513 HW2

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1 Problem 1 (Yash)

We wish to prove that $\mathfrak{sl}_n$ is simple, meaning it contains no proper nontrivial ideals. Note that ideals of $\mathfrak{sl}_n$ correspond to subrepresentations of the adjoint representation. So let $0 \neq V \subseteq \mathfrak{sl}_n$ be a subrepresentation. Then V contains a weight vector v, which is either contained in the Cartan subalgebra $\mathfrak{h}$ or v is of the form $v = cE_{i,j}$ for some $i \neq j$ and $c \neq 0$.

First suppose $v \in \mathfrak{h}$. Then there exists simple $a \in \Delta$ such that $a(h) \neq 0$. Therefore, taking $E_{i,j} = e_a$, we obtain $[E_{i,j}, h] = -a(h)E_{i,j}$, so $E_{i,j} \in V$, returning to the other case.

So assume $E_{i,j} \in V$. Then $E_{k,j} = [E_{k,i}, E_{i,j}] \in V$ as well for any $k \neq i, j$. Hence, all elements of the form $E_{j,j} - E_{k,k} = [E_{j,k}, E_{k,j}] \in V$. But these elements form a basis of $\mathfrak{h}$, so V contains $\mathfrak{h}$ in $\mathfrak{sl}_n$. Finally, note that for any $E_{l,m}$ and $E_{m,l}$ there exists some element in $t \in \mathfrak{h}$ such that $[E_{l,m}, t] = 2E_{l,m}$ and $[E_{m,l}, t] = -2E_{m,l}$ and we conclude that V contains all the basis elements of $\mathfrak{sl}_n$, so $V = \mathfrak{sl}_n$.

2 Problem 3b (Aaron)

For any element $h \in \mathfrak{h}$, by $\mathfrak{sl}_n$-invariance we have

$$\lambda(h)\operatorname{tr}(x, y) = \operatorname{tr}([h, x], y) = -\operatorname{tr}(x, [h, y]) = -\mu(h)\operatorname{tr}(x, y).$$

Hence, non-vanishing of $\operatorname{tr}(x, y)$ forces $\lambda = -\mu$.

3 Problem 3c

Observe directly that the trace form is nonzero. The radical of the form $\operatorname{tr} : \mathfrak{sl}_n \otimes \mathfrak{sl}_n \to \mathbb{C}$ is the kernel of the corresponding morphism of $\mathfrak{sl}_n$-representations $\mathfrak{sl}_n \to \mathfrak{sl}_n^*$. Since both $\mathfrak{sl}_n$ and $\mathfrak{sl}_n^*$ are simple, it follows that this morphism is an isomorphism. Hence the radical vanishes, so the trace form is non-degenerate. By a previous part, this form necessarily restricts to a non-degenerate pairing $(\mathfrak{sl}_n(\mathbb{C})_z \otimes \mathfrak{sl}_n(\mathbb{C})_{-z} \to \mathbb{C}$ at each $z \in \Phi \cup \{0\}$. Taking $z = 0$, we see that the trace form restricts to a nondegenerate form on the Cartan.

4 Problem 3d

By Schur's lemma, there is a unique map $\mathfrak{sl}_n \to \mathfrak{sl}_n^*$ up to scaling, so by adjunction (HW1,5b) we have

$$\dim \operatorname{Hom}_{\mathfrak{sl}_n}(\mathfrak{sl}_n \otimes \mathfrak{sl}_n, \mathbb{C}) = \dim \operatorname{Hom}_{\mathfrak{sl}_n}(\mathfrak{sl}_n, \mathfrak{sl}_n^*) = 1.$$

5 Problem 4

Suppose t is semisimple. Consider a basis $\{v_i\}$ consisting of t-eigenvectors in V. Then in the corresponding basis $M_{i,j} = (v_k \to \delta_{j,k}v_i)$ of $gl(V)$, the $M_{i,j}$ are eigenvectors for the adjoint action of t with all $M_{i,i}$ of eigenvalue 0. Hence $\mathfrak{sl}_n = span\{M_{i,j}, M_{i,i} - M_{j,j}\}$ has a basis of eigenvectors for the adjoint action of t.

Conversely, suppose t is not semisimple. Consider a basis $\{v_i\}$ under which t is in Jordan normal form. Consider any ordering on the corresponding $M_{i,j}$ and $M_{i,i} - M_{i+1,i+1}$ under which $M_{i,j} < M_{k,l}$ whenever $l - k < i - j$, and $M_{j,i} < M_{k,k} - M_{k+1,k+1} < M_{i,j}$ whenever $i < j$. Then in this ordered basis ad_t is upper triangular, but not diagonal. Indeed, there is an element $h = M_{i,i} - M_{i+1,i+1}$ such that $[t, h] = cM_{i,j}$. Hence ad_t is not semisimple.