513 HW3

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1 Problem 3.2 Humphreys

First assume L is solvable. Since $[L^{(i)}, L^{(i)}] \subseteq L^{(i+1)}$, the quotient $L^{(i)}/L^{(i+1)}$ is abelian, so we may choose the derived series to be chain in question. Conversely, assume there exists such a chain. Notice that $L^{(1)} \subset L$ since L_0/L_1 abelian implies that $[L, L] \subset L_1$. In a similar way, if $L^{(i)} \subset L_i$, then $L^{(i+1)} \subset L_{i+1}$, so by induction $L^{(i)} \subset L_i$ for all i. Hence, $L^{(n)} = 0$, so L is solvable.

2 Problem 3.6 Humphreys

Suppose I, J are nilpotent ideals. By induction, one can show that

$$(I + J)^n \subseteq \sum_{i=0}^{n} I^i \cap J^{n-i},$$

where $I^0 = J^0 = L$. So clearly $I + J$ is a nilpotent ideal as well.

3 Problem 3.7 Humphreys

Since K is a subalgebra of L, we have ad_K acting on the quotient vector space L/K. Since K is a proper subalgebra, $L/K \neq 0$, so there exists some vector $v \notin K$ such that $[K, v] \subseteq K$. That is, $v \in N_L(K)$, so $K \subset N_L(K)$.

4 Problem 3.10 Humphreys

Suppose (L/K) is nilpotent, say $(L/K)^n = 0$. This means $L^n \subseteq K$. Since $\mathrm{ad}_x|_K$ is nilpotent for all $x \in L$, it follows that $\mathrm{ad}_x|_{L^n}$ is nilpotent for all $x \in L$. Therefore, by Engel's theorem, L is nilpotent.

5 Problem 4.3 Humphreys

One can check that $[x, y] = x$ and that the eigenvectors of y are the canonical basis vectors. Also, none of these is an eigenvector of x (the action of x corresponds to shifting entries).

6 Problem 5.1 Humphreys

By Corollary 3.3 in Humphreys, there exists a basis of L such that $\mathrm{ad}_L \in \mathrm{End}(L)$ consists of strictly upper triangular matrices. So for $x, y \in L$, we have ad_x and ad_y both strictly upper triangular and thus $\mathrm{ad}_x \mathrm{ad}_y$ is strictly upper triangular nilpotent. Hence, $\kappa(x, y) = \mathrm{tr}(\mathrm{ad}_x \mathrm{ad}_y) = 0$.

7 Problem 5.5 Humphreys

We have

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and

$$\mathrm{ad}_h = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \mathrm{ad}_e = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \mathrm{ad}_f = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

with respect to the basis $\{e, h, f\}$ of $\mathfrak{sl}_2$. We compute the Killing form using this. For instance

$$\kappa(e, f) = \mathrm{tr}(\mathrm{ad}_e \, \mathrm{ad}_f) = \mathrm{tr} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 4.$$

We find $\kappa(h, h) = 8, \kappa(e, f) = 4, \kappa(f, e) = 4$ and all the others are 0, so the matrix of κ with respect to this basis is

$$K = \begin{pmatrix} \kappa(e,e) & \kappa(e,h) & \kappa(e,f) \\ \kappa(h,e) & \kappa(h,h) & \kappa(h,f) \\ \kappa(f,e) & \kappa(f,h) & \kappa(f,f) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{pmatrix}.$$

So the basis dual to $\{e, h, f\}$ is given by $\{\frac{1}{4}e, \frac{1}{8}h, \frac{1}{4}f\}$.

8 Problem 5.8 Humphreys

Let $x = x_1 + \cdots + x_t$, where $x_i \in L$. Apply the Jordan-Chevalley decomposition to each x_i to write $x_i = x_i^s + x_i^n$ with x_i^s semisimple and x_i^n nilpotent. Since $[x_i, x_j] = 0$ for $i \neq j$, it follows that ad_{x_i} and ad_{x_j} commute. Thus $\mathrm{ad}_{x_i^s}$ and $\mathrm{ad}_{x_j^s}$ commute and $\mathrm{ad}_{x_i^n}$ and $\mathrm{ad}_{x_j^n}$ commute. Hence $x_1^s + \ldots x_t^s$ is semisimple and $x_1^n + \ldots x_t^n$ is nilpotent. By the uniqueness part in Jordan-Chevalley we can conclude that $x_s = x_1^s + \ldots x_t^s$ and $x_n = x_1^n + \ldots x_t^n$.