513 HW1

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1 Problem 2

We need to show that the set
$$B = \{e^i f^j h^k \mid i, j, k \in \mathbb{N} \cup \{0\}\}$$
forms a basis of $U(\mathfrak{sl}_2)$. Consider an arbitrary element $a = x_1 x_2 \ldots x_k \in U(\mathfrak{sl}_2)$. We need to show that a can be written as a linear combination of elements in B. Let us use induction on k. If $k = 1$, there is nothing to show. So assume $k > 1$. Say a pair of indices (i, j) with $i < j$ is an inversion in a if one of the following three situations holds: $x_i = e$ and $x_j = f$, $x_i = e$ and $x_j = h$, $x_i = h$ and $x_j = f$. Continue by induction on the number of inversions in a. If there are no inversions, we are done (a is ordered). So fix an inversion $(j, j+1)$ (without loss of generality). Note that we can write

$$x_1 \ldots x_{i-1} x_i x_{i+1} \ldots x_k = x_1 \ldots x_{i-1} x_{i+1} x_i \ldots x_k + x_1 \ldots x_{i-1} [x_i, x_{i+1}] x_{i+2} \ldots x_k.$$

Observe that $[x_i, x_{i+1}] \in \{2e, -2e, 2f, -2f, h, -h\}$. Also, the second term in the sum has degree $k - 1$. The first term in the sum has one inversion fewer than the term on the left hand side. Done by induction.

To show linear independence, suppose that we have

$$a = \sum_{m=0}^{M} c_m e^{m_1} h^m f^{m_3} + \sum_{n_1, n_2, n_3} c_{n_1 n_2 n_3} e^{n_1} h^{n_2} f^{n_3} = 0$$

with $n_3 \geq m_3$ and $n_1 < m_1$ when $n_3 = m_3$ and that some $c_m \neq 0$.

Now consider $L(\lambda)$ for any λ such that $\lambda \geq \max(m_1, m_3)$. Choose nonzero vectors $v, v' \in L(\lambda)$ of weights $-\lambda + 2m_3$ and $-\lambda + 2m_1$, respectively. Then, for degree reasons, we have

$$a \cdot v = \left(\sum_{m=0}^{M} c_m e^{m_1} h^m f^{m_3} \right) \cdot v = \zeta \sum_{m=0}^{M} c_m (-\lambda)^m,$$

where ζ is the unique nonzero scalar such that $e^{m_1} f^{m_3} \cdot v = \zeta v'$.

On one hand, this must be 0 (by assumption), but on the other hand, $\sum_{m=0}^{M} c_m x^m$ is a polynomial of degree M, so it has finitely many zeros, but the above has to hold at infinitely many values and so we obtain a contradiction.

2 Problem 5

1. We need to show that
$$[x, y] \cdot \zeta = x \cdot (y \cdot \zeta) - y \cdot (x \cdot \zeta)$$
for all $x, y \in \mathfrak{g}$ and all $\zeta \in V^*$. For $v \in V$ we have
$$\begin{aligned} ([x, y] \cdot \zeta)(v) &= -\zeta([x, y] \cdot v) \\ &= -\zeta(x \cdot (y \cdot v) - y \cdot (x \cdot v)) \\ &= -(\zeta(x \cdot (y \cdot v)) - \zeta(y \cdot (x \cdot v)) \\ &= x \cdot (-\zeta(y \cdot v)) - y \cdot (-\zeta(x \cdot v)) \\ &= (x \cdot (y \cdot \zeta) - y \cdot (x \cdot \zeta))(v) \end{aligned} \tag{1}$$

2. Note that $W \otimes V^* \cong V^* \otimes W \cong \mathrm{Hom}_{\mathbb{C}}(V, W)$, so we need to find an isomorphism

$$\mathrm{Hom}_{\mathfrak{g}}(T \otimes V, W) \cong \mathrm{Hom}_{\mathfrak{g}}(T, \mathrm{Hom}_{\mathbb{C}}(V, W)).$$

Define maps

$$F : \mathrm{Hom}_{\mathfrak{g}}(T \otimes V, W) \to \mathrm{Hom}_{\mathfrak{g}}(T, \mathrm{Hom}_{\mathbb{C}}(V, W))$$

and

$$G : \mathrm{Hom}_{\mathfrak{g}}(T, \mathrm{Hom}_{\mathbb{C}}(V, W)) \to \mathrm{Hom}_{\mathfrak{g}}(T \otimes V, W)$$

by

$$F : \phi \mapsto \{t \mapsto \phi(t \otimes \_)\}$$

and

$$G : \psi \mapsto \{t \otimes v \mapsto \psi(t)(v)\}.$$

Then F and G are linear and are inverses of each other, so we obtain the isomorphism.