_L and A^3_R to be the linear endomorphisms of $u_+ \otimes u_q \otimes u_-$ defined by

$$A^3_L(\eta) = (T_+ \otimes 1 \otimes 1)(R_{13}R_{12}\eta Q_{12}Q_{13})Q^{-1}_{13}Q^{-1}_{12}(\Omega_L)^{-1}_{13}(\Omega_L)^{-1}_{12}$$

$$A^3_R(\eta) = (1 \otimes 1 \otimes T_-)(R_{13}R_{23}\eta Q_{13}Q_{23})Q^{-1}_{23}Q^{-1}_{13}(\Omega_L)^{-1}_{23}(\Omega_L)^{-1}_{13}.$$

The left and right 3-component ABRR equations are the equations $A^3_L(X) = X$ and $A^3_R(X) = X$.

Let us fix $J = J_{T,\mathtt{S}}$.

Lemma 3.11. *The elements $J_{1,23}J_{23}$ and $J_{12,3}J_{12}$ solve the left and right 3-component ABRR equations respectively.*

Proof. Take $T_1 = (T_+ \otimes 1 \otimes 1)$. We claim first that $A^3_L(J_{1,23}J_{23}) = A^3_L(J_{1,23})J_{23}$. Note that we may replace Q with $\Sigma = \mathtt{S}|\mathcal{L}^\perp \times \mathcal{L}$ in the equation for A^3_L, and that for any bicharacter B we have $B_{12}B_{13} = B_{1,23}$. Also recall that for any cocommutative element $h \in H$ we will have $R\Delta(h) = \Delta(h)R$, and that since $\Sigma \in \mathbb{C}[L^\perp \times L]$ we will have $\Sigma_{1,23} = (1 \otimes T^k_+ \otimes 1)(\Sigma_{1,23})$ for any nonnegative integer k. Using these facts together, along with the particular form of $J = J_{T,\mathtt{S}}$, one see that

$$\begin{aligned} T_1(J_{23}\Sigma_{1,23})\Sigma^{-1}_{1,23}(\Omega_L)^{-1}_{1,23} &= J_{23}(T_1(\Sigma))_{1,23}\Sigma^{-1}_{1,23}(\Omega_L)^{-1}_{1,23} \\ &= (T_1(\Sigma))_{1,23}\Sigma^{-1}_{1,23}(\Omega_L)^{-1}_{1,23}J_{23}, \end{aligned}$$

which implies $A^3_L(J_{1,23}J_{23}) = A^3_L(J_{1,23})J_{23}$. We now note that

$$J_{1,23} = (1 \otimes \Delta)(J) = (1 \otimes \Delta)(A^2_L(J)) = A^3_L(J_{1,23})$$

to see $A^3_L(J_{1,23}J_{23}) = J_{1,23}J_{23}$. The equality $A^3_R(J_{12,3}J_{12}) = J_{12,3}J_{12}$ is proved similarly. □

As was the case for the 2-component equations, one finds that solutions to the equations $A^3_L(X) = X$ and $A^3_R(X) = X$ are uniquely determined by their components in $\mathbb{C}[G] \otimes u_q \otimes u_-$ and $u_- \otimes u_q \otimes \mathbb{C}[G]$ respectively. (See also [14, Lem. 4.3].) We also consider the mixed ABRR equation

$$A^3_L A^3_R(X) = X.$$

Solutions to this equation are uniquely determined by their component in $\mathbb{C}[G] \otimes u_q \otimes \mathbb{C}[G]$, which we denote $X_{0,0}$. Note that

$$(J_{12,3}J_{12})_{0,0} = (J_{1,23}J_{23})_{0,0} = \mathtt{S}^{-1}_{12}\mathtt{S}^{-1}_{13}\mathtt{S}^{-1}_{23}(\Omega^{-1/2}_{L^\perp})_{12}(\Omega^{-1/2}_{L^\perp})_{13}(\Omega^{-1/2}_{L^\perp})_{23}. \quad (5)$$

So we would like to establish

Proposition 3.12. *Both $J_{12,3}J_{12}$ and $J_{1,23}J_{23}$ solve the mixed ABRR equation $A^3_L A^3_R(X) = X$.*

From this proposition one easily finds the proof of Theorem 3.1. We only prove the proposition for $J_{1,23}J_{23}$, the situation for $J_{12,3}J_{12}$ being completely analogous. Let us first give some technical lemmas. Recall $\mathrm{Z} = \mathtt{S}|\mathcal{L} \times \mathcal{L}^\perp$.

Lemma 3.13. *The element Z solves the following equations:*

(i) $(1\otimes T^{-1})(\mathtt{S}^{-1}\Omega_{L^\perp}^{1/2})=\mathtt{S}^{-1}\Omega_{L^\perp}^{-1/2}(1\otimes T^{-1})(\mathrm{Z}^{-1})\mathrm{Z}$.

(ii) $\big[(\Omega_L)_{13}(1\otimes 1\otimes T^{-1})(\mathrm{Z}_{13}^{-1})\mathrm{Z}_{13},(1\otimes 1\otimes T_-^k)(R_{23})\big]=0$ *for all* $k\geq 1$.

Proof. Equation (i) is equivalent to the equation

$$\Omega_{L^\perp}^{1/2}(1\otimes T^{-1})(\Omega_{L^\perp}^{1/2})=\mathtt{S}^{-1}(1\otimes T^{-1})(\mathtt{S})(1\otimes T^{-1})(\mathrm{Z}^{-1})\mathrm{Z},$$

which is seen to hold by (EQ–S), just as in the proof of Lemma 3.7. For (ii) first note that for any $\mu\in\mathcal{G}$ and $\nu\in\mathcal{G}_1$ we have

$$\begin{aligned}&\big(\Omega_L(1\otimes T^{-1})(\mathrm{Z}^{-1})\mathrm{Z}\big)(\mu,\nu)\\ &\qquad=\Omega(\bar\mu,\nu)\mathrm{Z}(\mu,\nu-T\nu)\\ &\qquad=\Omega(\bar\mu,\nu)\mathtt{S}(\bar\mu,\nu-T\nu)\\ &\qquad=\Omega(\bar\mu,\nu)\Omega^{-1/2}(\bar\mu,\nu+T\nu)=\Omega(\bar\mu,\nu)\Omega^{-1}(\bar\mu,\nu)=1,\end{aligned}$$

where $\bar\mu$ is the component of μ in $\mathcal{L}$ under the decomposition $\mathcal{G}=\mathcal{L}\times\mathcal{L}^\perp$. So we see that the bicharacter in question vanishes on $\mathcal{G}\times\mathcal{G}_1$, and hence

$$\Omega_L(1\otimes T^{-1})(\mathrm{Z}^{-1})\mathrm{Z}\in\mathbb{C}[G]\otimes\mathbb{C}[G_1^\perp].$$

It follows that all elements in $\mathbb{C}\otimes u_q\otimes(\mathbb{C}\langle G,F_\beta:\beta\in\Gamma_1\rangle)$ centralize $(\Omega_L)_{13}(1\otimes 1\otimes T^{-1})(\mathrm{Z}_{13}^{-1})\mathrm{Z}_{13}$. Since $(1\otimes 1\otimes T_-^k)(R_{23})$ is in this subspace we have (ii). □

We can now give the

Proof of Proposition 3.12. As noted above, we only prove that $J_{1,23}J_{23}$ solves the mixed ABRR equation. Since this element already satisfies $A_L^3(X)=X$ it suffices to show that it also solves $A_R^3(X)=X$. As in [14, Lem. 4.2], one checks that A_L^3 and A_R^3 commute so that $A_R^3(J_{1,23}J_{23})$ solves the left ABRR equation. By uniqueness of solutions we find that $A_R^3(J_{1,23}J_{23})=J_{1,23}J_{23}$ if and only if these elements have the same component in $\mathbb{C}[G]\otimes u_q\otimes u_-$. Let $A_R^3(J_{1,23}J_{23})_0$ and $(J_{1,23}J_{23})_0$ denote these components. Take $(T_-)_3=(1\otimes 1\otimes T_-)$ and $B=\mathtt{S}^{-1}\Omega_{L^\perp}^{-1/2}$.

Since J is in the subalgebra $\mathbb{C}\langle G\times G,E_\alpha\otimes F_\beta:\alpha,\beta\in\Gamma\rangle$ we see that $(J_{1,23}J_{23})_0=B_{1,23}J_{23}$, and we need to establish

$$A_R^3(J_{1,23}J_{23})_0=B_{1,23}J_{23}.$$

We have

$$\begin{aligned}A_R^3(J_{1,23}J_{23})_0&=(T_-)_3(\Omega_{13}R_{23}B_{1,23}J_{23}\mathrm{Z}_{12,3})\mathrm{Z}_{12,3}^{-1}(\Omega_L)_{12,3}^{-1}\\ &=(T_-)_3(\Omega_{13}B_{1,23}R_{23}J_{23}\mathrm{Z}_{12,3})\mathrm{Z}_{12,3}^{-1}(\Omega_L)_{12,3}^{-1}.\end{aligned}$$

Use the equality $B\Omega_{L^\perp}=\mathtt{S}^{-1}\Omega_{L^\perp}^{1/2}$ and Lemma 3.13 (i) to get

$$\begin{aligned}&A_R^3(J_{1,23}J_{23})_0\\ &=(T_-)_3((\Omega_L)_{13}B_{12}B_{13}(\Omega_{L^\perp})_{13}R_{23}J_{23}\mathrm{Z}_{12,3})\mathrm{Z}_{12,3}^{-1}(\Omega_L)_{12,3}^{-1}\\ &=(\Omega_L)_{13}B_{12}(T_-)_3(B_{13}(\Omega_{L^\perp})_{13}R_{23}J_{23}\mathrm{Z}_{12,3})\mathrm{Z}_{12,3}^{-1}(\Omega_L)_{12,3}^{-1}\\ &=(\Omega_L)_{13}B_{12}B_{13}(T_-)_3(\mathrm{Z}_{13}^{-1})\mathrm{Z}_{13}(T_-)_3(R_{23}J_{23}\mathrm{Z}_{12,3})\mathrm{Z}_{12,3}^{-1}(\Omega_L)_{12,3}^{-1}.\end{aligned}$$

Since J solves the 2-component ABRR equations this final expression reduces to

$$A_R^3(J_{1,23}J_{23})_0=(\Omega_L)_{13}B_{1,23}(T_-)_3(\mathrm{Z}_{13}^{-1})\mathrm{Z}_{13}J_{23}(T_-)_3(\mathrm{Z}_{13})\mathrm{Z}_{13}^{-1}(\Omega_L)_{13}^{-1}.$$

By Lemma 3.13 (ii) this final equation reduces to the desired equality

$$A_R^3(J_{1,23}J_{23})_0=B_{1,23}J_{23}=(J_{1,23}J_{23})_0.$$

This implies that $J_{1,23}J_{23}$ solves the right 3-component ABRR equation, and hence the mixed ABRR equation $A_L^3A_R^3(J_{1,23}J_{23})=J_{1,23}J_{23}$. □

Proof of Theorem 3.1. By uniqueness of solutions to the mixed ABRR equation, Proposition 3.12, and (5), we see that $J_{12,3}J_{12} = J_{1,23}J_{23}$. The remaning identity $(\epsilon \otimes 1)(J) = (1 \otimes \epsilon)(J) = 1$ follows from the identity $(\epsilon \otimes 1)(R) = (1 \otimes \epsilon)(R) = 1$ and the fact that ϵ commutes with $T_\pm$. □

4. Subalgebras from the R-matrix

We recall here some information from Radford's work [26]. We will let $D(H)$ denote the Drinfeld double of a Hopf algebra H. Recall that this is a quasitriangular Hopf algebra which, as a coalgebra, is simply the tensor coalgebra $D(H) = H \otimes (H^*)^{cop}$. Recall also that the two inclusions $H \to D(H)$ and $(H^*)^{cop} \to D(H)$ are Hopf algebra maps. This is all the information we will need about the Drinfeld double, and we invite the reader to see [23, Sect. 10.3] for more information.

4.1. The right and left subalgebras from R.

Let $H = (H, R)$ be a quasitriangular Hopf algebra. We can consider for any $Q \in H \otimes H$ the functions $t_Q : H^* \to H$, $f \mapsto (f \otimes 1)(Q)$ and $t'_Q : H^* \to H$, $f \mapsto (1 \otimes f)(Q)$. Indeed, for any $H_1, H_2 \subset H$ with $Q \in H_1 \otimes H_2$ we can restrict these functions to $t_Q : H_1^* \to H_2$, $t'_Q : H_2^* \to H_1$. For the R-matrix we have the right and left subspaces in H defined as follows.

Definition 4.1. For a quasitriangular Hopf algebra $H = (H, R)$ we take $R_{(r)} = t_R(H^*)$ and $R_{(l)} = t'_R(H^*)$.

We refer to these subalgebras as the Radford subalgebras associated to R. Using the properties of the R-matrix one shows

Proposition 4.2 ([26, Prop. 2])**.** *The subspaces $R_{(l)}$ and $R_{(r)}$ are Hopf subalgebras in H. Furthermore, the maps t_R and t'_R provide Hopf morphisms $(H^*)^{cop} \to H$ and $(H^*)^{op} \to H$, and Hopf isomorphisms $(R^*_{(l)})^{cop} \overset{\cong}{\to} R_{(r)}$ and $(R^*_{(r)})^{op} \overset{\cong}{\to} R_{(l)}$.*

Take $H_R = R_{(l)}R_{(r)}$. It turns out that this is a Hopf subalgebra in H, and that it is the minimal Hopf subalgebra in H with $R \in H_R \otimes H_R$. A quasitriangular Hopf algebra is called *minimal* if $H = H_R$. Strictly speaking, we will not be needing the following result. It does, however, inform the approach of the current work, and so we repeat it here.

Theorem 4.3 ([26, Thm. 2])**.** *For a minimal Hopf algebra H there is a (unique) surjective map of quasitriangular Hopf algebras $Y : D(R_{(l)}) \to H$ with $Y|R_{(r)}$ the inclusion and $Y|(R^*_{(l)})^{cop} = t_R$.*

Taking the dual of Y, we see that there is a algebra inclusion

$$H^* \to R^*_{(l)} \otimes R^{op}_{(l)} \cong R_{(r)} \otimes R^{op}_{(l)}$$

given as the composite $H^* \overset{\Delta}{\to} H^* \otimes H^* \overset{t_R \otimes t'_R}{\longrightarrow} R_{(r)} \otimes R^{op}_{(l)}$.

We note that although minimality is not preserved under twists, a stronger condition called factorizability is preserved under twists. Indeed, a finite dimensional quasitriangular Hopf algebra H is factorizable if and only if the Müger center of rep(H) is trivial [15]. Small quantum groups are examples of factorizable Hopf algebras, and so the twists u_q^J will be factorizable, and hence minimal as well.

4.2. **Bicharacters as R-matrces on abelian groups.**

Lemma 4.4. *Let Λ be a finite abelian group. Any bicharacter $B \in \mathbb{C}[\Lambda] \otimes \mathbb{C}[\Lambda]$ is an R-matrix for $\mathbb{C}[\Lambda]$.*

Proof. We need to check the equations $(\Delta \otimes 1)(B) = B_{13}B_{23}$, $(1 \otimes \Delta)(B) = B_{13}B_{12}$, and $B\Delta(\lambda)B^{-1} = \Delta(\lambda)$ for each $\lambda \in \Lambda$. The first two equations follow from the fact that B is a bicharacter, and the final equation follows from the fact that Λ is abelian. □

In the case of a bicharacter B giving an R-matrix for $\mathbb{C}[\Lambda]$, the two maps t_B and t'_B restrict to, and are specified by, the standard group maps $\Lambda^\vee \to \Lambda$ induced by B.

Definition 4.5. For a finite abelian group Λ and bicharacter $B \in \mathbb{C}[\Lambda] \otimes \mathbb{C}[\Lambda]$, we let $\Lambda_{(r)}$ and $\Lambda_{(l)}$ denote the images $t_B(\Lambda^\vee)$ and $t'_B(\Lambda^\vee)$ in Λ respectively.

We have $B_{(r)} = \mathbb{C}[\Lambda_{(r)}]$ and $B_{(l)} = \mathbb{C}[\Lambda_{(l)}]$.

5. Parabolic subalgebras in $u_q(\mathfrak{g})^J$ and Radford's subalgebras

For this section fix a Belavin-Drinfeld triple (Γ_1, Γ_2, T) and solution $\mathtt{S}$ to (EQ–S). Fix also $J = J_{T,\mathtt{S}}$ from Theorem 3.1. Recall $u_q = u_q(\mathfrak{g})$.

We saw in Section 4.1 that there are algebra surjections $t_{R^J} : (u_q^J)^* \to R^J_{(r)}$ and $t'_{R^J} : (u_q^J)^* \to (R^J_{(l)})^{op}$. Our main goal is to show that the map

$$\mathrm{Irrep}(R^J_{(r)}) \to \mathrm{Irrep}\left((u_q^J)^*\right)$$

induced by restriction is a bijection, modulo the action of a finite character group. In order to understand the irreducible representations of $R^J_{(r)}$, and to establish the proposed bijection, we need to understand the subalgebra $R^J_{(r)}$. The present section is dedicated to a study of the subalgebras $R^J_{(l)}$ and $R^J_{(r)}$.

For any root α we will take $\bar{\alpha}$ to be the component of α in $\mathcal{L}$, under the decomposition $\mathcal{G} = \mathcal{L} \times \mathcal{L}^\perp$.

5.1. **A preemptive change of coordinates.** Let us take

$$\mathbf{E}_\alpha = q^{\frac{1}{4}(\bar{\alpha},\bar{\alpha})} K_{\bar{\alpha}}^{-1/2} E_\alpha \quad \text{and} \quad \mathbf{F}_\beta = q^{\frac{1}{4}(\bar{\beta},\bar{\beta})} K_{\bar{\beta}}^{1/2} F_\beta.$$

These new generators satisfy the appropriate relations so that we have an algebra automorphism

$$\text{change of coord's} : u_q \overset{\cong}{\to} u_q, \quad \begin{cases} E_\alpha \mapsto \mathbf{E}_\alpha \\ F_\beta \mapsto \mathbf{F}_\beta \\ K_\mu \mapsto K_\mu. \end{cases}$$

Recall that each E_μ, $\mu \in \Phi^+$, is a linear combination of permutations of the monomial $E_{\alpha_{i_1}} \dots E_{\alpha_{i_m}}$, where $\mu = \alpha_{i_1} + \dots + \alpha_{i_m}$ with the α_{i_k} simple. So each E_μ is sent to $q^{\frac{1}{4}(\bar{\mu},\bar{\mu})} K_{\bar{\mu}}^{-1/2} E_\mu$ under the above change of coordinates. A similar statement holds for the F_ν, and we may adopt a consistent notation

$$\mathbf{E}_\mu = q^{\frac{1}{4}(\bar{\mu},\bar{\mu})} K_{\bar{\mu}}^{-1/2} E_\mu \quad \text{and} \quad \mathbf{F}_\nu = q^{\frac{1}{4}(\bar{\nu},\bar{\nu})} K_{\bar{\nu}}^{1/2} F_\nu,$$

for $\mu, \nu \in \Phi^+$. These bold elements produce a $\mathbb{C}[G]$-basis for $u_q = u_q^J$ just as in Lemma 1.1.

5.2. **The quantum parabolics in $u_q(\mathfrak{g})^J$ and the R-matrix.** For a fixed subset $\Sigma \subset \Gamma$ we let $\mathsf{p}_+ = \mathsf{p}_+(\Sigma)$ denote the corresponding positive parabolic in g and $u_q(\mathsf{p}_+)$ denote the Hopf subalgebra

$$u_q(\mathsf{p}_+) = \mathbb{C}\langle G, \mathbf{E}_\alpha, \mathbf{F}_\beta : \alpha \in \Gamma,\ \beta \in \Sigma\rangle \ \subset \ u_q(\mathsf{g}).$$

We have the negative analog

$$u_q(\mathsf{p}_-) = \mathbb{C}\langle G, \mathbf{E}_\beta, \mathbf{F}_\alpha : \beta \in \Sigma, \alpha \in \Gamma\rangle \ \subset \ u_q(\mathsf{g}).$$

We let $u_q(\mathsf{p}^{ss})$ denote the small quantum group associated to the (union of) Dynkin diagram(s) Σ in Γ. We suppose additionally that the perpendicular $G_\Sigma^\perp$ to the subgroup $G_\Sigma = \mathbb{Z}/l\mathbb{Z} \cdot \{K_\beta : \beta \in \Sigma\}$ in G is a complement to G_Σ.

Lemma 5.1. *Let Σ be a subset in Γ and p denote the corresponding positive (resp. negative) parabolic. There is an algebra surjection*

$$u_q(\mathsf{p}) \to \mathbb{C}[G_\Sigma^\perp] \otimes u_q(\mathsf{p}^{ss}), \quad \begin{cases} \mathbf{E}_\beta \mapsto \mathbf{E}_\beta & \text{when } \beta \in \Sigma \\ \mathbf{F}_\beta \mapsto \mathbf{F}_\beta & \text{when } \beta \in \Sigma \\ \mathbf{E}_\alpha \ (\textit{resp. } \mathbf{F}_\alpha) \mapsto 0 & \text{when } \alpha \in \Gamma - \Sigma \\ K_\gamma \mapsto K_\gamma \end{cases} \tag{6}$$

with kernel equal to the nilpotent ideal $N = (\mathbf{E}_\alpha : \alpha \in \Gamma - \Sigma)$ (resp. $N' = (\mathbf{F}_\alpha : \alpha \in \Gamma - \Sigma)$).

Proof. It suffices to prove the result for the positive parabolic. We arrive at the result for the negative quantum parabolic by considering the automorphism of $u_q(\mathsf{g})$ which exchanges the E_α and F_α, and inverts the K_γ, and hence exchanges the positive and negative parabolics. Simply by checking relations we see that there is a surjective algebra map $\mathbb{C}[G_\Sigma^\perp] \otimes u_q(\mathsf{p}^{ss}) \to u_q(\mathsf{p})/N$ defined on the generators in the obvious way. We will show that this map is injective by counting dimensions.

We have that the nonnegative part of $u_q(\mathsf{p})$ is all of u_+, and we see for grading reasons that $u_q(\mathsf{p})_-$ is free over $\mathbb{C}[G]$ with basis given by ordered monomials in the $\mathbf{F}_\nu$ with ν a positive root in the $\mathbb{Z}$-span on Σ (see Theorem 1.1). By the commutativity relation between the $\mathbf{E}$ and $\mathbf{F}$ we see that the restriction of the multiplication map $\theta : u(\mathsf{p})_+ \otimes_{\mathbb{C}[G]} u(\mathsf{p})_- \to u_q(\mathsf{p})$ is surjective. Since this map is given by restricting the isomorphism $u_q(\mathsf{g})_+ \otimes_{\mathbb{C}[G]} u_q(\mathsf{g})_- \to u_q(\mathsf{g})$, and since all modules are flat over $\mathbb{C}[G]$, we see that θ is injective as well. It follows that $u_q(\mathsf{p})$ has the obvious basis consisting of ordered monomials in the $\mathbf{E}_\mu$ and $\mathbf{F}_\nu$, where ν is as above.

When we take the quotient we now see that $u_q(\mathsf{p})/N$ has a $\mathbb{C}[G]$-basis of orderend monomials in the $\mathbf{E}_{\mu'}$ and $\mathbf{F}_\nu$, with $\mu', \nu \in \Phi^+ \cap (\mathbb{Z} \cdot \Sigma)$. Since $\Psi = \Phi \cap (\mathbb{Z} \cdot \Sigma)$ is the root system for p^{ss}, we find by Lusztig's basis for $u_q(\mathsf{p}^{ss})$ that $u_q(\mathsf{p})/N$ and $\mathbb{C}[G_\Sigma^\perp] \otimes u_q(\mathsf{p}^{ss})$ have the same dimension. Whence our surjection is an isomorphism. The inverse is given by the same formulas as (6), and implies the existence of (6).

As for nilpotence of N, when we grade by the group $\mathbb{Z}\{\Gamma - \Sigma\}$ we see that N^k is in degrees $\mathbb{Z}_{\geq k}\{\Gamma - \Sigma\}$. Since $u_q(\mathsf{p})$ is finite dimensional it has no nonzero elements in degrees $\mathbb{Z}_{\geq k}\{\Gamma - \Sigma\}$ for large k. □

5.3. **Quantum parabolics and BD triples.**

Definition 5.2. For any Belavin-Drinfeld triple (Γ_1, Γ_2, T) we let $u_q(\mathsf{p}_1)$ and $u_q(\mathsf{p}_2)$ denote the positive and negative quantum parabolics in $u_q(\mathsf{g})$ corresponding to Γ_1 and Γ_2 respectively.

So $u_q(\mathsf{p}_1)$ contains only the $\mathbf{F}_\alpha$ with $\alpha \in \Gamma_1$, and $u_q(\mathsf{p}_2)$ contains only the $\mathbf{E}_\beta$ with $\beta \in \Gamma_2$. Recall our twist $J = J_{T,\mathtt{S}}$ and the definition $R^J = J_{21}^{-1} R J$. We have also

$$\begin{aligned} &J_{21}^{-1} \\ &= (1 \otimes T)(\Omega) \dots (1 \otimes T^n)(\Omega)\mathtt{S}^{-1}\Omega_{L^\perp}^{1/2}(S^{-1} \otimes T_+^n)(R_{21}) \dots (S^{-1} \otimes T_+)(R_{21}) \\ &= (T^{-1} \otimes 1)(\Omega) \dots (T^{-n} \otimes 1)(\Omega)\mathtt{S}^{-1}\Omega_{L^\perp}^{1/2}(T_-^n \otimes S)(R_{21}) \dots (T_- \otimes S)(R_{21}). \end{aligned} \tag{7}$$

Lemma 5.3. *There are containments $R^J_{(r)} \subset u_q(\mathsf{p}_2)$ and $R^J_{(l)} \subset u_q(\mathsf{p}_2)$.*

Proof. This is immediate from the form of J and R, and the fact that $(T_+^k \otimes 1)(R) = (1 \otimes T_-^k)(R)$. □

We consider $\mathbb{C}[G]$ as a quasitriangular Hopf algebra with R-matrix Ω. Then $\mathtt{S}^{-1}$ provides a twist for $\mathbb{C}[G]$ and new R-matrix $\mathtt{S}_{21}\Omega\mathtt{S}^{-1} = \mathtt{S}^{-2}\Omega$. We take $G_{(r)}$ and $G_{(l)}$ the right and left subgroups associated to $\mathtt{S}^{-2}\Omega$, as in Section 4.2. Note that by the duality $t_{\Omega^{\mathtt{S}^{-1}}} : G_{(l)}^\vee \overset{\cong}{\to} G_{(r)}$ we know that these two groups have the same order. We want to prove

Proposition 5.4. *The inclusions $R^J_{(r)} \subset u_q(\mathsf{p}_2)$ and $R^J_{(l)} \subset u_q(\mathsf{p}_1)$ are equalities exactly when $G_{(r)} = G_{(l)} = G$. In general, we have that*

$$R^J_{(r)} = \mathbb{C}\langle G_{(r)}, \mathbf{E}_\beta, \mathbf{F}_\gamma : \beta \in \Gamma_2, \gamma \in \Gamma\rangle$$

and

$$R^J_{(l)} = \mathbb{C}\langle G_{(l)}, \mathbf{E}_\gamma, \mathbf{F}_\alpha : \alpha \in \Gamma_1, \gamma \in \Gamma\rangle$$

in $u_q(\mathsf{g})$.

Section 6 is dedicated to a proof of Proposition 5.4. As a corollary we will have

Corollary 5.5. *Let $\mathcal{N} \subset R^J_{(r)}$ denote the preimage of the ideal $N = (\mathbf{F}_\beta : \beta \in \Gamma - \Gamma_2)$ in $u_q(\mathsf{p}_2)$ along the inclusion $R^J_{(r)} \to u_q(\mathsf{p}_2)$. Take also $\Lambda = G_{(r)} \cap G_2^\perp$. Then we have a canonical algebra isomorphism*

$$R^J_{(r)}/\mathcal{N} \overset{\cong}{\to} \mathbb{C}[\Lambda] \otimes u_q(\mathsf{p}_2^{ss}).$$

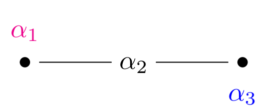
For the analogously defined $\mathcal{N}' \subset R^J_{(l)}$ and $\Lambda' \subset G_{(l)}$ we have also

$$R^J_{(l)}/\mathcal{N}' \overset{\cong}{\to} \mathbb{C}[\Lambda'] \otimes u_q(\mathsf{p}_1^{ss}).$$

Proof. The isomorphisms come from restricting the isomorphisms of Lemma 5.1 along the inclusion $R^J_{(*)} \to u_q(\mathsf{p}_*)$. □

5.4. **An example.** It seems, from considering examples, that the subalgebra $R^J_{(r)}$ will often be the full parabolic $u_q(\mathsf{p}_2)$. This will always be the case, for example, when considering twists associated to maximal triples (Γ_1, Γ_2, T) on A_n (see the discussion following Lemma 3.2). To construct an example for which the containment $R^J_{(r)} \subset u_q(\mathsf{p}_2)$ is proper we need only construct an example for which the containment $G_{(r)} \subset G$ is proper.

We claim that in the following example we will have $G_{(r)} \subsetneq G$ and $R^J_{(r)} \subsetneq u_q(\mathsf{p}_2)$: Take $l = 3$ and consider the tiple on A_3



with $\Gamma_1 = \{\alpha_1\}$, $\Gamma_2 = \{\alpha_3\}$, $T(\alpha_1) = \alpha_3$. We have here

$$\mathcal{L} = \mathbb{Z}/3\mathbb{Z} \cdot \{\alpha_2, \alpha_1 + \alpha_3\}, \quad \mathcal{G}_2^\perp = \mathbb{Z}/3\mathbb{Z} \cdot \{\alpha_1, \alpha_2 + \frac{1}{2}\alpha_3\}$$

and the Killing form on $\mathcal{G}_2^\perp$ is given (in multiplicative notation) by

$$(\alpha_1, \alpha_1) = q^2, \ (\alpha_1, \alpha_2 + \frac{1}{2}\alpha_3) = q^{-1}, \ (\alpha_2 + \frac{1}{2}\alpha_3, \alpha_2 + \frac{1}{2}\alpha_3) = 1.$$

The (unique) solution $\mathtt{S}$ with $\mathtt{S}(\alpha_1 + \alpha_3, \alpha_2) = q^{-1}$ is such that

$$\begin{array}{ll} \mathtt{S}^{-2}\Omega(\alpha_1 + \alpha_3, \alpha_1) = q^{-1}, & \mathtt{S}^{-2}\Omega(\alpha_2, \alpha_1) = q \\ \mathtt{S}^{-2}\Omega(\alpha_1 + \alpha_2, \alpha_2 + \frac{1}{2}\alpha_3) = q, & \mathtt{S}^{-2}\Omega(\alpha_2, \alpha_2 + \frac{1}{2}\alpha_3) = q^{-1} \end{array}$$

and hence

$$t_{\mathtt{S}^{-2}\Omega}(\alpha_1 + \alpha_3) = t_{\mathtt{S}^{-2}\Omega}(\alpha_2)^{-1} = K_{\alpha_1}^{-1} K_{\alpha_2 + \frac{1}{2}\alpha_3}^{-1} \mod G_2.$$

It follows that $G_{(r)} = \mathbb{Z}/3\mathbb{Z} \oplus G_2 = (\mathbb{Z}/3\mathbb{Z})^2$ is a proper subgroup of $G = (\mathbb{Z}/3\mathbb{Z})^3$. Alternatively, the solution with $\mathtt{S}(\alpha_1 + \alpha_3, \alpha_2) = 1$ yields $G_{(r)} = G$.

6. A proof of Proposition 5.4

Fix $J = J_{T,\mathtt{S}}$. We will prove by direct calculation that $R^J_{(r)}$ is as proposed in Proposition 5.4. Let J' be the twist associated to the triple $\Gamma_1' = \Gamma_2$, $\Gamma_2' = \Gamma_1$, $T' = T^{-1}$, and solution $\mathtt{S}' = \mathtt{S}^{-1}$. The result for $R^J_{(l)}$ can subsequently be deduced from the fact that the algebra automorphism $\phi : u_q \to u_q$, which exchanges E_α with F_α and sends K_γ to K_γ^{-1}, is such that $\phi(R^J_{(l)}) = R^{J'}_{(r)}$.

6.1. Some supporting results.

Lemma 6.1. *$G_{(r)} \subset R^J_{(r)}$ and $G_2 \subset G_{(r)}$.*

Proof. We have the (u_-, u_+)-bimodule isomorphism $u_- \otimes_{\mathbb{C}[G]} u_+ \to u_q$ given by multiplication. The two projections $u_\pm \to \mathbb{C}[G]$ then give $u_- \otimes_{\mathbb{C}[G]} u_+ \to \mathbb{C}[G]$ and hence a bimodule projection $\Pi : u_q \to \mathbb{C}[G]$. Taking the dual gives an embedding $\Pi^* : \mathbb{C}[G]^* \to u_q^*$. We have

$$(\Pi \otimes 1)(R^J) = \mathtt{S}^{-1}\Omega_{L^\perp}^{1/2}\Omega\mathtt{S}^{-1}\Omega_{L^\perp}^{-1/2} = \mathtt{S}^{-2}\Omega \in \mathbb{C}[G] \otimes u_q.$$

Note that $\mathtt{S}^{-2}\Omega = (\mathtt{S}_{21}^{-1})^{-1}\Omega\mathtt{S}^{-1}$ so that for any character $\mu \in \mathcal{G}$ we have

$$(\mu\Pi \otimes 1)(R^J) = t_{\Omega\mathtt{S}^{-1}}(\mu)$$

and hence $t_{R^J}(\mathcal{G}\Pi) = G_{(r)}$. This gives the proposed inclusion $G_{(r)} \subset R^J_{(r)}$.

As for the inclusion $G_2 \subset G_{(r)}$, note that for $\alpha \in \Gamma_1$ we have

$$\mathtt{S}^{-2}\Omega(\alpha - T\alpha, ?) = \Omega^{-1}(\alpha + T\alpha, ?)\Omega(\alpha - T\alpha, ?) = \Omega^{-2}(T\alpha, ?) = K_{T\alpha}^2.$$

Since 2 is a unit in $\mathbb{Z}/l\mathbb{Z}$ we see that each $K_{T\alpha} \in G_{(r)}$ and hence $G_2 \subset G_{(r)}$. □

The inclusion $G_2 \subset G_{(r)}$ and splitting $G = G_2^\perp \times G_2$ implies that $G_{(r)}$ splits as $G_{(r)} = \Lambda \times G_2$, where $\Lambda = G_2^\perp \cap G_{(r)}$.

In the following lemma we use the fact that for any bicharacter $B \in \mathbb{C}[G] \otimes \mathbb{C}[G]$ we have

$$B = \sum_{\mu,\nu \in \mathcal{G}} B(\mu, \nu) P_\mu \otimes P_\nu,$$

where $P_\mu = \frac{1}{|G|}\sum_{\gamma\in\mathcal{G}}\mu(K_\gamma^{-1})K_\gamma$ is the idempotent associated to μ. We have $P_\mu P_\nu = \delta_{\mu,\nu}P_\mu$ and $\mu(P_\nu)=\delta_{\mu,\nu}$. For any bicharacter B and $\mu\in\mathcal{G}$ we take

$$B(\mu) = \text{the unique character on } G \text{ with } B(\mu,\nu)=q^{(B(\mu),\nu)}\ \forall\ \nu\in\mathcal{G}.$$

Lemma 6.2. *For any bicharacter B, and $\alpha,\beta\in\Gamma$, we have*

$$(E_\alpha\otimes F_\beta)B = B(K_{B_{21}(\beta)}\otimes K_{B^{-1}(\alpha)})(E_\alpha\otimes F_\beta)$$

and

$$(F_\beta\otimes E_\alpha)B = B(K_{B_{21}^{-1}(\alpha)}\otimes K_{B(\beta)})(F_\beta\otimes E_\alpha).$$

Proof. We have

$$E_\alpha K_\gamma = q^{-(\alpha,\gamma)}K_\gamma E_\alpha \Rightarrow E_\alpha P_\mu = P_{\mu+\alpha}E_\alpha$$

and $F_\beta P_\nu = P_{\nu-\beta}F_\beta$. So for any bicharacter B we have

$$\begin{aligned}
&(E_\alpha\otimes F_\beta)B = (\textstyle\sum_{\mu,\nu}B(\mu,\nu)P_{\mu+\alpha}\otimes P_{\nu-\beta})(E_\alpha\otimes F_\beta)\\
&= (\textstyle\sum_{\mu,\nu}B(\mu-\alpha,\nu+\beta)P_\mu\otimes P_\nu)(E_\alpha\otimes F_\beta)\\
&= B(\textstyle\sum_{\mu,\nu}B(\mu,\beta)B^{-1}(\alpha,\nu)P_\mu\otimes P_\nu)(E_\alpha\otimes F_\beta)\\
&= B(K_{B_{21}(\beta)}\otimes K_{B^{-1}(\alpha)})E_\alpha\otimes F_\beta.
\end{aligned}$$

We arrive at the equation for $F_\beta\otimes E_\alpha$ similarly. □

Considering the case $B=\Omega_L^{1/2}$, and $\beta\in\Gamma_2$, gives

$$\begin{aligned}
(E_\beta\otimes F_{T^{-k}\beta})\Omega_L^{1/2} &= \Omega_L^{1/2}(K_{T^{-k}\beta}^{-1/2}E_\beta\otimes K_{\bar\beta}^{1/2}F_{T^{-k}\beta})\\
&= \Omega_L^{1/2}(K_{\bar\beta}^{-1/2}E_\beta\otimes K_{\bar\beta}^{1/2}F_{T^{-k}\beta}) = q^{-\frac12(\bar\beta,\bar\beta)}\Omega_L^{1/2}(\mathbf{E}_\beta\otimes\mathbf{F}_{T^{-k}\beta}).
\end{aligned}$$

Similarly $(F_\alpha\otimes E_{T^k\alpha})\Omega_L^{1/2} = q^{-\frac12(\bar\alpha,\bar\alpha)}\Omega_L^{1/2}(\mathbf{F}_\alpha\otimes\mathbf{E}_{T^k\alpha})$ for $\alpha\in\Gamma_1$.

6.2. **Proof of Proposition 5.4.** As explained in the beginning of the section, we need only prove the proposition for $R^J_{(r)}$. We prove the proposition in two parts. First we establish the containment $\mathbb{C}\langle G_{(r)},\mathbf{E}_\alpha,\mathbf{F}_\beta:\alpha\in\Gamma_2,\beta\in\Gamma\rangle\subset R^J_{(r)}$, then we establish the opposite containment.

Proof of Proposition 5.4. **Part I:** Take

$$\Omega(k,m) = \prod_{k\le i\le m}(T^i\otimes 1)(\Omega)\quad\text{and}\quad \Omega'(k,m) = \prod_{k\le j\le m}(1\otimes T^j)(\Omega),$$

with the empty product equal to 1. Now J appears as

$$(1\otimes T_-)(R)\dots(1\otimes T_-^n)(R)\mathtt{S}^{-1}\Omega_{L^\perp}^{-1/2}\Omega(1,n)^{-1}$$

and J_{21}^{-1} appears as

$$\Omega'(1,n)\mathtt{S}^{-1}\Omega_{L^\perp}^{1/2}(S^{-1}\otimes T_+^n)(R_{21})\dots(S^{-1}\otimes T_+)(R_{21}).$$

It suffices to prove that each of the $\mathbf{E}_\alpha$ and $\mathbf{F}_\beta$ are in $R^J_{(l)}$, by Lemma 6.1.

From our $\mathbb{C}[G]$-basis for u_q we have the $\mathbb{C}[G]$-linear projection

$$\pi^E_\beta : u_q\to\mathbb{C}[G]E_\beta$$

which annihilates each of the basis elements from Theorem 1.1, save for E_β. More specifically, we take π^E_β to be the obvious projection composed with the scaling by $q(1-q^2)^{-1}$. Then we have

$$\begin{aligned}
&(\pi^E_\beta\otimes 1)(R^J)\\
&=\textstyle\sum_{k=0}^{m(\beta)} \mathtt{S}^{-1}\Omega^{1/2}_{L^\perp}\Omega(0,k-1)(E_\beta\otimes F_{T^{-k}\beta})\Omega(k,n)\mathtt{S}^{-1}\Omega^{-1/2}_{L^\perp}\Omega(1,n)^{-1}\\
&=\textstyle\sum_k \mathtt{S}^{-1}\Omega^{1/2}_{L^\perp}\Omega(0,k-1)(E_\beta\otimes F_{T^{-k}\beta})\Omega(1,k-1)^{-1}\mathtt{S}^{-1}\Omega^{-1/2}_{L^\perp}\\
&=\textstyle\sum_k \mathtt{S}^{-1}\Omega^{1/2}_{L^\perp}\Omega(0,k-1)(E_\beta\otimes F_{T^{-k}\beta})\Omega(0,k-1)^{-1}\mathtt{S}^{-1}\Omega^{1/2}\Omega^{1/2}_{L},
\end{aligned}$$

where $m(\beta)=0$ when $\beta\notin\Gamma_2$ and otherwise $m(\beta)$ is minimal with $T^{-m(\beta)}\beta\notin\Gamma_2$ and $T^{-i}\beta\in\Gamma_2$ for $0\le i<m(\beta)$. We have

$$\Omega(0,k-1)^{-1}_{21}(T^{-k}\beta)=\sum_{i=1}^{k}T^{-i}\beta\quad\text{and}\quad \Omega(0,k-1)(\alpha)=-\sum_{j=0}^{k-1}T^j\beta$$

so that the final expression reduces to

$$\begin{aligned}
&\textstyle\sum_k \mathtt{S}^{-1}\Omega^{1/2}_{L^\perp}(K_{\sum_{i=1}^k T^{-i}\beta}\otimes K^{-1}_{\sum_{j=0}^{k-1}T^j\beta})(E_\beta\otimes F_{T^{-k}\beta})\mathtt{S}^{-1}\Omega^{1/2}\Omega^{1/2}_L\\
&=\textstyle\sum_k \mathtt{S}^{-2}\Omega_{L^\perp}\Omega^{1/2}_L(K^{1/2}_{\mathtt{S}^2\Omega(T^{-k}\beta)}K_{\sum_i T^{-i}\beta}\otimes K^{-1/2}_{\mathtt{S}^{-2}\Omega(\beta)}K^{-1}_{\sum_j T^j\beta})(E_\beta\otimes F_{T^{-k}\beta})\Omega^{1/2}_L\\
&=q^{-\frac12(\bar\beta,\bar\beta)}\textstyle\sum_k \mathtt{S}^{-2}\Omega(K^{1/2}_{\mathtt{S}^2\Omega(T^{-k}\beta)}K_{\sum_i T^{-i}\beta}\otimes K^{-1/2}_{\mathtt{S}^{-2}\Omega(\beta)}K^{-1}_{\sum_j T^j\beta})(\mathbf{E}_\beta\otimes\mathbf{F}_{T^{-k}\beta}).
\end{aligned}$$

For $\epsilon_\beta:\mathbb{C}[G]E_\beta\to\mathbb{C}$, $gE_\beta\mapsto q^{\frac12(\bar\beta,\bar\beta)}$, we then have

$$(\epsilon_\beta\pi^E_\beta\otimes 1)(R^J)=\sum_{k=0}^{m(\beta)}K^{-1/2}_{\mathtt{S}^{-2}\Omega(\beta)}K^{-1}_{\sum_{j=0}^{k-1}T^j\beta}\mathbf{F}_{T^{-k}\beta}\in R^J_{(r)}. \tag{8}$$

Note that the coefficients $K^{-1/2}_{\mathtt{S}^{-2}\Omega(\beta)}K^{-1}_{\sum_{j=0}^{k-1}T^j\beta}$ are all in $G_{(r)}$.

When $m(\beta)=0$, i.e. when $\beta\in\Gamma-\Gamma_2$, the sum (8) is just the element $K^{-1/2}_{\mathtt{S}^{-2}\Omega(\beta)}\mathbf{F}_\beta$. Since $K^{-1/2}_{\mathtt{S}^{-2}\Omega(\beta)}\in G_{(r)}\subset R^J_{(r)}$ this implies $\mathbf{F}_\beta\in R^J_{(r)}$. Since $m(\beta)=m(T^{-1}\beta)+1$ when $\beta\in\Gamma_2$, it now follows from (8) and induction on $m(\beta)$ that all $\mathbf{F}_\beta\in R^J_{(r)}$.

The computation for the $\mathbf{E}_\beta$, $\beta\in\Gamma_2$, is quite similar. Namely, one show for $\alpha\in\Gamma_1$ that $(\pi^F_\alpha\otimes 1)(R^J)$ is the a sum

$$q^{-\frac12(\bar\alpha,\bar\alpha)}\sum_{k=1}^{m'(\alpha)}\mathtt{S}^{-2}\Omega(g_k\otimes K^{1/2}_{\mathtt{S}^{-2}\Omega(\alpha)}K^{-1}_{T^k\alpha+\sum_{j=1}^k T^j\alpha})(\mathbf{F}_\alpha\otimes\mathbf{E}_{T^k\alpha}),$$

where π^F_α is a scaling of the obvious projection and $g_k\in G$, then proceeds by induction on $m'(\alpha)$ just as above.

Part II: We now give the opposite containment $R^J_{(r)}\subset\mathbb{C}\langle G_{(r)},\mathbf{E}_\alpha,\mathbf{F}_\beta:\alpha\in\Gamma_2,\beta\in\Gamma\rangle$ to complete the proof. We adopt the same notation $\Omega(k,m)$ and $\Omega'(k,m)$ as above. We have that R^J is a $\mathbb{C}[G]\otimes\mathbb{C}[G_{(r)}]$-linear combination of elements of the form $M_1M_2M_3$ with

$$\begin{aligned}
M_1&=\Omega'(1,n)\mathtt{S}^{-1}\Omega^{1/2}_{L^\perp}(F_{\xi_n}\otimes E_{T^n\xi_n})(1\otimes T^n)(\Omega)^{-1}\dots(F_{\xi_1}\otimes E_{T\xi_1})(1\otimes T)(\Omega)^{-1},\\
M_2&=(E_\zeta\otimes F_\zeta)\Omega,\\
M_3&=(E_{\eta_1}\otimes F_{T^{-1}\eta_1})(T\otimes 1)(\Omega)\dots(E_{\eta_n}\otimes F_{T^{-n}\eta_n})\Omega(0,n-1)^{-1}\mathtt{S}^{-1}\Omega^{1/2}\Omega^{1/2}_L.
\end{aligned}$$

Here the ξ_k are in $\mathbb{Z}_{\ge0}\Gamma_1$ with $T^i(\xi_k)\in\mathbb{Z}_{\ge0}\Gamma_1$ for each $0\le i<k$. We take a similar restriction for the $\eta_j\in\mathbb{Z}_{\ge0}\Gamma_2$ and let ζ be arbitrary in the positive root

lattice. For $\tau = \alpha_1 + \cdots + \alpha_m$ with the α_i simple roots, by E_τ (resp. F_τ) we simply mean some permutation of the monomial $E_{\alpha_1} \dots E_{\alpha_n}$ (resp. $F_{\alpha_1} \dots F_{\alpha_n}$). So we are deviating from the notation of Theorem 1.1 here.

One simply moves all the bicharacters from the right to left, in order, using Lemma 6.2, to find

$$\begin{aligned} &M_1M_2M_3 \\ &= q^\epsilon(g_1 \otimes g_2)\mathsf{S}^{-2}\Omega\left((\textstyle\prod_i \mathbf{F}_{\xi_i})\mathbf{E}_\zeta(\prod_j \mathbf{E}_{\eta_j}) \otimes (\prod_i \mathbf{E}_{T^i\xi_i})\mathbf{F}_\zeta(\prod_j \mathbf{F}_{T^{-j}\eta_j})\right)\end{aligned}$$

with $g_1 \in G$ and $g_2 \in G_{(r)}$. Hence for any $f \in u_q^*$ we will have, for some constant $c_f \in \mathbb{C}$,

$$\begin{aligned} &(f \otimes 1)(M_1M_2M_3) \\ &= c_f g_2 t_{\mathsf{S}^{-2}\Omega}(f)(\textstyle\prod_i \mathbf{E}_{T^i\xi_i})\mathbf{F}_\zeta(\prod_j \mathbf{F}_{T^{-j}\eta_j}) \in \mathbb{C}\langle G_{(r)}, \mathbf{E}_\alpha, \mathbf{F}_\beta : \alpha \in \Gamma_2, \beta \in \Gamma\rangle.\end{aligned}$$

Since R^J is a sum of such monomials $M_1M_2M_3$ we find

$$R^J_{(r)} = t_{R^J}(u_q^*) \subset \mathbb{C}\langle G_{(r)}, \mathbf{E}_\alpha, \mathbf{F}_\beta : \alpha \in \Gamma_2, \beta \in \Gamma\rangle.$$

□

7. Representation theory of the dual $(u_q(\mathsf{g})^J)^*$

In this section we describe the irreducible representations of the dual $(u_q(\mathsf{g})^J)^*$, for $J = J_{T,\mathsf{S}}$ as in Theorem 3.1. As always $u_q = u_q(\mathsf{g})$.

7.1. Grouplikes and the parabolic subalgebras.

Lemma 7.1. *Each $K_\mu \in L$ is grouplike in the twist u_q^J.*

Proof. We claim that $K_\mu \otimes K_\mu$ commutes with J, so that $\Delta^J(K_\mu) = J^{-1}(K_\mu \otimes K_\mu)J = K_\mu \otimes K_\mu$. From the particular form on J, we see that it suffices to show that $K_\mu \otimes K_\mu$ commutes with $T_+^k E_\nu \otimes F_\nu$ and $F_\nu \otimes T_+^k E_\nu$ for ν a positive root in $\mathbb{Z}\Gamma_1$ with $T^i\nu \in \mathbb{Z}\Gamma_1$ for all $0 \leq i < k$. But this is clear since $\nu - T^k\nu \in \mathcal{L}^\perp$ and $\mu \in \mathcal{L}$. □

Take $\heartsuit$ equal to either (r) or (l). By Lemma 7.1 we now see that the restriction of the multiplication map $\mathbb{C}[L] \otimes R^J_\heartsuit \to u_q^J$ is a coalgebra map, where we just give $\mathbb{C}[L]$ its usual group ring structure. If we let L act on $R^J_\heartsuit$ by conjugation this gives a Hopf map $\mathbb{C}[L] \ltimes R^J_\heartsuit \to u_q^J$. According to the particular form of $R^J_\heartsuit$ given in Proposition 5.4, and Lemma 1.3, we see that this map has image equal to the corresponding quantum parabolic $u_q(\mathsf{p}_i)$. So we find

Lemma 7.2. *The quantum parabolics $u_q(\mathsf{p}_i)$ are both Hopf subalgebras in the twist $u_q(\mathsf{g})^J$.*

From the Hopf map $\mathbb{C}[L] \ltimes R^J_{(l)} \to u_q^J$ we also get a dual Hopf map

$$(u_q^J)^* \to \mathbb{C}[\mathcal{L}] \otimes (R^J_{(l)})^* \overset{1\otimes t_{R^J}}{\to} \mathbb{C}[\mathcal{L}] \otimes R^J_{(r)} \tag{9}$$

which extends to an algebra map

$$(u_q^J)^* \to \mathbb{C}[\mathcal{L}] \otimes R^J_{(r)}/\mathcal{N} \cong \mathbb{C}[\mathcal{L}] \otimes \mathbb{C}[\Lambda] \otimes u_q(\mathsf{p}_2^{ss}),$$

by Corollary 5.5. (Recall our subgroup $\Lambda = G_{(r)} \cap G_2^\perp$ from Corollary 5.5.) Below we will need the following lemma.

Lemma 7.3. *Take $\mathscr{C} = L/(G_{(l)} \cap L)$. The subgroup Λ in $G_{(r)}$ is isomorphic to the dual $(G_{(l)} \cap L)^\vee$, and we have an exact sequence $0 \to \mathscr{C}^\vee \to \mathcal{L} \to \Lambda \to 0$.*

Proof. The dual $\Lambda^\vee$ gives the character group of $(R^J_{(l)})^*$, by Corollary 5.5. The character group is identified with the group of grouplikes in $R^J_{(l)}$. Since the intersection $G_{(l)} \cap L$ provides exactly $|\Lambda| = |G_{(r)}/G_2| = |G_{(l)}/G_1|$ grouplike elements in $R^J_{(l)}$ we see that $\Lambda = (G_{(l)} \cap L)^\vee$. Whence we have an exact sequence $0 \to \mathscr{C}^\vee \to \mathcal{L} \to \Lambda \to 0$. □

7.2. **Irreducible representations of** $(u_q^J)^*$**.** We take p^{ss} to be either of the (isomorphic) Lie algebras p_1^{ss} or p_2^{ss}. In this section we prove

Theorem 7.4. *There is a bijection*

$$\mathrm{Irrep}\,(\mathbb{C}[\mathcal{L}] \otimes u_q(\mathsf{p}^{ss})) \overset{\cong}{\to} \mathrm{Irrep}\,\big((u_q^J)^*\big)$$

given by restricting along an algebra surjection $(u_q^J)^* \to \mathbb{C}[\mathcal{L}] \otimes u_q(\mathsf{p}^{ss})$.

Remark 7.5. In Theorem 7.4 we take advantage of the existence of an abstract algebra isomorphism $\mathbb{C}[\mathscr{C}^\vee \times \Lambda] \cong \mathbb{C}[\mathcal{L}]$, where $\mathscr{C}$ is as in Lemma 7.3. Such an isomorphism exists simply because both groups are abelian of the same order, by Lemma 7.3. However, as we'll see below, the character group of the dual $(u_q^J)^*$ is naturally identified with L, so that the appearance of $\mathcal{L}$ is appropriate.

Before giving the proof we establish some background material.

Lemma 7.6. *The subcoalgebra A in $R^J_{(l)}$ dual to the quotient $R^J_{(r)}/\mathcal{N}$, under the Hopf isomorphism $t_{R^J} : (R^J_{(l)})^* \to R^J_{(r)}$, is exactly the subalgebra $\mathbb{C}\langle G_{(l)}, \mathbf{E}_\alpha, \mathbf{F}_\beta : \alpha \in \Gamma_2,\ \beta \in \Gamma_1\rangle$.*

From the statement it is clear that A is actually a Hopf subalgebra. We are claiming that A is the minimal subspace in $R^J_{(l)}$ admitting a factoring $(R^J_{(l)})^* \to A^* \to R^J_{(r)}/\mathcal{N}$.

Proof. Recall $\mathcal{N}$ is generated by all the $\mathbf{F}_\alpha$ with $\alpha \in \Gamma - \Gamma_2$. For π the projection $R^J_{(r)} \to R^J_{(r)}/\mathcal{N}$, one sees directly from the form of R^J that $(1 \otimes \pi)(R^J)$ is in the product $A' \otimes (R^J_{(r)}/\mathcal{N})$ where $A' = \mathbb{C}\langle G, \mathbf{E}_\alpha, \mathbf{F}_\beta : \alpha \in \Gamma_2,\ \beta \in \Gamma_1\rangle$. But $(1 \otimes \pi)(R^J)$ is also in $R^J_{(l)} \otimes (R^J_{(r)}/\mathcal{N})$ so that

$$(1 \otimes \pi)(R^J) \in \Big(R^J_{(l)} \otimes (R^J_{(r)}/\mathcal{N})\Big) \cap \Big(A' \otimes (R^J_{(r)}/\mathcal{N})\Big).$$

By flatness of everything over $\mathbb{C}$, this intersection is exactly $A \otimes (R^J_{(r)}/\mathcal{N})$. So the surjective map $(R^J_{(l)})^* \to R^J_{(r)}/\mathcal{N}$ factors through A^*. Since the dimensions of A and $R^J_{(r)}/\mathcal{N}$ agree we must have that A is in fact dual to $R^J_{(r)}/\mathcal{N}$. □

The Hopf subalgebra A is strongly related to the intersection of the quantum parabolics $u_q(\mathsf{p}_1) \cap u_q(\mathsf{p}_2)$, which we denote `Int`. From considering bases of the two quantum parabolics, as in Theorem 1.1, one arrives at the presentation

$$\mathtt{Int} = u_q(\mathsf{p}_1) \cap u_q(\mathsf{p}_2) = \mathbb{C}\langle G, \mathbf{E}_\alpha, \mathbf{F}_\beta : \alpha \in \Gamma_2,\ \beta \in \Gamma_1\rangle.$$

Note that since the quantum parabolics are Hopf subalgebras, the intersection will be a Hopf subalgebra as well.

Lemma 7.7. *Take* $\mathscr{C} = L/(G_{(l)} \cap L)$. *There is a coalgebra isomorphism* $\mathbb{C}[\mathscr{C}] \otimes A \to$ `Int` *given by multiplication.*

Proof. We have the multiplication map $\mathbb{C}[L] \otimes A \to u_q^J$ which is a surjection onto the intersection `Int`. Choose for each $\bar{\xi} \in \mathscr{C}$ a representative $\xi \in L$ and restrict the above multiplication map to get an coalgebra embedding

$$\mathbb{C}[\mathscr{C}] \otimes A = \bigoplus_{\bar{\xi} \in \mathscr{C}} \mathbb{C}\bar{\xi} \otimes A \to u_q^J, \quad \bar{\xi} \otimes a \mapsto \xi \cdot a,$$

with image exactly `Int`. □

We have now the

Proof of Theorem 7.4. Let K be the kernel of the projection $(u_q^J)^* \to \mathtt{Int}^*$ dual to the inclusion $\mathtt{Int} \to u_q^J$. Note that K will be a Hopf ideal in the dual. We have the Hopf maps $(u_q^J)^* \to \mathbb{C}[\mathcal{L}] \otimes R^J_{\heartsuit}$ of (9) which factor

$$(u_q^J)^* \overset{\Delta}{\to} (u_q^J)^* \otimes (u_q^J)^* \overset{(?)|L \otimes t_{R^J}}{\longrightarrow} \mathbb{C}[\mathcal{L}] \otimes R^J_{\heartsuit}.$$

We claim that the induced maps

$$F : (u_q^J)^* \to \mathbb{C}[\mathcal{L}] \otimes R^J_{(r)}/\mathcal{N} \quad \text{and} \quad F' : (u_q^J)^* \to \mathbb{C}[\mathcal{L}] \otimes R^J_{(l)}/\mathcal{N}'$$

factor through $\mathtt{Int}^*$. Equivalently, we claim that K is in their kernels. Let π and π' be the projections $\pi : R^J_{(r)} \to R^J_{(r)}/\mathcal{N}$ and $\pi' : R^J_{(l)} \to R^J_{(l)}/\mathcal{N}'$.

We prove the result for $R^J_{(r)}$. Recall that $\mathcal{N}$ is the ideal generated by all the $\mathbf{F}_\alpha$ with $\alpha \in \Gamma - \Gamma_2$, and that K consists of all functions vanishing on `Int`. Note that the intersection contains all of $\mathbb{C}[G]$, so that $K|L = 0$. Hence for each $f \in K$ we have

$$F(f) = \sum_i (f_{i_1}|L) \otimes \pi t_{R^J}(f_{i_2}) = \sum_i (f_{i_1}|L) \otimes \big((f_{i_2} \otimes \pi)(R^J)\big)$$

for some $f_{i_2} \in K$. So it suffices to show $(K \otimes \pi)(R^J) = 0$. However, we have already seen in Lemma 7.6 that $(1 \otimes \pi)(R^J)$ lies in $A \otimes R^J_{(r)}/\mathcal{N}$, and $A \subset \mathtt{Int}$. Hence $(f \otimes \pi)(R^J) = 0$ for each $f \in K$, and we find $(K \otimes \pi)(R^J) = 0$. So the map F factors through $\mathtt{Int}^*$, and a completely analogous argument shows that F' factors through $\mathtt{Int}^*$ as well.

Since $(u_q^J)^* \to \mathtt{Int}^*$ is a Hopf map the factorizations of F and F' imply that the map

$$(u_q^J)^* \overset{\Delta}{\to} (u_q^J)^* \otimes (u_q^J)^* \overset{F \otimes F'}{\to} (\mathbb{C}[\mathcal{L}] \otimes R^J_{(r)}/\mathcal{N}) \otimes (\mathbb{C}[\mathcal{L}] \otimes R^J_{(l)}/\mathcal{N}') \tag{10}$$

factors

$$(u_q^J)^* \to \mathtt{Int}^* \to (\mathbb{C}[\mathcal{L}] \otimes R^J_{(r)}/\mathcal{N}) \otimes (\mathbb{C}[\mathcal{L}] \otimes R^J_{(l)}/\mathcal{N}'). \tag{11}$$

We note that that the map

$$(u_q^J)^* \overset{\Delta}{\to} (u_q^J)^* \otimes (u_q^J)^* \to (\mathbb{C}[\mathcal{L}] \otimes R^J_{(r)}) \otimes (\mathbb{C}[\mathcal{L}] \otimes R^J_{(l)})$$

is an embedding, since its dual is a surjection, so that the kernels of (10) and (11) are nilpotent. It follows that the kernel of the projection $(u_q^J)^* \to \mathtt{Int}^*$ must be nilpotent as well.

We have from Lemmas 7.6 and 7.7 that $\mathtt{Int}^* \cong \mathbb{C}[\mathscr{C}^\vee] \otimes R^J_{(r)}/\mathcal{N}$. Recall from Corollary 5.5 that $R^J_{(r)}/\mathcal{N}$ is isomorphic to $\mathbb{C}[\Lambda] \otimes u_q(\mathsf{p}^{ss})$ and that $\mathbb{C}[\mathscr{C}^\vee \times \Lambda] \cong \mathbb{C}[\mathcal{L}]$, abstractly, to arrive at a surjection $(u_q^J)^* \to \mathbb{C}[\mathcal{L}] \otimes u_q(\mathsf{p}^{ss})$ with nilpotent kernel. Restricting then gives the proposed bijection on irreducible representations. □

Corollary 7.8. *The set of grouplikes $G(u_q^J)$ is exactly L.*

Proof. Since all the elements in L are grouplike, by Lemma 7.1, we need only know that $|G(u_q^J)| = |L|$. But this just follows from the theorem, since grouplikes in u_q^J are identified with one dimensional representations of $(u_q^J)^*$. □

To compare u_q to u_q^J let us consider a maximal BD triple on A_n. In this case there is a unique solution $\mathtt{S}$ to (EQ–S) and, as mentioned previously, the algebras $R_\heartsuit^J$ will be the full parabolics.[3] We will have, for $n = 2$ and $l = 5$ for example, a following variation in the dimensions of the coradicals:

$$\dim \mathrm{Corad}(u(\mathsf{sl}_3)) = 25, \quad \dim \mathrm{Corad}(u(\mathsf{sl}_3)^J) = 105.$$

The difference is made more stark from the fact that corepresentation theory of $u_q(\mathsf{sl}_{n+1})$ is essentially trivial, at least when we restrict our attention to the irreducibles and fusion rule, while the corepresentation theory of $u_q(\mathsf{sl}_{n+1})^J$ should be at least as complicated as the representation theory of $u_q(\mathsf{sl}_n)$.

8. The Drinfeld element and properties of the antipode

Here we discuss preservation of the Drinfeld element under twisting. Basic information on the Drinfeld element in a quasitriangular Hopf algebra, and its relation to the antipode, can be found in [23, 15]. We fix a Belavin-Drinfeld triple (Γ_1, Γ_2, T), solution $\mathtt{S}$, and twist $J = J_{T,\mathtt{S}}$ of $u_q(\mathfrak{g})$.

Let $\rho \in \mathcal{G}$ be the sum $\rho = \sum_{\mu \in \Phi^+} \mu$. Then we have $(\rho, \alpha) = 2$ for each simple root α [18, Sect. 10.2]. This gives $S^2 = \mathrm{ad}_{K_\rho}$ and the Drinfeld element for $u_q(\mathfrak{g})$ thus factors $u = K_\rho v$, where v is a central element with $\Delta(v) = (v \otimes v)(R_{21}R)^{-1}$, i.e. a ribbon element. Note that ρ is in $\mathcal{L}$ as $(\rho, \alpha - T(\alpha)) = 2 - 2 = 0$ for each $\alpha \in \Gamma$. So K_ρ remains grouplike in the twist $u_q(\mathfrak{g})^J$.

Recall that under an arbitrary twist J of a quasitriangular Hopf algebra H the Drinfeld element for H^J is the product $u^J = Q_J^{-1} S(Q_J) u$ (see e.g. [11]). In our case this means that the Drinfeld element for $u_q(\mathfrak{g})^J$ is given by

$$u^J = Q_J^{-1} S(Q_J) K_\rho v.$$

Centrality of v implies

$$\Delta^J(v) = (v \otimes v) J^{-1} (R_{21}R)^{-1} J = (v \otimes v)(R_{21}^J R^J)^{-1}.$$

So v is still a ribbon element for the twist, and $Q_J^{-1} S(Q_J) K_\rho$ is grouplike in the twist. Since K_ρ itself is grouplike we conclude that $Q_J^{-1} S(Q_J)$ is grouplike as well.

Proposition 8.1. *The twists $J = J_{T,\mathtt{S}}$ are such that $Q_J^{-1} S(Q_J) = 1$.*

In the proof of the proposition we employ what we call a T-grading on u_q^J. We define this as any algebra $\mathbb{Z}$-grading with the following properties:

(a) $\mathbb{C}[G]$ is homogeneous of degree 0.

[3] This basically follows from the fact that $\mathcal{G}_2^\perp$ will be a free $\mathbb{Z}/l\mathbb{Z}$-module so that $\mathtt{S}(\mu, \nu) = 0$ for any $\mu, \nu \in \mathcal{G}_2^\perp$, by antisymmetry. Thus $\mathtt{S}^{-2}\Omega|\mathcal{G}_2^\perp \times \mathcal{G}_2^\perp = \Omega_{\mathcal{G}_2^\perp}$ and we must have all of $G_2^\perp$ in $G_{(r)}$.

(b) The E_α are of positive degree and the F_α are of negative degree with $\deg(F_\alpha) = -\deg(E_\alpha)$.
(c) $\deg(E_{T\alpha}) > \deg(E_\alpha)$ for each $\alpha \in \Gamma_1$.

It is easy to construct such a grading. For example, one can construct the acyclic directed graph $\mathtt{Graph}(\Gamma, T)$ with vertices Γ and an arrow from α to $T(\alpha)$ for each $\alpha \in \Gamma_1$. One then takes

$$\deg(E_\alpha) = -\deg(F_\alpha) = |\mathtt{Graph}_{\leq\alpha}|,$$

where $\mathtt{Graph}_{\leq\alpha}$ is the collection of all vertices with a path to α in $\mathtt{Graph}(\Gamma, T)$, including α. Note that the antipode preserves degree under any T-grading.

Proof. Under any T-grading on u_q we will have that J and J^{-1} both lie in non-negative degree in $u_q \otimes u_q$, where $\deg(a \otimes b) = \deg(a) + \deg(b)$ for $a, b \in u_q$. This is clear from the explicit forms of the twist and its inverse given at Theorem 3.1 and (7). We have also $J_0 = \mathtt{S}^{-1}\Omega_{L^\perp}^{-1/2}$ and $(J^{-1})_0 = \mathtt{S}\Omega_{L^\perp}^{1/2}$. It follows, from the expressions of Q_J and Q_J^{-1} given in Section 2, that both Q_J^{-1} and $S(Q_J)$ lie in nonnegative degree with

$$(Q_J^{-1})_0 = m(\mathtt{S}^{-1}\Omega_{L^\perp}^{-1/2}), \quad S(Q_J)_0 = m(\mathtt{S}\Omega_{L^\perp}^{1/2}),$$

where m is multiplication. We have now $\left(Q_J^{-1}S(Q_J)\right)_0 = (Q_J^{-1})_0 S(Q_J)_0$ and since the multiplication map on any commutative algebra, such as $\mathbb{C}[G]$, is a ring map

$$(Q_J^{-1})_0 S(Q_J)_0 = m(\mathtt{S}^{-1}\Omega_{L^\perp}^{-1/2}\mathtt{S}\Omega_{L^\perp}^{1/2}) = 1.$$

Finally we note that since $Q_J^{-1}S(Q_J)$ is grouplike it must lie in degree 0. Therefore $Q_J^{-1}S(Q_J) = \left(Q_J^{-1}S(Q_J)\right)_0 = 1.$ □

As an immediate corollary we have

Corollary 8.2. *The Drinfeld element for $u_q(\mathtt{g})^J$ is equal to the Drinfeld element for $u_q(\mathtt{g})$.*

8.1. **Implications for the antipode.** In [24] the question was posed as to whether or not the order of the antipode and the traces of the powers of the antipode are preserved under twisting. The question was answered positively for Hopf algebras with the Chevalley property. Using the expression of the Chevalley property given in [1, Prop. 4.2, 5] it is relatively easy to see that no small quantum group has the Chevalley property. We can, however, verify the proposed invariance for Belavin-Drinfeld twists.

Corollary 8.3. *For S the antipode on $u_q(\mathtt{g})$ and S_J the antipode on the twist $u_q(\mathtt{g})^J$, and $J = J_{T,\mathtt{S}}$, we have $\mathrm{Tr}(S_J^m) = \mathrm{Tr}(S^m)$ for all $m \in \mathbb{Z}$ and $\mathrm{ord}(S_J) = \mathrm{ord}(S)$.*

Proof. Since $Q_J^{-1}S(Q_J) = 1$ the proof of [24, Thm. 4.3] still works to get $\mathrm{Tr}(S_J^m) = \mathrm{Tr}(S^m)$. Since S and S_J are semisimple operators invariance of order follows from invariance of the traces. □

We can also get invariance of the so-called regular object of [28, Sect. 5.4] using the condition of [24, Prop. 7.3 (ii)]. This positively answers [28, Question (5.12)] for the twists $J_{T,\mathtt{S}}$ on small quantum groups.

9. Twisted automorphisms and group actions on $\mathrm{rep}(u_q(\mathsf{g}))$

We use below the notion of a 2-group. A 2-group is simply a monoidal category in which all morphisms are invertible and all objects have a weak inverse, i.e. an inverse up to isomorphism. For a tensor category $\mathscr{C}$ we let $\underline{\mathrm{Aut}}(\mathscr{C})$ denote the 2-group of autoequivalences of $\mathscr{C}$ as a tensor category, with natural isomorphisms, and $\mathrm{Aut}(\mathscr{C})$ denote the associated group of isoclasses of autoequivalences.

Following Davydov [6], for a Hopf algebra H we call a pair (ϕ, J) of a twist and a Hopf isomorphism $\phi : H \to H^J$ a *twisted automorphism* of H. Each twisted automorphism can be identified with the tensor autoequivalence of $\mathrm{rep}(H)$ given by composing

$$\mathrm{rep}(H) \overset{J}{\to} \mathrm{rep}(H^J) \overset{\mathrm{res}_\phi}{\to} \mathrm{rep}(H).$$

Indeed, twisted automorphisms form a 2-subgroup in the 2-group of autoequivalences $\underline{\mathrm{Aut}}(\mathrm{rep}(H))$ with product $(\phi', J')\cdot(\phi, J) = (\phi\phi', J\phi^{\otimes 2}(J'))$. The induced isomorphisms between twisted automorphisms are gauge equivalences (see 9.3 below). Furthermore, Ng and Schauenburg have shown that when H is finite dimensional any autoequivalence of $\mathrm{rep}(H)$ will be isomorphic to a twisted automorphism [25, Thm. 2.2].

We take g simple and simply laced, q a primitive lth root of unity, for l as in Section 1.2, and $u_q = u_q(\mathsf{g})$. In this final section we introduce twists J_α^λ of the small quantum group u_q which are paired with automorphisms $\exp_\alpha^\lambda$ so that each pair $(\exp_\alpha^\lambda, J_\alpha^\lambda)$ provides a twisted automorphism of u_q. We then relate a canonical algebraic group action on $\mathrm{rep}(u_q)$ to the twisted automorphisms $(\exp_\alpha^\lambda, J_\alpha^\lambda)$, and pose a question regarding a set of "generators" for the collection of all twists of u_q.

9.1. Twists via exponentiation: an extended quantum coadjoint action.

Recall that u_q embeds as a Hopf subalgebra in Lusztig's divided powers quantum group

$$U_q = U_q(\mathsf{g}) = \mathbb{C}\langle K_\alpha^{\pm 1}, E_\alpha, F_\alpha, E_\alpha^{(l)}, F_\alpha^{(l)} : \alpha \in \Gamma\rangle/(\text{relations}).$$

We do not recall the specific construction of U_q here, and refer the reader instead to [21, 20], and in particular [21, Sect. 6.5], for the details.

According to [20, Lem. 4.5] the commutator

$$\mathrm{ad}_{E_\alpha^{(l)}} : U_q \to U_q, \quad x \mapsto [E_\alpha^{(l)}, x]$$

preserves the subalgebra u_q and the restriction $\mathrm{ad}_{E_\alpha^{(l)}}|u_q$ is a nilpotnent operator. The same is true if we scale by any $\lambda \in \mathbb{C}$. Hence we can exponentiate this operator to produce an algebra automorphism

$$\exp_\alpha^\lambda := \exp(\mathrm{ad}_{\lambda E_\alpha^{(l)}}|u_q)$$

of the small quantum group u_q. We can similarly define

$$\exp_{-\alpha}^\lambda := \exp(\mathrm{ad}_{\lambda F_\alpha^{(l)}}|u_q).$$

If we consider $u_q(\mathsf{sl}_2)$ for example, and $\exp_+^\lambda$ corresponding to the positive simple root, we have $\exp_+^\lambda(E) = \exp_+^\lambda(K) = 0$ and

$$\exp_+^\lambda(F) = F + \lambda\left(\frac{qK + q^{-1}K^{-1}}{q - q^{-1}}\right)E^{(l-1)}.$$

As $E_\alpha^{(l)}$ fails to be primitive, the automorphism $\exp_\alpha^\lambda$ fails to be a Hopf map. We have, in the ambient algebra U_q,

$$\Delta(E_\alpha^{(l)}) = E_\alpha^{(l)} \otimes 1 + 1 \otimes E_\alpha^{(l)} + \sum_{1\le i\le l-1} q^{-i(l-i)} K^i E_\alpha^{(l-i)} \otimes E_\alpha^{(i)}$$

and can define the element

$$\Theta(E_\alpha) = \Delta(E_\alpha^{(l)}) - (E_\alpha^{(l)} \otimes 1 + 1 \otimes E_\alpha^{(l)})$$

in $u_q \otimes u_q$. Note that $\Theta(E_\alpha)$ is square zero, and hence we can exponentiate any scaling $\lambda\Theta(E_\alpha)$ to arrive at a unit

$$J_\alpha^\lambda = \exp(\lambda\Theta(E_\alpha)) \in u_q \otimes u_q.$$

We define similarly $J_{-\alpha}^\lambda = \exp(\lambda\Theta(F_\alpha))$ for $\Theta(F_\alpha) = \Delta(F_\alpha^{(l)}) - (F_\alpha^{(l)} \otimes 1 - 1 \otimes F_\alpha^{(l)})$. One can check easily from the expressions of $\Theta(E_\alpha)$ and $\Theta(F_\alpha)$ that

$$(\epsilon \otimes 1)(J_{\pm\alpha}^\lambda) = (1 \otimes \epsilon)(J_{\pm\alpha}^\lambda) = 1.$$

Theorem 9.1. *For an arbitrary simple root α, and $\lambda \in \mathbb{C}$, the unit $J_{\pm\alpha}^\lambda$ is a twist for $u_q(\mathsf{g})$. Furthermore, each pair $(\exp_{\pm\alpha}^\lambda, J_{\pm\alpha}^\lambda)$ is a twisted automorphism of $u_q(\mathsf{g})$.*

We will only prove the result for positive α, the computation for $-\alpha$ being completely similar. Let us first give a technical lemma.

Lemma 9.2. *The elements $(\Delta \otimes 1)(\Theta(E_\alpha))$ and $\Theta(E_\alpha) \otimes 1$ commute, as do the elements $(1 \otimes \Delta)(\Theta(E_\alpha))$ and $1 \otimes \Theta(E_\alpha)$.*

Proof. Since E_α is in the Hopf subalgebra $U_q(\mathsf{sl}_2) \subset U_q(\mathsf{g})$ generated by K_α and the $E_\alpha^{(n)}$, $F_\alpha^{(n)}$, we may assume $\mathsf{g} = \mathsf{sl}_2$. We may further restrict to the positive Borel U_+, in which $E^{(l)}$ is central. Take $\Theta = \Theta(E)$ and $pE^{(l)} = E^{(l)} \otimes 1 + 1 \otimes E^{(l)}$. We have now

$$\begin{aligned}
&(\Delta \otimes 1)(\Theta)(\Theta \otimes 1)\\
&= (\Delta \otimes 1)(\Theta)(\Delta E^{(l)} \otimes 1) - (\Delta \otimes 1)(\Theta \otimes 1)(pE^{(l)} \otimes 1)\\
&= (\Delta \otimes 1)\left(\Theta(E^{(l)} \otimes 1)\right) - (\Delta \otimes 1)(\Theta \otimes 1)(pE^{(l)} \otimes 1)\\
&= (\Delta \otimes 1)\left((E^{(l)} \otimes 1)\Theta\right) - (pE^{(l)} \otimes 1)(\Delta \otimes 1)(\Theta \otimes 1)\\
&= \left((\Delta \otimes 1)(E^{(l)} \otimes 1) - (pE^{(l)} \otimes 1)\right)(\Delta \otimes 1)(\Theta)\\
&= (\Theta \otimes 1)(\Delta \otimes 1)(\Theta).
\end{aligned}$$

This gives the first proposed commutativity

$$(\Delta \otimes 1)(\Theta)(\Theta \otimes 1) = (\Theta \otimes 1)(\Delta \otimes 1)(\Theta).$$

The verification of the relation

$$(1 \otimes \Delta)(\Theta)(1 \otimes \Theta) = (1 \otimes \Theta)(1 \otimes \Delta)(\Theta)$$

is completely similar. □

Since all of the elements in the statement of Lemma 9.2 are nilpotent in $u_q^{\otimes 3}$ we can now exponentiate to get

$$\begin{aligned}
\exp\left((\Delta \otimes 1)(\lambda\Theta(E_\alpha)) + (\lambda\Theta(E_\alpha) \otimes 1)\right) &= \exp\left((\Delta \otimes 1)(\lambda\Theta(E_\alpha))\right)\exp\left(\lambda\Theta(E_\alpha) \otimes 1\right)\\
&= (\Delta \otimes 1)\left(\exp(\lambda\Theta(E_\alpha))\right)\exp\left(\lambda\Theta(E_\alpha) \otimes 1\right)\\
&= (\Delta \otimes 1)\left(J_\alpha^\lambda\right)\left(J_\alpha^\lambda \otimes 1\right)
\end{aligned} \tag{12}$$

and

$$\exp\left((1 \otimes \Delta)(\lambda\Theta(E_\alpha)) + (1 \otimes \lambda\Theta(E_\alpha))\right) = (1 \otimes \Delta)(J_\alpha^\lambda)(1 \otimes J_\alpha^\lambda), \tag{13}$$

for arbitrary $\lambda \in \mathbb{C}$.

Proof of Theorem 9.1. Again, we may assume $\mathfrak{g} = \mathfrak{sl}_2$. By the above observations (12, 13) the dual cocycle condition for J_α^λ is equivalent to the equality

$$(\Delta \otimes 1)(\lambda\Theta) + (\lambda\Theta \otimes 1) = (1 \otimes \Delta)(\lambda\Theta) + (1 \otimes \lambda\Theta),$$

where $\Theta = \Theta(E)$. By dividing by λ on both sides we may take $\lambda = 1$. We then see directly

$$(\Delta \otimes 1)(\Theta) + (\Theta \otimes 1) = \sum_{\substack{0 \le i,j,k < l \\ i+j+k=l}} q^{i(j+k)+jk} K^{i(j+k)} E^{(i)} \otimes K^k E^{(j)} \otimes E^{(k)}$$

$$= (1 \otimes \Delta)(\Theta) + (1 \otimes \Theta).$$

Hence J_α^λ is a twist.

As for compatibility with the automorphism $\exp_\alpha^\lambda$, we have the diagram

$$\begin{array}{ccc} u_q & \xrightarrow{\mathrm{ad}_{\lambda E_\alpha^{(l)}}} & u_q \\ \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\Delta} \\ u_q \otimes u_q & \xrightarrow[\mathrm{ad}_{\Delta\lambda E_\alpha^{(l)}}]{} & u_q \otimes u_q \end{array}$$

which implies the diagram

$$\begin{array}{ccc} u_q & \xrightarrow{\exp_\alpha^\lambda} & u_q \\ \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\Delta} \\ u_q \otimes u_q & \xrightarrow[\exp(\mathrm{ad}_{\Delta\lambda E_\alpha^{(l)}})]{} & u_q \otimes u_q. \end{array}$$

Since $\Delta E_\alpha^{(l)} = E_\alpha^{(l)} \otimes 1 + 1 \otimes E_\alpha^{(l)} + \Theta(E_\alpha)$, we have

$$\exp(\mathrm{ad}_{\Delta\lambda E_\alpha^{(l)}}) = (\exp_\alpha^\lambda \otimes \exp_\alpha^\lambda)\mathrm{Ad}_{J_\alpha^\lambda},$$

where $\mathrm{Ad}_u(x) = uxu^{-1}$, and the above diagram gives on elements

$$(\exp_\alpha^\lambda \otimes \exp_\alpha^\lambda)\Delta^{J_\alpha^{-\lambda}}(x) = \Delta(\exp_\alpha^\lambda(x)).$$

Replace x with $\exp_\alpha^{-\lambda}(x)$, compose with $(\exp_\alpha^{-\lambda} \otimes \exp_\alpha^{-\lambda})$, and swap λ for $-\lambda$ to find that $\Delta^{J_\alpha^\lambda}(\exp_\alpha^\lambda(x)) = (\exp_\alpha^\lambda \otimes \exp_\alpha^\lambda)\Delta(x)$. So we see $\exp_\alpha^\lambda : u_q \to u_q^{J_\alpha^\lambda}$ is a Hopf map. $\square$

One can check easily

$$(\exp_{\pm\alpha}^\lambda, J_{\pm\alpha}^\lambda) \cdot (\exp_{\pm\alpha}^{\lambda'}, J_{\pm\alpha}^{\lambda'}) = (\exp_{\pm\alpha}^{\lambda'+\lambda}, J_{\pm\alpha}^{\lambda'+\lambda}).$$

It follows that the assignment $\lambda \mapsto (\exp_{\pm\alpha}^{-\lambda}, J_{\pm\alpha}^{-\lambda})$ gives a 1-parameter subgroup in the 2-group of twisted automorphisms for u_q, and hence a 1-parameter subgroup $\mathbb{C} \to \underline{\mathrm{Aut}}(\mathrm{rep}(u_q))$ into the 2-group of autoequivalences $\underline{\mathrm{Aut}}(\mathrm{rep}(u_q))$. The negation here appears for technical reasons, but intuitively corrects the fact that the multiplication of twisted automorphisms defined above appears to be backwards. We denote this 1-parameter subgroup $\omega_{\pm\alpha}$.

Remark 9.3. The algebra automorphisms appearing in the 1-parameter subgroups $\omega_{\pm\alpha}$ can be recovered alternatively from the quantum coadjoint action of De Concini and Kac, via the reduction $U_q^{\mathrm{DK}} \to u_q$ from the non-divided-powers quantum group [8, Prop. 3.5]. So we are saying above that the induced quantum coadjoint action on u_q extends naturally to an action on the tensor category $\mathrm{rep}(u_q)$.

9.2. **Identification with the Arkhipov-Gaitsgory action.** Take $\mathbb{G}$ the connected, simply connected, semisimple algebraic group with Lie algebra $\mathfrak{g}$. As a set we identify $\mathbb{G}$ with its $\mathbb{C}$-points. Taking the (finite) dual of the exact sequence of Hopf algebras $\mathbb{C} \to u_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \to U(\mathfrak{g}) \to \mathbb{C}$ produces an exact sequence

$$\mathbb{C} \to \mathscr{O}(\mathbb{G}) \to \mathscr{O}_q(\mathbb{G}) \to u_q(\mathfrak{g})^* \to \mathbb{C}$$

with $\mathscr{O}(\mathbb{G})$ lying in the center of the quantum function algebra [9, Thm. 6.3, Lem. 6.1]. According now to [3, Thm. 2.8] and [2, Prop. 4.1] we have a tensor equivalence between the de-equivariantization $\mathrm{corep}\,(\mathscr{O}_q(\mathbb{G}))_{\mathbb{G}}$ and $\mathrm{rep}(u_q(\mathfrak{g}))$. We take $\mathscr{O} = \mathscr{O}(\mathbb{G})$ and $\mathscr{O}_q = \mathscr{O}_q(\mathbb{G})$.

Recall that the de-equivariantization is the category of finitely generated left $\mathscr{O}$-modules with a compatible right $\mathscr{O}_q$-coaction [2, Def. 3.7]. This category is monoidal under the product $\otimes_{\mathscr{O}}$. The action of $\mathbb{G}$ on itself by left translation, and pushing forward by the corresponding automorphisms of $\mathscr{O}$, gives an action of $\mathbb{G}$ on the de-equivariantization by tensor functors. Rather, we have a canonical monoidal functor from $\mathbb{G}$ to the 2-group of tensor autoequivalences of $\mathrm{corep}\,(\mathscr{O}_q)_{\mathbb{G}}$. The equivalence $\mathrm{corep}\,(\mathscr{O}_q)_{\mathbb{G}} \overset{\sim}{\to} \mathrm{rep}(u_q)$ of [3] is given by taking the fiber at the identity $?|_\epsilon = \mathbb{C}\otimes_{\mathscr{O}}?$, and via this equivalence we get an action of $\mathbb{G}$ on $\mathrm{rep}(u_q)$.

We let $\gamma_{\pm\alpha}$ denote the 1-parameter subgroup in $\mathbb{G}$ given by exponentiating the root space $\mathfrak{g}_{\pm\alpha}$.

Proposition 9.4. *For any simple root $\alpha \in \Gamma$, the composite*

$$\mathbb{C} \overset{\gamma_{\pm\alpha}}{\longrightarrow} \mathbb{G} \to \underline{\mathrm{Aut}}\left(\mathrm{corep}\,(\mathscr{O}_q)_{\mathbb{G}}\right) \overset{\mathrm{Ad}_{?|_\epsilon^{-1}}}{\to} \underline{\mathrm{Aut}}(\mathrm{rep}(u_q))$$

is isomorphic to the 1*-parameter subgroup* $\omega_{\pm\alpha} : \lambda \mapsto (\exp_{\pm\alpha}^{-\lambda}, J_{\pm\alpha}^{-\lambda})$.

What one should mean by a general isomorphism of 1-parameter subgroups is not exactly clear. From our perspective we would like a family of natural isomorphisms between the two functors $\mathbb{C} \to \underline{\mathrm{Aut}}(\mathrm{rep}(u_q))$ which satisfy all obvious commutativity and additivity relations. We will focus here only on the production of a natural family of natural isomorphisms which vary with λ.

Proof. We consider only the positive root α. Since high powers of $E_\alpha^{(l)}$ annihilate any finite dimensional representation, each function f in the finite dual $\mathscr{O}_q$ will vanish on high powers of $E_\alpha^{(l)}$ [20, Prop. 5.1]. Hence the exponent

$$\exp(\lambda E_\alpha^{(l)}) : \mathscr{O}_q \to \mathbb{C}$$

is a well-defined function. Restricting along the inclusion $\mathscr{O} \to \mathscr{O}_q$ recovers the point $\gamma_\alpha(\lambda) = \exp(\lambda e_\alpha)$ in $\mathbb{G}$. Let us fix $x^\lambda = \gamma_\alpha(\lambda)$ and $v^\lambda = \exp(\lambda E_\alpha^{(l)})$.

We aim to produce a family of isomorphisms between the autoequivalence of $\mathrm{rep}(u_q)$ induced by the pushforward isomorphisms

$$x_*^\lambda : \mathrm{corep}(\mathscr{O}_q)_{\mathbb{G}} \to \mathrm{corep}(\mathscr{O}_q)_{\mathbb{G}}$$

and the autoequivalences given by the twisted isomorphisms $(\exp_\alpha^{-\lambda}, J_\alpha^{-\lambda})$. To be precise, x^λ will act by left translation $x^\lambda : \mathscr{O} \to \mathscr{O}$, $\chi \mapsto x^\lambda(\chi_1)\chi_2$, and the pushforward $x^\lambda_* V$ of an object V in the de-equivariantization is equal to V along with the restricted $\mathscr{O}$-action and the unaltered $\mathscr{O}_q$-coaction.

Take $\mathrm{Ad}_{v^\lambda} : \mathscr{O}_q \to \mathscr{O}_q$ the linear automorphism $f \mapsto v^{-\lambda}(f_1)f_2v^\lambda(f_3)$. We note that Ad_{v^λ} is a Hopf isomorphism from the cocycle twist of $\mathscr{O}_q$ via the 2-cocycle $J_\alpha^{-\lambda} : \mathscr{O}_q \otimes \mathscr{O}_q \to \mathbb{C}$ to $\mathscr{O}_q$, and so the sequence

$$\mathrm{corep}(\mathscr{O}_q) \overset{J_\alpha^{-\lambda}}{\to} \mathrm{corep}((\mathscr{O}_q)_{J_\alpha^{-\lambda}}) \overset{\mathrm{res}_{\mathrm{Ad}_v}}{\to} \mathrm{corep}(\mathscr{O}_q)$$

is an equivalence. This equivalence induces an equivalence on the de-equivariantization, where we additionally restrict the action of $\mathscr{O}$ along $\mathrm{Ad}_{x^{-\lambda}}$. We denote this autoequivalence by $F^\lambda : \mathrm{corep}(\mathscr{O}_q)_\mathbb{G} \to \mathrm{corep}(\mathscr{O}_q)_\mathbb{G}$.

We can produce an isomorphism of functors $x^\lambda_* \overset{\cong}{\to} F^\lambda$ which is given on objects as the composite

$$x^\lambda_* V \overset{\text{comult}}{\longrightarrow} (x^\lambda_* V) \otimes \mathscr{O}_q \overset{1\otimes v^\lambda}{\longrightarrow} F^\lambda V.$$

Rather, we multiply on the left by the function v^λ. We denote the isomorphism simply by v^λ.[4]

We now examine the equivalences on $\mathrm{rep}(u_q)$ induced by the pushforwards x^λ_*. The quasi-inverse to the reduction $?|_\epsilon : \mathrm{corep}\,(\mathscr{O}_q)_\mathbb{G} \to \mathrm{rep}(u_q)$ is the induction-like functor $\mathrm{Ind} = (\mathscr{O}_q \otimes ?)^{u_q}$, where u_q acts on a product $\mathscr{O}_q \otimes V$ diagonally by $h \cdot (f \otimes v) = (fS(h_1) \otimes h_2 v)$. The equivalence on $\mathrm{rep}(u_q)$ induced by each x^λ is then the composition $(?|_\epsilon) \circ x^\lambda_* \circ \mathrm{Ind}$. From the v^λ we get an induced isomorphism of 1-parameter subgroups

$$\dot{v}^\lambda = (?|_\epsilon) \circ v^\lambda \circ \mathrm{Ind} : (?|_\epsilon) \circ x^\lambda_* \circ \mathrm{Ind} \to (?|_\epsilon) \circ F^\lambda \circ \mathrm{Ind}.$$

Since $\mathrm{Ad}_x : \mathscr{O} \to \mathscr{O}$ preserves the counit, and Ad_v induces the automorphism $\exp_\alpha^{-\lambda}$ on the quotient u_q^* (or rather its dual), taking the fiber at the identity gives

$$(F^\lambda \mathrm{Ind}(V))|_\epsilon = ({}_{\exp_\alpha^{-\lambda}} u_q^* \otimes V)^{u_q} = {}_{\exp_\alpha^{-\lambda}}(u_q^* \otimes V)^{u_q}$$

for each u_q-representation V. The subscript of $\exp_\alpha^{-\lambda}$ here means that we are restricting the action of u_q along this automorphism. But now the natural isomorphism of u_q-modules $ev_1 \otimes 1 : (u_q^* \otimes V)^{u_q} \to V$ given by the counit of u_q^* produces the desired family of natural isomorphism

$$\ddot{v}^\lambda : (?|_\epsilon) \circ x^\lambda_* \circ \mathrm{Ind} \overset{\dot{v}^\lambda}{\longrightarrow} (?|_\epsilon) \circ F^\lambda \circ \mathrm{Ind} \overset{ev_1 \otimes 1}{\longrightarrow} (\exp_\alpha^{-\lambda}, J_\alpha^{-\lambda}).$$

□

Recall that $\mathbb{G}$ is generated by the 1-parameter subgroups $\gamma_{\pm\alpha}$ [19, Thm. 27.5]. Hence, after taking isoclasses, the group map $\mathbb{G} \to \mathrm{Aut}(\mathrm{rep}(u_q))$ is determined completely by its value on these 1-parameter subgroups. One can also show that the action of $\mathbb{G}$ on $\mathrm{rep}(u_q)$ is determined up to unique isomorphism by these 1-parameter subgroups, but this more limited information is enough for us to formulate Question 9.5 below, which proposes a set of generators for the groupoid of twists of $u_q(\mathfrak{g})$.

[4]The interested reader can check that the family of isomorphisms $\{v^\lambda\}_\lambda$ satisfies all desired commutativity and additivity relations to give an isomorphism between these two 1-parameter subgroups in $\underline{\mathrm{Aut}}\,(\mathrm{corep}\,(\mathscr{O}_q)_\mathbb{G})$.

9.3. **Autoequivalences of** $\mathrm{rep}(u_q(\mathfrak{g}))$ **and the classification of twists.** Question 9.5 below refers to gauge equivalence of twists. We say two twists J and J' are gauge equivalent if there is a unit v in H with $J' = \Delta(v)J(v^{-1} \otimes v^{-1})$. We let $\mathsf{TW}(H)$ denote the groupoid of twists of H, with morphisms given by gauge equivalences.

We note that the information of an isomorphism from the autoequivalence specified by a twisted automorphisms (ϕ, J) to that of (ϕ', J') is exactly the data of a unit $v \in H$ so that $J' = \Delta(v)J(v^{-1} \otimes v^{-1})$ and $\phi' = \mathrm{Ad}_v\phi$. We call such a unit a gauge equivalence of twisted automorphisms. Hence we have naturally a 2-group of twisted automorphisms with gauge equivalences.

As is noted in [6], the groupoid $\mathsf{TW}(H)$ admits a well-defined right action of the 2-group of twisted automorphisms. This action is defined simply by $J' \cdot (\phi, J) = J\phi^{\otimes 2}(J')$. Take

$$\tilde{\mathbb{G}} = \left\{ \begin{array}{c} \text{The 2-subgroup of all twisted automorphisms which are iso-} \\ \text{morphic to an element in the image of } \mathbb{G}\text{, in } \underline{\mathrm{Aut}}(\mathrm{rep}(u_q)) \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \text{The 2-subgroup of twisted automorphisms which are} \\ \text{gauge equivalent to a product of the } (\exp^{\lambda}_{\pm\alpha}, J^{\lambda}_{\pm\alpha}) \end{array} \right\}.$$

Take also $\mathsf{BD}(u_q) \subset \mathsf{TW}(u_q)$ the full subcategory of Belavin-Drinfeld twists $\{J_{T,\mathsf{s}}\}_{T,\mathsf{s}}$. The following question is also raised in [7], where the authors investigate the algebraic structure of $\mathrm{Aut}(\mathrm{rep}(u_q))$, and autoequivalence groups of finite tensor categories in general.

Question 9.5. Is the groupoid of twists of the small quantum group generated by $\mathsf{BD}(u_q)$ and the 1-paramater subgroups $\{(\exp^{\lambda}_{\pm\alpha}, J^{\lambda}_{\pm\alpha})\}_\lambda$? Equivalently, is the inclusion $\mathsf{BD}(u_q) \cdot \tilde{\mathbb{G}} \to \mathsf{TW}(u_q)$ an equivalence?

References

[1] N. Andruskiewitsch, P. Etingof, and S. Gelaki. Triangular Hopf algebras with the Chevalley property. *Michigan Math. J.*, 49(2):277–298, 2001.

[2] I. Angiono, C. Galindo, and M. Pereira. De-equivariantization of Hopf algebras. *Algebr. Represent. Theory*, pages 1–20, 2014.

[3] S. Arkhipov and D. Gaitsgory. Another realization of the category of modules over the small quantum group. *Adv. Math.*, 173(1):114–143, 2003.

[4] D. Arnaudon, E. Buffenoir, E. Ragoucy, and P. Roche. Universal solutions of quantum dynamical Yand-Baxter equations. *Lett. Math. Phys.*, 44(3):2–1–214, 1998.

[5] A. Belavin and V. Drinfeld. *Triangle equations and simple Lie algebras.* Hardwood Academic Publishers, 1998.

[6] A. Davydov. Twisted automorphisms of Hopf algebras. In *Noncommutative structures in mathematics and physics*, pages 103–130. K. Vlaam. Acad. Belgie Wet. Kunsten (KVAB), Brussels, 2010.

[7] A. Davydov, P. Etingof, and D. Nikshych. Autoequivalences of tensor categories attached to quantum groups at roots of 1. *in preparation.*

[8] C. De Concini and V. G. Kac. Representations of quantum groups at roots of 1. In *Operator algebras, unitary representations, enveloping algebras, and invariant theory (Paris, 1989)*, volume 92 of *Progr. Math.*, pages 471–506. Birkhäuser Boston, Boston, MA, 1990.

[9] C. DeConcini and V. Lyubashenko. Quantum function algebra at roots of 1. *Adv. Math.*, 108(2):205–262, 1994.

[10] V. Drinfeld. Almost commutative Hopf algebras. *Leningrad Math. J.*, 1:321–342, 1990.

[11] P. Etingof and S. Gelaki. On the quasi-exponent of finite-dimensional Hopf algebras. *Math. Res. Lett.*, 9(2-3):277–287, 2002.

[12] P. Etingof and D. Kazhdan. Quantization of Lie bialgebras, i. *Selecta Math.*, 2(1):1–41, 1996.

[13] P. Etingof and D. Nikshych. Dynamical quantum groups at roots of 1. *Duke Math. J.*, 108(1):135–168, 2001.
[14] P. Etingof, T. Schedler, and O. Schiffmann. Explicit quantization of dynamical r-matrices for finite dimensional semisimple Lie algebras. *J. Amer. Math. Soc.*, 13(3):595–609, 2000.
[15] P. I. Etingof, S. Gelaki, D. Nikshych, V. Ostrik, S. Gelaki, and D. Nikshych. *Tensor categories*, volume 205. American Mathematical Society, 2015.
[16] S. Gelaki. Semisimple triangular hopf algebras and tannakian categories, arithmetic fundamental groups and noncommutative algebra (berkeley, ca, 1999), 497–515. In *Proc. Sympos. Pure Math*, volume 70, pages 497–515, 2002.
[17] P. Guillot, C. Kassel, and A. Masuoka. Twisting algebras using non-commutative torsors: explicit computations. *Math. Z.*, 271(3-4):789–818, 2012.
[18] J. E. Humphreys. *Introduction to Lie algebras and representation theory*, volume 9. Springer Science & Business Media, 1972.
[19] J. E. Humphreys. *Linear algebraic groups*, volume 21. Springer Science & Business Media, 2012.
[20] G. Lusztig. Modular representations and quantum groups. *Contemp. Math*, 82(1080):59–78, 1989.
[21] G. Lusztig. Finite dimensional Hopf algebras arising from quantized universal enveloping algebras. *J. Amer. Math. Soc.*, 3(1):257–296, 1990.
[22] M. Mombelli Module categories over pointed Hopf algebras. *Math. Z.*, 2(266):319–344, 2010.
[23] S. Montgomery. *Hopf algebras and their actions on rings*, volume 82 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1993.
[24] C. Negron and S.-H. Ng. Gauge invariants from the powers of antipodes. *preprint* arXiv:1609.00994.
[25] S.-H. Ng and P. Schauenburg. Central invariants and higher indicators for semisimple quasi-Hopf algebras. *Trans. Amer. Math. Soc.*, 360(4):1839–1860, 2008.
[26] D. E. Radford. Minimal quasitriangular Hopf algebras. *J. Algebra*, 157(2):285–315, 1993.
[27] D. E. Radford. *Hopf Algebras*. World Scientific, 2011.
[28] K. Shimizu. The pivotal cover and Frobenius–Schur indicators. *J. Algebra*, 428:357–402, 2015.
[29] V. G. Turaev. *Quantum invariants of knots and 3-manifolds*, volume 18 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1994.
[30] D. P. Zhelobenko. Extremal cocycles of Weyl groups. *Funct. Anal. Appl.*, 21(3):183–192, 1987.

E-mail address: negronc@mit.edu

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139